A note on Bar Induction in Constructive Set Theory*

Michael Rathjen
Department of Mathematics
The Ohio State University
Columbus, OH 43210, USA
and
School of Mathematics
University of Leeds
Leeds, LS2 9JT, UK
Email: rathjen@math.ohio-state.edu

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Abstract

Bar Induction occupies a central place in Brouwerian mathematics. This note is concerned with the strength of Bar Induction on the basis of Constructive Zermelo-Fraenkel Set Theory, **CZF**. It is shown that **CZF** augmented by decidable Bar Induction proves the 1-consistency of **CZF**. This answers a question of P. Aczel who used Bar Induction to give a proof of the Lusin Separation Theorem in the constructive set theory **CZF**.

Key words: Constructive set theory, Brouwerian principles, Bar Induction, proof-theoretic strength

1 Introduction

Intuitionistic mathematics diverges from other types of constructive mathematics in its interpretation of the continuum whose formative principles spring from Brouwer's ideas about choice sequences. The central tenets of Brouwerian mathematics are the principle of Continuous Choice (**CC**), the Fan Theorem (**FT**), and Bar Induction (**BI**).

The intent of the paper is to study Constructive Zermelo-Fraenkel Set Theory, **CZF**, augmented by Bar Induction. **CZF** is a standard reference theory for developing constructive predicative mathematics. The general topic of Constructive Set Theory originated in the seminal 1975 paper of John Myhill (cf. [12]), where a specific axiom system *CST* was introduced. Myhill developed constructive set theory with the aim of isolating the principles underlying Bishop's conception of what sets and functions are. Moreover, he wanted "these principles to be such as to make the process of formalization completely trivial, as it is in the classical case" ([12], p. 347). Indeed, while he uses other primitives in his set theory *CST* besides the notion of set, it can be viewed as a subsystem of **ZF**. The advantage of this is that the ideas, conventions and practise of the set theoretical presentation of ordinary mathematics can be employed in the set theoretical development of constructive mathematics, too. Constructive Set Theory provides a standard set theoretical framework

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for the development of constructive mathematics in the style of Errett Bishop and Douglas Bridges [6] and is one of several such frameworks for constructive mathematics that have been considered. Aczel slightly modified Myhill's **CST** and the resulting theory was called *Zermelo-Fraenkel set theory*, **CZF**. A hallmark of this theory is that it possesses a type-theoretic interpretation (cf. [1, 2, 3]). Specifically, **CZF** has a scheme called Subset Collection Axiom (which is a generalization of Myhill's Exponentiation Axiom) whose formalization was directly inspired by the type-theoretic interpretation.

It follows from [18] that the addition of **CC** and **FT** to **CZF** does not change the proof-theoretic strength. In point of fact, **CZF** + **FT** + **CC** is conservative over **CZF** with respect to Π^0_2 statements of arithmetic. By contrast, it will be shown that **CZF** augmented by decidable Bar Induction, $\mathbf{BI_D}$, is a stronger theory than **CZF** in that **CZF** + $\mathbf{BI_D}$ deduces the 1-consistency of **CZF**.

This answers a question of P. Aczel who used Bar Induction to give a proof of the Lusin Separation Theorem in the constructive set theory **CZF** (see [4]).

As it will turn, the proof that $\mathbf{BI_D}$ entails the 1-consistency of **CZF** can already be carried out on the basis of a restriction of **CZF** which has Replacement instead of Strong Collection and Exponentiation in place of Subset Collection. This system is denoted by $\mathbf{CZF}_{R,E}$. The central technical tool of this paper is Theorem 4.6, asserting that $\mathbf{CZF}_{R,E}+\mathbf{BI_D}$ proves that the well-founded part of a decidable partial ordering on $\mathbb{N}$ is a set. In conjunction with results from [17] this will establish the main claim of this paper.

2 Constructive Zermelo-Fraenkel Set Theory

We briefly recall the axioms of *Constructive Zermelo-Fraenkel Set Theory*, **CZF**.

Definition: 2.1 (Axioms of **CZF**) The language of **CZF** is the first order language of Zermelo-Fraenkel set theory, *LST*, with the non logical primitive symbol $\in$. **CZF** is based on intuitionistic predicate logic with equality. The set theoretic axioms of axioms of **CZF** are the following:

1. **Extensionality** $\forall a\,\forall b\,(\forall y\,(y\in a\leftrightarrow y\in b)\;\rightarrow\; a=b)$.
2. **Pair** $\forall a\,\forall b\,\exists x\,\forall y\,(y\in x\leftrightarrow y=a\vee y=b)$.
3. **Union** $\forall a\,\exists x\,\forall y\,(y\in x\leftrightarrow \exists z\in a\,y\in z)$.
4. **Restricted Separation scheme** $\forall a\,\exists x\,\forall y\,(y\in x\leftrightarrow y\in a\;\wedge\;\varphi(y))$,

   for every *restricted* formula $\varphi(y)$, where a formula $\varphi(x)$ is restricted, or Δ_0, if all the quantifiers occurring in it are restricted, i.e. of the form $\forall x\in b$ or $\exists x\in b$.
5. **Subset Collection scheme**

   $$\begin{array}{l}\forall a\,\forall b\,\exists c\,\forall u\,(\forall x\in a\,\exists y\in b\;\varphi(x,y,u)\;\rightarrow\\ \exists d\in c\,(\forall x\in a\,\exists y\in d\;\varphi(x,y,u)\;\wedge\;\forall y\in d\,\exists x\in a\;\varphi(x,y,u)))\end{array}$$

   for every formula $\varphi(x,y,u)$.
6. **Strong Collection scheme**

   $$\begin{array}{l}\forall a\,(\forall x\in a\,\exists y\;\varphi(x,y)\;\rightarrow\\ \exists b\,(\forall x\in a\,\exists y\in b\;\varphi(x,y)\;\wedge\;\forall y\in b\,\exists x\in a\;\varphi(x,y)))\end{array}$$

   for every formula $\varphi(x,y)$.

7. **Infinity**

$$\exists x \forall u [u \in x \;\leftrightarrow\; (0 = u \;\vee\; \exists v \in x (u = v \cup \{v\}))]$$

   where $y+1$ is $y \cup \{y\}$, and 0 is the empty set, defined in the obvious way.

8. **Set Induction scheme**

$$(IND_{\in}) \qquad \forall a\, (\forall x \in a\, \varphi(x) \;\rightarrow\; \varphi(a)) \;\rightarrow\; \forall a\, \varphi(a),$$

   for every formula $\varphi(a)$.

From Infinity, Set Induction, and Extensionality one can deduce that there exists exactly one set x such that $\forall u[u \in x \;\leftrightarrow\; (0 = u \;\vee\; \exists v \in x(u = v \cup \{v\}))]$; this set will be denoted by ω.

2.1 The strength of CZF

In what follows we shall use the notions of proof-theoretic equivalence of theories and proof-theoretic strength of a theory whose precise definitions one can find in [14]. For our purposes here we take proof-theoretic equivalence of set theories T_1 and T_2 to mean that these theories prove the same Π^0_2 statements of arithmetic and that this insight can be obtained on the basis of a weak theory such as primitive recursive arithmetic, **PRA**.

Theorem: 2.2 *Let* **KP** *be Kripke-Platek Set Theory (with the Infinity Axiom) (see [5]). Let* $\mathbf{CZF}^-$ *be* **CZF** *without Subset Collection.*

(i) **CZF** *and* $\mathbf{CZF}^-$ *are of the same proof-theoretic strength as* **KP** *and the classical theory* $\mathbf{ID_1}$ *of non-iterated positive arithmetical inductive definitions. These systems prove the same* Π^0_2 *statements of arithmetic.*

(ii) **CZF** *does not prove the Power Set axiom.*

Proof: Let **Pow** denote the Power Set axiom. (i) follows from [13] Theorem 4.14. Also (ii) follows from [13] Theorem 4.14 as one easily sees that 2-order Heyting arithmetic has a model in $\mathbf{CZF} + \mathbf{Pow}$. Since second-order Heyting arithmetic is of the same strength as classical second-order arithmetic it follows that $\mathbf{CZF} + \mathbf{Pow}$ is stronger than classical second-order arithmetic (which is much stronger than **KP**). □

3 Bar Induction

Intuitionistic mathematics diverges from other types of constructive mathematics in its interpretation of the term ‘sequence’. This led Brouwer to adopt the **principle of continuous choice**, **CC**. The other tool central to Brouwerian mathematics is the so-called **Fan Theorem**, **FT**, which is also classically valid and equivalent to König’s Lemma. Brouwer justified the Fan Theorem by appealing to his principle of **Bar Induction**.

Definition: 3.1 Let $\mathbb{N}^{\mathbb{N}}$ be the set of all functions $\alpha : \mathbb{N} \rightarrow \mathbb{N}$ and let $\mathbb{N}^*$ be the set of finite sequences of natural numbers. For $s, t \in \mathbb{N}^*$ we write $s \subseteq t$ to mean that s is an initial segment of t. A **bar of** $\mathbb{N}^*$ is subset R of $\mathbb{N}^*$ such that the following property holds:

$$\forall \alpha \in \mathbb{N}^{\mathbb{N}}\, \exists n\, \bar{\alpha}(n) \in R.$$

The bar R is **decidable** if it also satisfies

$$\forall s \in \mathbb{N}^* \, (s \in R \vee s \notin R). \tag{1}$$

$\mathbf{BI_D}$ is asserts that for every decidable bar R of $\mathbb{N}^*$ and arbitrary class Q,

$$\begin{aligned}
&\forall s \in \mathbb{N}^* \, (s \in R \to s \in Q) \; \wedge \\
&\forall s \in \mathbb{N}^* \, [(\forall k \in \mathbb{N} \, s * \langle k \rangle \in Q) \to s \in Q] \; \to \\
&\langle\rangle \in Q.
\end{aligned}$$

Monotone Bar Induction, $\mathbf{BI_M}$, asserts that for every bar R of $\mathbb{N}^*$ and arbitrary class Q,

$$\begin{aligned}
&\forall s, t \in \mathbb{N}^* \, (s \in R \to s * t \in R) \; \wedge \\
&\forall s \in \mathbb{N}^* \, (s \in R \to s \in Q) \; \wedge \\
&\forall s \in \mathbb{N}^* \, [(\forall k \in \mathbb{N} \, s * \langle k \rangle \in Q) \to s \in Q) \; \to \\
&\langle\rangle \in Q.
\end{aligned}$$

It is easy to see that $\mathbf{BI_M}$ entails $\mathbf{BI_D}$ (cf. [10], Theorem 3.7).

Corollary: 3.2 $\mathbf{BI_D}$ *implies* $\mathbf{FT_D}$ *and* $\mathbf{BI_M}$ *implies* $\mathbf{FT}$.

Proof: See [11], Ch.I,§6.10 (or exercise). □

Remark: 3.3 By 'the Bar Theorem' Brouwer actually meant the unrestricted principle of the validity of Bar Induction, which may be stated by omitting the requirement (1) that R be decidable from $\mathbf{BI_D}$. Brouwer's argument for the Bar Theorem makes no appeal to any restriction on the bar R. Kleene, however, showed that a restriction is necessary (be it in the form $\mathbf{BI_D}$ or $\mathbf{BI_M}$) lest one should forsake the cherished principle of continuous choice. More precisely, unrestricted Bar Induction implies undesirable instances of excluded middle such as

$$\forall \alpha \in \mathbb{N}^{\mathbb{N}} \; [\forall n \in \mathbb{N} \, \alpha(n) = 0 \; \vee \; \neg \forall n \in \mathbb{N} \, \alpha(n) = 0]$$

which give rise to the existence of discontinuous functions from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$ (see [11], section 7.14 and [10], pp. 68-75).

4 $\mathbf{BI_D}$ strengthens CZF

This section will show that the addition of $\mathbf{BI_D}$ to **CZF** is not a conservative extension of **CZF** with respect to Π^0_1 sentences in that we shall prove that $\mathbf{CZF} + \mathbf{BI_D}$ proves the 1-consistency of **CZF**.

To set the stage for the proof we will be needing some definitions.

Definition: 4.1 Myhill's [12] **Exponentiation Axiom** is the statement that, given two sets A and B, the class of all functions from A to B is a set. Let $\mathbf{CZF}^0$ be the system **CZF** without Subset Collection. $\mathbf{CZF}^0_R$ results from **CZF** by deleting Subset Collection and replacing Strong Collection with Replacement. Let $\mathbf{CZF}_{R,E}$ be obtained from **CZF** by replacing Strong Collection with Replacement and Subset Collection with Exponentiation, respectively.

Note that Strong Collection implies Replacement and that Subset Collection implies Exponentiation (see [3] for details). Thus, both $\mathbf{CZF}^0_R$ and $\mathbf{CZF}_{R,E}$ are subtheories of **CZF**.

Definition: 4.2 If R is a binary relation, that is a set of ordered pairs, we write xRy for $\langle x, y\rangle \in R$.

A *partial ordering* is a pair $\langle A, R\rangle$ such that R partially orders A - that is, A is a set, R is a binary relation, R is *transitive on* A: $\forall x, y, z \in A\,[xRy \ \wedge \ yRz \rightarrow xRz]$, and R is *irreflexive on* A: $\forall x \in A\ \neg xRx$.

Note that we do not assume $R \subseteq A \times A$. So if $\langle A, R\rangle$ is a partial ordering, then for any $B \subseteq A$, $\langle B, R\rangle$ is a partial ordering.

$\langle A, R\rangle$ is a *total ordering* if it is a partial ordering and *trichotomy* holds: $\forall x, y \in A\,[xRy \ \vee \ x = y \ \vee \ yRx]$.

Theorem: 4.3 ($\mathbf{CZF}^0_R$) *If $\langle A, R\rangle$ is a partial ordering then there is a smallest class X such that for all $a \in A$, whenever $\forall u \in A\,(uRa \rightarrow u \in X)$ then $a \in X$. This class will be denoted by $\mathbf{WF}(A, R)$ (the well-founded part of $\langle A, R\rangle$).*

Proof: [17] Theorem 2.6. □

Definition: 4.4 Suppose $\langle A, R\rangle$ is a partial ordering and $\phi(u)$ is a set-theoretic formula. Let

$$\mathbf{Prog}_{\langle A,R\rangle}(\phi) \quad \text{iff} \quad (\forall u \in A)\,[(\forall v \in A)\,(vRu \rightarrow \phi(v)) \rightarrow \phi(u)].$$

If X is a class $\{u \mid \phi(u)\}$ we also write $\mathbf{Prog}_{\langle A,R\rangle}(X)$ instead of $\mathbf{Prog}_{\langle A,R\rangle}(\phi)$.

Corollary: 4.5 *Let $\langle A, R\rangle$ be a partial ordering and $\phi(u)$ be a set-theoretic formula.*

(i) $(\forall x \in A)\,[x \in \mathbf{WF}(A, R) \ \leftrightarrow \ (\forall u \in A)(uRx \rightarrow u \in \mathbf{WF}(A, R))]$.

(ii) $\mathbf{Prog}_{\langle A,R\rangle}(\phi) \ \rightarrow \ (\forall x \in \mathbf{WF}(A, R))\,\phi(x)$.

Proof: (i) and (ii) are immediate consequences of Theorem 4.3. □

Theorem: 4.6 ($\mathbf{CZF}_{R,E} + \mathbf{BI_D}$) *Let $\langle A, \prec\rangle$ be a partial ordering such that $A \subseteq \mathbb{N}$, $\prec \subseteq A \times A$, and A and $\prec$ are decidable, i.e., $\forall n \in \mathbb{N}\ (n \in A \ \vee \ n \notin A)$ and $\forall n, m \in \mathbb{N}\,(n \prec m \ \vee \ n \not\prec m)$. Then $\mathbf{WF}(A, \prec)$ is a set.*

Proof: Set

$$\mathcal{X} \ = \ \{x \in A \mid \forall f \in {}^{\mathbb{N}}\mathbb{N}\,[f(0) \prec x \rightarrow \exists n \in \mathbb{N}\, f(n+1) \not\prec f(n)]\}. \tag{2}$$

Note that $\mathcal{X}$ is a set owing to Exponentiation and Bounded Separation. We claim that

$$\mathbf{WF}(A, \prec) \subseteq \mathcal{X}. \tag{3}$$

According to Corollary 4.5 (ii) it suffices to show $\mathbf{Prog}_{\langle A,R\rangle}(\mathcal{X})$. So suppose $a \in A$ and for all $u \prec a$, $u \in \mathcal{X}$. To show $a \in \mathcal{X}$ suppose $f \in {}^{\mathbb{N}}\mathbb{N}$ and $f(0) \prec a$. We have to find a $k \in \mathbb{N}$ such that $f(k+1) \not\prec f(k)$. Owing to the decidability of $\prec$ we either have $f(1) \prec f(0)$ or $f(1) \not\prec f(0)$. In the latter case choose $k = 0$. In the former case define $f^* : \mathbb{N} \rightarrow \mathbb{N}$ by $f^*(n) = f(n+1)$. As $f^*(0) = f(1) \prec f(0)$ and $f(0) \in \mathcal{X}$ there exists $m \in \mathbb{N}$ such that $f^*(m+1) \not\prec f^*(m)$. Thus with $k = m+1$ it holds that $f(k+1) \not\prec f(k)$.

As a result of the foregoing we have $a \in \mathcal{X}$.

Next we show

$$\mathcal{X} \subseteq \mathbf{WF}(A, \prec). \tag{4}$$

This part will require $\mathbf{BI_D}$. Fix $a \in \mathcal{X}$. For a finite sequence σ of natural numbers we use $S_a(\sigma)$ to convey that $\sigma = \langle\rangle$ or $\sigma = \langle \sigma_0, \ldots, \sigma_r \rangle$ with $r \geq 0$, $\sigma_0 \prec a$, and $\sigma_{i+1} \prec \sigma_i$ for all $0 \leq i < r$. Define $R_a(\sigma)$ by $\neg S_a(\sigma)$. Note that R_a is decidable because $\prec$ is decidable.

Let $f : \mathbb{N} \to \mathbb{N}$. $f(0) \not\prec a$ yields $R_a(\bar{f}(1))$. If $f(0) \prec a$ then there exists $m \in \mathbb{N}$ such $f(m+1) \not\prec f(m)$ because $a \in \mathcal{X}$; and hence $R_a(\bar{f}(m+2))$. Therefore we can conclude that

$$\forall f \in {}^{\mathbb{N}}\mathbb{N}\ \exists k \in \mathbb{N}\ R_a(\bar{f}(k)). \tag{5}$$

We say that $\mathcal{W}(\tau)$ holds if τ is of the form $\langle \tau_0, \ldots, \tau_r \rangle$ with $r \geq 0$ and $\tau_r \in \mathbf{WF}(A, \prec)$. Further define

$$Q_a(\sigma) \quad \leftrightarrow \quad R_a(\sigma)\ \vee\ \mathcal{W}(\langle a \rangle * \sigma).$$

Clearly,

$$R_a(\sigma) \quad \to \quad Q_a(\sigma). \tag{6}$$

Suppose that $\forall m \in \mathbb{N}\ Q_a(\sigma * \langle m \rangle)$. We want to show that $Q_a(\sigma)$. In view of (6) and the fact that R_a is decidable we may assume that $\neg R_a(\sigma)$. Then $\langle a \rangle * \sigma$ is of the form $\langle \tau_0, \ldots, \tau_r \rangle$ with $r \geq 0$, $\tau_0 = a$, and $\tau_0 \succ \ldots \succ \tau_r$. For all $u \prec \tau_r$ we then also have $\neg R_a(\sigma * \langle u \rangle)$ by definition of R_a. Thus, as $Q_a(\sigma * \langle m \rangle)$ holds for all m by supposition, we conclude that $\mathcal{W}(\langle a \rangle * \sigma * \langle u \rangle)$ holds for all $u \prec \tau_r$. This entails that $u \in \mathbf{WF}(A, \prec)$ for all $u \prec \tau_r$, yielding $\tau_r \in \mathbf{WF}(A, \prec)$; and consequently, $\mathcal{W}(\langle a \rangle * \sigma)$ and $Q_a(\sigma)$. As a result, we have shown that

$$\forall m \in \mathbb{N}\ Q_a(\sigma * \langle m \rangle) \quad \to \quad Q_a(\sigma). \tag{7}$$

In view of (5), (6), and (7), $\mathbf{BI_D}$ allows us to conclude that $Q_a(\langle\rangle)$, i.e. $a \in \mathbf{WF}(A, \prec)$. This completes the proof of (4).

From (3) and (4) we get $\mathcal{X} = \mathbf{WF}(A, \prec)$, showing that $\mathbf{WF}(A, \prec)$ is a set. □

Definition: 4.7 If T is a theory, the 1-*consistency of* T is the schema

$$\forall u \in \mathbb{N}[Pr_T(\ulcorner F(\dot{u}) \urcorner) \to F(u)]$$

for Σ^0_1 formulae $F(u)$ with one free variable u (for the 'dot' notation see [19], section 3.2.2).

Corollary: 4.8 $\mathbf{CZF}_{R,E} + \mathbf{BI_D}$ *proves the 1-consistency of* $\mathbf{CZF}$ *and* $\mathbf{KP}$.

Proof: From Theorem 4.6 it follows that the class **Acc** of [17] Definition 4.2 is a set. The order-type of this set is the Bachmann-Howard ordinal which is the proof-theoretic ordinal of **CZF** and **KP** (see [15] Theorem 2.1 (i)). As a result, the ordinal analyses of **CZF** and **KP** can be carried in $\mathbf{CZF}_{R,E} + \mathbf{BI_D}$, yielding the provability of the 1-consistency of these theories in $\mathbf{CZF}_{R,E} + \mathbf{BI_D}$. For more details see [17] remark 4.14. □

Another important axiom for constructive set theory is the so-called **Regular Extension Axiom**, **REA** (see [2, 15]).

Definition: 4.9 A set c is said to be *regular* if it is transitive, inhabited (i.e. $\exists u\; u \in c$) and for any $u \in c$ and set $R \subseteq u \times c$ if $\forall x \in u\, \exists y\; \langle x, y\rangle \in R$ then there is a set $v \in c$ such that

$$\forall x \in u\, \exists y \in v\; \langle x, y\rangle \in R \;\wedge\; \forall y \in v\, \exists x \in u\; \langle x, y\rangle \in R.$$

We write $\mathbf{Reg}(a)$ for 'a is regular'.

REA is the principle

$$\forall x\, \exists y\; (x \in y \;\wedge\; \mathbf{Reg}(y)).$$

It turns out that $\mathbf{BI_D}$ and $\mathbf{BI_M}$ do not engender more strength when considered on the basis of $\mathbf{CZF} + \mathbf{REA}$. Furthermore, conservativity still holds good if one also adds the axioms of Relativized Dependent Choices, **RDC**, and choice for families of sets indexed over $\mathbb{N}^{\mathbb{N}}$, dubbed $\mathbf{AC}_2$ (see [18]).

Theorem: 4.10 $\mathbf{CZF} + \mathbf{REA}$ *and* $\mathbf{CZF} + \mathbf{REA} + \mathbf{CC} + \mathbf{BI_M} + \mathbf{AC}_2 + \mathbf{RDC}$ *have the same proof-theoretic strength and prove the same* Π^0_2 *sentences of arithmetic.*

Proof: [18] Theorem 9.10. □

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