3. Coordinate geometry in the (x, y)-plane

3.1. Lines

Recall that the **gradient** or **slope** of a line is

$$m = \frac{\text{change in } y}{\text{change in } x}.$$

If the line passes through the points (x_1, y_1) and (x_2, y_2) then

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Negative gradient means that it slopes down.

If a line passes through (x_1, y_1) and has slope m, then a general point (x, y) satisfies $m = (y - y_1)/(x - x_1)$, or $y - y_1 = m(x - x_1)$. You can write this as

$$y = mx + c$$

where $c = y_1 - mx_1$. This is the general equation of a straight line. If we put $x = 0$ then $y = c$, so the line meets the y-axis at $(0, c)$; the number c is called the **intercept** (on the y-axis).

3.2. Special cases

The line through $(1, 1)$ and $(3, 1)$ has gradient $(1 - 1)/(3 - 1) = 0$, that is, it is horizontal. It has equation $y = 0x + 1$, that is, $y = 1$.
The line through $(3, 1)$ and $(3, 6)$ has gradient $(6 - 1)/(3 - 3) = \infty$, that is, it is vertical. Vertical lines cannot be written in the form $y = mx + c$, but rather in the form $x =$ constant. Here the equation is $x = 3$.

3.3. Parallel and perpendicular lines

Given two lines with gradients m_1, m_2, they are
parallel if $m_1 = m_2$
perpendicular if $m_1 m_2 = -1$.
For example, if $m_1 = 2$, rotate the picture a right angle to get $m_2 = -1/2$.

Example 3.1. Let L be the line with equation $y = 2x + 1$.
Find the equation of the line parallel to L which passes through $(2, 1)$.
It has slope 2, so the equation is $y - 1 = 2(x - 2)$, i.e. $y = 2x - 3$.
Find the equation of the line perpendicular to L which passes through $(4, 3)$.
It has slope $-1/2$, so the equation is $y - 3 = -\frac{1}{2}(x - 4)$, i.e. $y = -\frac{1}{2}x + 5$.

3.4. **Distance**

By Pythagoras's Theorem, the distance between two points (x_1, y_1) and (x_2, y_2) is

$$\text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

For example, the distance between $(1, 2)$ and $(4, 6)$ is

$$\sqrt{(4-1)^2 + (6-2)^2} = \sqrt{9 + 16} = \sqrt{25} = 5.$$

3.5. **Circles**

Consider the circle with centre (a, b) and radius r. If a point (x, y) is on the circle, then the distance between (x, y) and (a, b) is equal to r. Therefore

$$(x - a)^2 + (y - b)^2 = r^2.$$

This is the equation of a circle. Expanding, it becomes

$$x^2 - 2ax + a^2 + y^2 - 2by + b^2 = r^2$$

so

$$x^2 + y^2 - 2ax - 2by + c = 0$$

where $c = a^2 + b^2 - r^2$.

Example 3.2. The circle with centre $(2, -3)$ and radius 5 has equation

$$(x - 2)^2 + (y + 3)^2 = 25,$$

or equivalently

$$x^2 + y^2 - 4x + 6y - 12 = 0.$$

Example 3.3. A circle has equation $x^2 + y^2 + 2x - 6y + 6 = 0$. Find its centre and radius.

Write it as

$$x^2 + 2x + y^2 - 6y = -6.$$

Now complete the square:

$$x^2 + 2x + 1 + y^2 - 6y + 9 = -6 + 1 + 9 = 4.$$

We can write this as

$$(x + 1)^2 + (y - 3)^2 = 4.$$

Therefore the centre is at $(-1, 3)$ and the radius is 2.

Example 3.4. Find the points where the line with equation $y = 3x - 4$ meets the circle with centre $(2, -3)$ and radius 5.

The circle has equation $(x - 2)^2 + (y + 3)^2 = 25.$

Solve these simultaneous equations by substitution.

$$\begin{aligned}(x-2)^2+((3x-4)+3)^2 &= 25\\(x-2)^2+(3x-1)^2 &= 25\\x^2-4x+4+9x^2-6x+1 &= 25\\10x^2-10x+5 &= 25\\10x^2-10x-20 &= 0\\x^2-x-2 &= 0\\(x+1)(x-2) &= 0.\end{aligned}$$

Therefore $x=-1$ or $x=2$.
If $x=-1$ then $y=3x-4=-7$.
If $x=2$ then $y=3x-4=2$.
The points of intersection are $(-1,-7)$ and $(2,2)$.

NOTE. A line may meet a circle at 0, 1 or 2 points. If it meets a circle at only one point A, it means that it is tangent to the circle at that point. Another condition for tangency is that the line is perpendicular to the radius CA, where C is the centre of the circle.

3.6. Different types of equations for curves

A curve may be specified by a Cartesian equation or by parametric equations.

(a) **Cartesian equation**. The curve is given by an equation of the form $h(x,y)=0$ for some function $h(x,y)$.
For example a straight line is given by $y=mx+c$ (so $y-mx-c=0$), and a circle is given by $x^2+y^2-2ax-2by+c=0$.
It may be difficult to draw a curve given by such an equation, unless you can rearrange it in the form $y=f(x)$ or $x=f(y)$.

EXAMPLE 3.5. Plot the curve given by the equation

$$x^2-y^2=1.$$

Solve for y giving $y^2=x^2-1$, so that $y=\pm\sqrt{x^2-1}$. Can you plot this?

(b) **Parametric equations**. The curve is given by two equations, $x=f(t)$ and $y=g(t)$, for suitable functions $f(t)$ and $g(t)$. These give the coordinates of any point of the curve in terms of a new independent variable, t, called the **parameter**.
To plot the curve, calculate $f(t)$ and $g(t)$ for a range of values of t, and plot the corresponding points.
Alternatively, try to eliminate t.

EXAMPLE 3.6. Show that the curve given by the parametric equations $x=2-2t$, $y=3+7t$ is a line, and find its slope.

We eliminate t: from the first equation, $2t = 2 - x$, so $t = (2-x)/2$, so $y = 3 + 7(2-x)/2$, so $y = -\frac{7}{2}x + 10$. This is the equation of a line with slope $-7/2$.

3.7. **Worked examples**

Example 3.7. What is the slope of the line $3x + 2y + 5 = 0$, and where does it meets the y-axis?

Example 3.8. The point A has coordinates $(7, 4)$. The straight lines with equations $x + 3y + 1 = 0$ and $2x + 5y = 0$ intersect at the point B. Show that one of these two lines is perpendicular to AB.

Example 3.9. The line joining the points A $(0, 5)$ and B $(4, 1)$ is tangent to a circle whose centre C is at the point $(5, 4)$.

(a) Find the equation of the line AB.
(b) Find the equation of the line through C which is perpendicular to AB.
(c) Find the coordinates of the point of contact of the line AB with the circle.
(d) Find the equation of the circle.

Example 3.10. Find the Cartesian equation for the curve given by the parametric equations $x = 2\cos\theta$, $y = \sin\theta$.
[Hint. Eliminate θ by using the identity $\sin^2\theta + \cos^2\theta = 1$.]

Example 3.11. Sketch the curve given by the parametric equations $x = t^2$, $y = 2t$, and find a Cartesian equation for it.

Example 3.12. Find the centre and radius of the circle with equation $x^2 + y^2 = 6x$. The line $x + y = k$ is a tangent to this circle. Find the two possible values of the constant k.