## 10. **Ordinary Differential Equations**

A **first order differential equation** for $y$ is one only involving $x$, $y$ and $dy/dx$, and no higher derivatives: $d^2y/dx^2$, etc.
A **first order differential equation with separable variables** is one of the form

$$dy/dx = f(x)g(y)\,.$$

To find the general solution, collect the terms involving $x$ on one side, those involving $y$ on the other and integrate

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx.$$

EXAMPLE 10.1. *Find the general solution of the differential equation*

$$\frac{dy}{dx} = (y^2+1)x.$$

Separate the variables:

$$\int \frac{dy}{y^2+1} = \int x\,dx.$$

Putting the constant of integration on the right hand side, this gives

$$\tan^{-1} y = \frac{1}{2}x^2 + c.$$

Therefore

$$y = \tan\left(\frac{1}{2}x^2 + c\right)$$

for some constant $c$.

EXAMPLE 10.2. *Find the general solution of the differential equation*

$$\frac{dy}{dx} = y \sin x.$$

Separate the variables:

$$\int \frac{dy}{y} = \int \sin x\,dx$$

Putting the constant of integration on the RHS this gives

$$\ln y = -\cos x + c\,.$$

Therefore,

$$y = e^{-\cos x + c} = e^c e^{-\cos x}.$$

It is neater to rewrite the constant $e^c$ as a new constant, say $C$. Then

$$y = Ce^{-\cos x}.$$

**Particular solutions**. One often wants to pick out a solution satisfying an extra condition, for example the value of $y$ when $x = 0$ (which is called an **initial condition**).
To find a particular solution, first find the general solution, involving a constant, and then use the extra condition to determine the constant.

EXAMPLE 10.3. *Find the general solution of the differential equation*

$$dy/dx = x^2/y$$

*and the particular solution with $y = 2$ when $x = 0$.*
Separate the variables:

$$\int y\,dy = \int x^2\,dx, .$$

Integrating gives

$$y^2/2 = x^3/3 + c$$

so that

$$y^2 = \frac{2}{3}x^3 + C$$

where $C = 2c$, so

$$y = \sqrt{\frac{2}{3}x^3 + C}\,.$$

This is the general solution. For the particular solution with $y = 2$ when $x = 0$, we want $2 = \sqrt{C}$, so $C = 4$. Therefore

$$y = \sqrt{\frac{2}{3}x^3 + 4}\,.$$

EXAMPLE 10.4. *Find the general solution of the differential equation*

$$dy/dx = y(1-y)$$

*and the particular solution with $y = \frac{1}{4}$ when $x = 0$.*
Separate the variables:

$$\int \frac{1}{y(1-y)}dy = \int 1\,dx\,.$$

Write as partial fractions:

$$\frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y}\,.$$

Integrating, gives

$$\ln y - \ln(1-y) = x + c\,.$$

Therefore

$$\ln\left(\frac{y}{1-y}\right) = x + c\,.$$

Therefore

$$\frac{y}{1-y} = e^{x+c}.$$

It is neater to write this as

$$\frac{y}{1-y} = Ce^x$$

where $C = e^c$.
Now make $y$ the subject of this formula:

$$y = Ce^x(1-y) = Ce^x - Ce^x y$$

so

$$y(1 + Ce^x) = Ce^x$$

hence

$$y = \frac{Ce^x}{1 + Ce^x}\,.$$

This is the general solution.
Now suppose $y = \frac{1}{4}$ when $x = 0$. Then $\frac{1}{4} = C/(1+C)$. Therefore $\frac{1}{4}(1+C) = C$, so $\frac{3}{4}C = \frac{1}{4}$ and hence $C = 3$. Therefore the particular solution is

$$y = \frac{3e^x}{1 + 3e^x}.$$

## 10.1. **Worked examples**

EXAMPLE 10.5. Find the general solution to the differential equation

$$y' = y^2 x e^x\,;$$

then find the particular solution with $y = -1$ when $x = 1$.