9. Integration

9.1. Standard integrals

If u is a function of x, then the **indefinite integral**

$$\int u\,dx$$

is a function whose derivative is u. The standard integrals are:
(1) If a is a constant, then

$$\int a\,dx = ax + c$$

where c is a constant, called the **constant of integration**; this means that the derivative of $ax + c$ is a.
(2) $\displaystyle\int x^n dx = \frac{x^{n+1}}{n+1} + c$ for $n \neq -1$.
For example,

$$\int \frac{1}{\sqrt{x}}\,dx = \int x^{-1/2}dx = \frac{x^{1/2}}{1/2} + c = 2\sqrt{x} + c\,.$$

(3) $\displaystyle\int \sin x\,dx = -\cos x + c$; $\displaystyle\int \cos x\,dx = \sin x + c.$
(4) $\displaystyle\int e^x dx = e^x + c.$
(5) $\displaystyle\int \frac{1}{x}\,dx = \ln x + c$, or to allow for the possibility of x being negative, $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + c.$
(6) The inverse trigonometric functions also give some integrals, like

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + c\,.$$

9.2. Basic rules

The rules for differentiation imply the following basic rules for integration.
1. If a is a constant then

$$\int au\,dx = a\int u\,dx\,;$$

for example, $\int 3\cos x\,dx = 3\int \cos x\,dx = 3\sin x + c$.
Note that you don't need to write the constant of integration until the final answer.
2.

$$\int (u \pm v)\,dx = \int u\,dx \pm \int v\,dx$$

For example

$$\int (e^x + x^3)\,dx = \int e^x\,dx + \int x^3\,dx = e^x + \frac{x^4}{4} + c.$$

EXAMPLE 9.1.

$$\int \Big(x^2 + \frac{1}{x}\Big)^2 dx = \int \Big((x^2)^2 + 2(x^2)\frac{1}{x} + (\frac{1}{x})^2\Big) dx =$$
$$\int \big(x^4 + 2x + x^{-2}\big) dx = \frac{x^5}{5} + x^2 - x^{-1} + c\,.$$

9.3. Definite integrals

The **definite integral** of $f(x)$ from a to b is denoted by

$$\int_a^b f(x)\,dx,$$

and it is equal to

$$[F(x)]_a^b = F(b) - F(a),$$

where $F(x) = \int f(x)\,dx$ is the indefinite integral.
You don't need to include the constant of integration in $F(x)$ — if you do it cancels out so **there is no constant of integration in the final answer**.

EXAMPLE 9.2.

$$\int_0^{\pi/2} \cos x\,dx = \Big[\sin x\Big]_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1.$$

EXAMPLE 9.3.

$$\int_{-2}^2 x^2 dx = \Big[\frac{x^3}{3}\Big]_{-2}^2 = \frac{2^3}{3} - \frac{(-2)^3}{3} = \frac{8}{3} - \frac{-8}{3} = \frac{16}{3}\,.$$

The **Fundamental Theorem of Calculus** says that the indefinite integral $\int_a^b f(x)\,dx$ measures the area under the curve $y = f(x)$ in the region $a \le x \le b$.

EXAMPLE 9.4. *What is the area of the triangular region under $y = mx$ between $x = 0$ and $x = b$?*
It is

$$\int_0^b mx\,dx = \Big[\frac{mx^2}{2}\Big]_0^b = \frac{mb^2}{2}\,.$$

Note that this is just half the base times the height, as expected.

9.4. Integration by substitution

To integrate $\int y\,dx$ by **substitution**, do the following:
(a) Choose a suitable substitution, say $u = g(x)$. Use it to write x in terms of u if necessary.
(b) Find dx/du, either directly, or from the formula $dx/du = 1/(du/dx)$.
(c) Rewrite the integral using the formula

$$\int y\,dx = \int y\frac{dx}{du}\,du$$

(this is valid because of the chain rule).
(d) Try to express the integrand $y(dx/du)$ as a function of u alone, not involving x, so that the integral becomes of the form $\int h(u)du$. If you can't do that, the substitution is no use.
(e) Compute the integral $\int h(u)\,du$, then use the substitution $u = g(x)$ to write it in terms of x.

EXAMPLE 9.5. *Find* $\displaystyle\int \cos 2x\,dx$ *using the substitution* $u = 2x$.

We have $x = \frac{1}{2}u$, so $dx/du = \frac{1}{2}$. Therefore

$$\int \cos 2x\,dx = \frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + c = \frac{1}{2}\sin 2x + c$$

EXAMPLE 9.6. *Find* $\displaystyle\int \frac{\sin x}{\cos x + 1}\,dx$ *using the substitution* $u = \cos x + 1$.

We have $du/dx = -\sin x$, so $dx/du = -1/\sin x$. Then

$$\int \frac{\sin x}{\cos x + 1}\,dx = -\int \frac{\sin x}{u}\,\frac{1}{\sin x}\,du = -\int \frac{1}{u}\,du = -\ln|u| + c = -\ln|\cos x + 1| + c\,.$$

EXAMPLE 9.7. *Find* $\displaystyle\int \frac{1}{\sqrt{9 - x^2}}\,dx$.

Let $x = 3u$, so that $u = x/3$. We have $dx/du = 3$, so

$$\int \frac{1}{\sqrt{9 - x^2}}\,dx = \int \frac{1}{\sqrt{9 - x^2}} 3\,du$$

$$\int \frac{1}{\sqrt{9 - 9u^2}} 3\,du = \int \frac{1}{\sqrt{1 - u^2}}\,du = \sin^{-1} u + c$$

$$= \sin^{-1}(x/3) + c.$$

EXAMPLE 9.8. *Using the substitution* $u = x^2 + 1$*, find the area under the curve* $y = x/(x^2 + 1)$ *between* $x = 0$ *and* $x = 3$.

It is

$$\int_0^3 \frac{x}{x^2 + 1}\,dx\,.$$

To find the indefinite integral

$$\int \frac{x}{x^2 + 1}\,dx,$$

use the substitution $u = x^2 + 1$. Then $du/dx = 2x$, so $dx/du = 1/(2x)$. Therefore

$$\int \frac{x}{x^2 + 1}\,dx = \int \frac{x}{x^2 + 1}.\frac{1}{2x}\,du$$

$$= \int \frac{1}{2u}\,du = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|x^2 + 1| + c\,.$$

Therefore
$$\int_0^3 \frac{x}{x^2+1}\,dx = \left[\frac{1}{2}\ln|x^2+1|\right]_0^3 = \frac{1}{2}\ln 10 - \frac{1}{2}\ln 1 = \frac{1}{2}\ln 10.$$
Special case.
If $\int f(x)\,dx = F(x) + c$, and a and b are constants, then
$$\int f(ax+b)dx = \frac{1}{a}F(ax+b) + c$$
(this follows by putting $u = ax + b$).

EXAMPLE 9.9. $\displaystyle\int \frac{1}{3x+5}dx = \frac{1}{3}\ln(3x+5) + c$

EXAMPLE 9.10. $\int (4x+1)^5 dx = \dfrac{1}{4}\,\dfrac{(4x+1)^6}{6} + c.$

9.5. Integration by parts

To integrate $\int y\,dx$ **by parts**, do the following:
(a) Write y as a product of two terms, and choose one of them that you can integrate.
(b) Let dv/dx be the term you can integrate, and let v be the integral.
(c) Let u be the other term, and compute du/dx.
(d) Rewrite the integral using the formula
$$\int y\,dx = \int u\frac{dv}{dx}\,dx = uv - \int \frac{du}{dx}v\,dx$$
(this follows from the product formula).

EXAMPLE 9.11. *Find* $\displaystyle\int x\cos x\,dx$ *by parts.*
The integrand $x\cos x$ is the product of x and $\cos x$.
We can integrate both terms, but if you choose x, it won't help — in fact, it makes the integral more complicated. Instead we choose to integrate $\cos x$.
Let $dv/dx = \cos x$, so that $v = \sin x$. (You don't need to include the $+c$ here)
Let $u = x$. Then $du/dx = 1$.
Then
$$\int x\cos x\,dx = \int u\frac{dv}{dx}\,dx = uv - \int \frac{du}{dx}v\,dx = x\sin x - \int \sin x\,dx\,.$$
We know this last integral, so we get
$$\int x\cos x\,dx = x\sin x + \cos x + c\,.$$

EXAMPLE 9.12. *Find the definite integral* $\int_0^2 (2x+1)e^{-x}dx$.
First we want the indefinite integral
$$\int (2x+1)e^{-x}dx.$$

Take $dv/dx = e^{-x}$ and $u = 2x+1$. Then $v = -e^{-x}$ and $du/dx = 2$. Therefore

$$\int (2x+1)e^{-x}dx = \int u\frac{dv}{dx}dx = uv - \int \frac{du}{dx}v\,dx$$

$$= -(2x+1)e^{-x} + 2\int e^{-x}dx = -(2x+1)e^{-x} - 2e^{-x} + c$$

$$= -(2x+3)e^{-x} + c\,.$$

Then

$$\int_0^2 (2x+1)e^{-x}dx = \left[-(2x+3)e^{-x}\right]_0^2 = 3 - 7e^{-2}.$$

EXAMPLE 9.13. *Find* $\int x^2 \sin x\,dx$.

One needs to use integration by parts twice.
Take $dv/dx = \sin x$, so $v = -\cos x$, and $u = x^2$, so $du/dx = 2x$. Then

$$\int x^2 \sin x\,dx = \int u\frac{dv}{dx}\,dx = uv - \int \frac{du}{dx}v\,dx = -x^2\cos x + \int 2x\cos x\,dx\,.$$

Now we already found

$$\int x\cos x\,dx = x\sin x + \cos x + c$$

using integration by parts, so we get

$$\int x^2\sin x\,dx = -x^2\cos x + 2x\sin x + 2\cos x + c.$$

EXAMPLE 9.14. *Find* $\int \ln x\,dx$.

This doesn't look like a product, but write it as

$$\ln x = \ln x \times 1$$

with $dv/dx = 1$ and $u = \ln x$. Then $v = x$ and $du/dx = 1/x$. Therefore

$$\int \ln x\,dx = uv - \int \frac{du}{dx}v\,dx = x\ln x - \int 1\,dx = x\ln x - x + c.$$

9.6. Integration of rational functions

To compute the integral of a rational function, convert it into partial fractions. Then use the following integrals:

(1) $\displaystyle\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$. Namely, letting $u = ax+b$ we have $\dfrac{du}{dx} = a$, so $\dfrac{dx}{du} = \dfrac{1}{a}$. Then $\displaystyle\int \frac{1}{ax+b}\,dx = \int \frac{1}{u}\times\frac{1}{a}\,du = \frac{1}{a}\ln|u| + c = \frac{1}{a}\ln|ax+b| + c.$

(2) $\displaystyle\int \frac{1}{(ax+b)^2}\,dx = -\frac{1}{a}\Big(\frac{1}{ax+b}\Big) + c$. To see this, let $u = ax+b$; then we have

$$\int \frac{1}{(ax+b)^2}\,dx = \int \frac{1}{u^2} \times \frac{1}{a}\,du = -\frac{1}{a}\Big(\frac{1}{u}\Big) + c = -\frac{1}{a}\Big(\frac{1}{ax+b}\Big) + c\,.$$

Example 9.15. *What is*

$$\int \frac{1}{(x+1)^2(x+2)}\,dx\;?$$

Write

$$\frac{1}{(x+1)^2(x+2)} = \frac{1}{(x+1)^2} - \frac{1}{x+1} + \frac{1}{x+2}\,.$$

Then

$$\begin{aligned}\int \frac{1}{(x+1)^2(x+2)}\,dx &= \int \frac{1}{(x+1)^2}\,dx - \int \frac{1}{x+1}dx + \int \frac{1}{x+2}\,dx \\ &= -\frac{1}{x+1} - \ln|x+1| + \ln|x+2| + c\,.\end{aligned}$$

To integrate rational functions with the quadratic x^2+a^2 in the denominator, use

(3) $\displaystyle\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\,;$

(4) $\displaystyle\int \frac{x}{x^2+a^2}\,dx = \frac{1}{2}\ln(x^2+a^2) + c\,.$

Example 9.16.

$$\int \frac{5x+1}{x^2+3}\,dx = 5\int \frac{x}{x^2+3}\,dx + \int \frac{1}{x^2+3}\,dx.$$

This fits the formulae with $a = \sqrt{3}$. Therefore

$$\int \frac{2x+1}{x^2+3}\,dx = \frac{5}{2}\ln(x^2+3) + \frac{1}{\sqrt{3}}\tan^{-1}\Big(\frac{x}{\sqrt{3}}\Big) + c\,.$$

To integrate rational functions with a more general quadratic in the denominator, complete the square, and use a substitution.

Example 9.17. *Find*

$$\int \frac{x+5}{x^2+6x+18}\,dx\,.$$

We complete the square:

$$x^2+6x+25 = (x^2+6x+9) + 16 = (x+3)^2 + 16$$

so

$$\int \frac{x+5}{x^2+6x+18}\,dx = \int \frac{x+5}{(x+3)^2+16}\,dx\,.$$

Now use the substitution $u = x + 3$. Then $x = u - 3$, so $dx/du = 1$. Therefore

$$\int \frac{x+5}{x^2+6x+25}\,dx = \int \frac{u+2}{u^2+16}\,du = \int \frac{u}{u^2+16}\,du + 2\int \frac{1}{u^2+16}\,du$$
$$= \frac{1}{2}\ln(u^2+16) + \frac{2}{4}\tan^{-1}(u/4) + c$$
$$= \frac{1}{2}\ln(x^2+6x+25) + \frac{1}{2}\tan^{-1}((x+3)/4) + c.$$

9.7. **Worked examples**

EXAMPLE 9.18. Calculate the indefinite integrals:

(i) $\int \tan x\,dx$

(ii) $\int x \ln x\,dx$

(iii) $\int \sin^2(x)\,dx$

(iv) $\int e^{2x}\,dx$

EXAMPLE 9.19. Calculate the integral

$$\int_1^2 \frac{1}{x(x+1)^2}\,dx\,.$$

EXAMPLE 9.20. Calculate the integral

$$\int_0^{\pi/4} \frac{\cos x - \sin x}{\cos x + \sin x}\,dx\,.$$