An ordinal analysis for theories of self-referential truth

Graham Emil Leigh and Michael Rathjen*

December 19, 2009

Abstract

The first attempt at a systematic approach to axiomatic theories of truth was undertaken by Friedman and Sheard [11]. There twelve principles consisting of axioms, axiom schemata and rules of inference, each embodying a reasonable property of truth were isolated for study. Working with a base theory of truth conservative over PA, Friedman and Sheard raised the following questions. Which subsets of the Optional Axioms are consistent over the base theory? What are the proof-theoretic strengths of the consistent theories?

The first question was answered completely by Friedman and Sheard; all subsets of the Optional Axioms were classified as either consistent or inconsistent giving rise to nine maximal consistent theories of truth. They also determined the proof-theoretic strength of two subsets of the Optional Axioms. The aim of this paper is to continue the work begun by Friedman and Sheard. We will establish the proof-theoretic strength of all the remaining seven theories and relate their arithmetic part to well-known theories ranging from PA to the theory of Σ_1^1 dependent choice.

MSC: 03F03, 03F25, 03F05, 03B42

Key words: theories of truth, Relative consistency and interpretations, ordinal analysis, cut elimination

1 Introduction

In the face of the paradoxes, there are three possible routes that have been taken to develop a consistent theory of truth, namely by restriction of language, logic, or truth principles. The first is exemplified by Tarski's theory of hierarchical truth [29]. Examples for the second are logics based on more than two truth values, partial logics or paraconsistent logics. Feferman has raised concerns over the adoption of non-standard logics, i.e. logics other than classical or intuitionistic logic, for it has not been established whether "sustained ordinary reasoning can be carried out" ([9, p. 95]) in such logics, not only in mathematics and the

*Research of the first author was partially supported by a University of Leeds Research Scholarship and the Marie Curie Early Stage Training Network MATHLOGAPS (MEST-CT-2004- 504029). Research of the second author was supported by a Royal Society International Joint Projects award 2006/R3. The second author would like to thank the *Swedish Collegium for Advanced Study* in Uppsala for providing an excellent research environment for the completion of this paper.

sciences but also in everyday reasoning. This paper will be concerned with the third type of theories, more precisely non-hierarchical axiomatic theories of truth which intend to capture the notion of truth within one language. They allow for self-referential applications of a truth predicate T while assuming that the underlying logic is classical logic.[1] Of course, the axiomatic approach does not commit us to the interpretation of 'T' as truth. "One may reasonably take the attitude that what we are really exploring here is the axiomatic properties of concepts which lie somewhere between 'provability' (which is well understood but somehow insufficient) and full 'truth' (which is mysterious and perhaps inherently unstable). Possible interpretations of T might be 'intuitively provable' or 'knowably true'." ([11, p. 3]).

All axiomatic truth theories considered here will be extensions of Peano Arithmetic, PA. The addition of a truth predicate 'T' to the language of PA, together with a menu of reasonable axioms about T, but avoiding the faulty $A \leftrightarrow T(\ulcorner A \urcorner)$. As in [11], we take a philosophically neutral standpoint and are not concerned with how well each theory resembles our intuitive understanding of truth, but instead how accepting the various axioms of truth enriches the theory we work in.[2] Theories of truth will be compared directly with some well known mathematical systems to reveal possibly surprising connections.

The first attempt at a systematic approach to axiomatic theories of truth was undertaken by Friedman and Sheard [11]. There twelve principles (called Optional Axioms) consisting of axioms, axiom schemata and rules of inference, each embodying a reasonable property of truth were isolated for study. Working with a base theory of truth conservative over PA, Friedman and Sheard raised the following questions. Which subsets of the Optional Axioms are consistent over the base theory? What are the proof-theoretic strengths of the consistent theories?

The first question was answered completely by Friedman and Sheard; all subsets of the Optional Axioms were classified as either consistent or inconsistent giving rise to nine maximal consistent theories of truth (Theorem 2.6). Section 7 of [11] provides the beginnings of an answer to their second question. They determined the proof-theoretic strength of two subsets of the Optional Axioms showing one proves the same arithmetical consequences as ID_1, the theory of one inductive definition. This work was subsequently extended by Cantini [6] and Halbach [14] with the result that the proof-theoretic strength of two of the nine theories was settled.

The aim of this paper is to continue the work begun by Friedman and Sheard. We will establish the proof-theoretic strength of the remaining seven theories. For three this will be simply a matter of formalising the model constructions found in [11]; however, in order to provide upper bounds for the other four we shall need to employ the techniques of infinitary proof theory, namely perform cut elimination for an infinitary proof system in which our finite theories of truth can be embedded. We show that the strengths of the remaining theories vary from being a conservative extension of PA to the strength of the Σ^1_1 dependent choice axiom, i.e., the theory $\Sigma^1_1\text{-DC}_0$.

[1]The other canonical choice would have been intuitionistic logic. It seems that nobody has explored this natural route so far.

[2]Some of the truth theories can also be viewed as reflective closures of PA by means of a truth predicate the aim of which is to spell out what principles one might want to accept after a certain reflective process if one accepts the basic notions and principles of PA.

We shall assume the reader is familiar with the notation and definitions of [11].

1.1 Preliminaries

Let $\mathcal{L}$ be the language of Peano Arithmetic, $\mathcal{L}_T$ be $\mathcal{L}$ augmented by an additional predicate symbol 'T', and $\mathcal{L}^2$ be the language of second-order arithmetic. We use the symbols A, B, etc. to range over formulae and s, t, etc., to range over terms which are built up in the usual way. n, m, k, l, etc. range over natural numbers. If n is a natural number then we denote by $\bar{n}$ the corresponding numeral. An $\mathcal{L}_T$-formula A is called *arithmetical* if $A \in \mathcal{L}$, that is, if A does not contain the predicate T.

We may assume that we have a fixed primitive recursive Gödel numbering $\ulcorner . \urcorner$ of terms and formulae of $\mathcal{L}_T \cup \mathcal{L}^2$, a pairing function $\langle . \rangle : \mathbb{N}^2 \to \mathbb{N}$, with projections $\pi_k : \mathbb{N} \to \mathbb{N}$ for $k < 2$ such that $\langle \pi_0(x), \pi_1(x) \rangle = x$. We extend the pairing function and its inverses to encode sequences of natural numbers as follows. Define $\langle \rangle = 0$, let $\langle x_0, \ldots, x_n \rangle$ denote $\langle x_0, \langle x_1, \ldots, x_n \rangle \rangle$, and let $(\langle x_0, \ldots, x_n \rangle)_k = x_k$ if $k \leq n$. We may also define a primitive recursive function lh so that $lh(\langle x_0, \ldots, x_n \rangle) = n + 1$.

Furthermore, we may assume we have, for each logical connective and the predicate T, a primitive recursive function symbol, **imp**, **neg**, **T**, etc., such that for all formulae A, B, $\mathbf{imp}(\ulcorner A \urcorner, \ulcorner B \urcorner) = \ulcorner A \to B \urcorner$, $\mathbf{neg}(\ulcorner A \urcorner) = \ulcorner \neg A \urcorner$, $\mathbf{T}(x) = \ulcorner T(x) \urcorner$, etc.. It is assumed that the above functions (except for **T**) will only output codes for formulae if all of the arguments are codes of formulae; for example, $\mathbf{imp}(x, y)$ is the code of an $\mathcal{L}_T$-formula only if both x and y are codes for $\mathcal{L}_T$-formulae. Also present is a function **sub** such that, for each term t and formula $A(x)$ with at least x free, $\mathbf{sub}(\ulcorner A(x) \urcorner, \ulcorner x \urcorner, \ulcorner t \urcorner) = \ulcorner A(t) \urcorner$, that is, the Gödel number of the formula obtained from A by replacing each occurrence of the free variable x by the term t. As with the previous functions it is assumed that **sub** is defined so that if either x is not the Gödel number of a formula, y is not the code of a variable or z is not the code of a term of the language then $\mathbf{sub}(x, y, z)$ is not the code of an $\mathcal{L}_T$-formula. We use the convention that $\ulcorner A(\dot{y}) \urcorner$ abbreviates $\mathbf{sub}(\ulcorner A(x) \urcorner, \ulcorner x \urcorner, \mathbf{num}(y))$, where $\mathbf{num}(y)$ is the Gödel number of $\bar{y}$. Finally **ucl** is a primitive recursive function which takes the Gödel number of a formula and outputs the Gödel number of its universal closure. For more details we refer the reader to [27].

Let $\mathsf{Sent}_{\mathcal{L}}(x)$ denote that 'x is the Gödel number of a sentence of the language $\mathcal{L}$' and $\mathsf{Prov}_S(x)$ denote that 'x is the Gödel number of a formula provable in the theory S'.

We append '.' to a quantifier to denote it having the largest possible scope. For example, $\forall x\, A \to B$ is to be read as $(\forall x\, A) \to B$ whereas $\forall x.\ A \to B$ is to read as $\forall x (A \to B)$.

Let Δ^0_0 be the smallest set containing the atomic formulae in $\mathcal{L}$ and closed under logical connectives and bounded quantification, that is if $A \in \Delta^0_0$ then $(\forall x.\ x < t \to A(x)) \in \Delta^0_0$ and $(\exists x.\ x < t \wedge A(x)) \in \Delta^0_0$ for any term t of the language. Π^0_0 and Σ^0_0 also denote the set Δ^0_0. Define recursively the sets Π^0_{n+1} and Σ^0_{n+1} by $A \in \Pi^0_{n+1}$ (Σ^0_{n+1}) if A is of the form $\forall x\ A_0(x)$ ($\exists x\ A_0(x)$) for some formula $A_0 \in \Sigma^0_n$ (Π^0_n).

2 Theories of truth

2.1 A base for truth

We denote by PA_T the theory PA formulated in the language $\mathcal{L}_T$. That is, with the induction schema extended to all formulae of $\mathcal{L}_T$.

2.1 Definition *Let $Base_T$ be the theory containing PA_T with the additional axioms*

$$\forall x \forall y.\ (T(x) \wedge T(\mathbf{imp}(x,y))) \to T(y) \tag{1}$$

$$\forall x.\ \mathsf{val}(x) \to T(\mathbf{ucl}(x)) \tag{2}$$

$$\forall x.\ \mathsf{Ax}_{\mathrm{PRA}}(x) \to T(\mathbf{ucl}(x)) \tag{3}$$

where $\mathsf{val}(x)$ expresses 'x is the Gödel number of a logically valid (in first order predicate logic) formula of $\mathcal{L}_T$' and $\mathsf{Ax}_{\mathrm{PRA}}(x)$ denotes that x is (the Gödel number of) an axiom of PRA*, the theory of primitive recursive arithmetic, that is x is the code of a true equation involving constants, free variables and primitive recursive function symbols.*

2.2 Proposition *Suppose $Base_T \vdash T\ulcorner A_1 \wedge A_2 \wedge \ldots \wedge A_n\urcorner$ and that B follows from $A_1, A_2, \ldots, A_n$ in predicate logic. Then $Base_T \vdash T\ulcorner B\urcorner$.*

Proof Since B follows from $A_1, A_2, \ldots, A_n$ in predicate logic it follows that $(A_1 \wedge A_2 \wedge \cdots \wedge A_n) \to B$ is a tautology. Thus Axiom 2 implies $Base_T \vdash T\ulcorner(A_1 \wedge A_2 \wedge \ldots \wedge A_n) \to B\urcorner$ from which Axiom 1 yields the required conclusion. ■

It is preferable to introduce some notation for the purpose of quantifying over (Gödel numbers of) formulae and sentences.

2.3 Notation *Let $\forall\ulcorner A\urcorner B(\ulcorner A\urcorner)$ and $\exists\ulcorner A\urcorner B(\ulcorner A\urcorner)$ be abbreviations for the formulae $\forall y.\ \mathsf{Sent}_{\mathcal{L}_T}(y) \to B(y)$ and $\exists y.\ \mathsf{Sent}_{\mathcal{L}_T}(y) \wedge B(y)$ respectively. We also have a need to quantify over formulae with free variables for which we introduce $\forall\ulcorner A(x)\urcorner B(\ulcorner A\urcorner)$ to denote the formula $\forall y.\ \mathsf{Sent}_{\mathcal{L}_T}(\mathbf{sub}(y, \ulcorner x\urcorner, \ulcorner 0\urcorner)) \to B(y)$ and $\exists\ulcorner A(x)\urcorner B(\ulcorner A\urcorner)$ to denote $\exists y.\ \mathsf{Sent}_{\mathcal{L}_T}(\mathbf{sub}(y, \ulcorner x\urcorner, \ulcorner 0\urcorner)) \wedge B(y)$ respectively.*

2.2 Axioms of truth

We now consider possible extensions to our base theory. As remarked before we will work with a list of twelve axioms, axiom schemata and rules of inference as proposed in [11]. These Optional Axioms, as they shall be referred to, are listed in Table 1 below. Each Optional Axiom embodies some intuitive property that one may desire our interpretation of truth to satisfy.

2.4 Remark *First, a note on T-Completeness: We could take in its place the unrestricted (and arguably more natural) version $\forall x. T(x) \vee T(\mathbf{neg}(x))$, and suppose that if x is not the code of an $\mathcal{L}_T$ sentence then $\mathbf{neg}(x) = \ulcorner \bar{0} = \bar{0}\urcorner$. This would not affect any of the results involving T-Completeness (both here and in [11]), however, for the purposes of Section 3 this definition of **neg** would unnecessarily complicate matters and so it is preferable to take T-Completeness in the form given in Table 1.*

Second, the rules of inference stated above are given in parametric form, however for the purposes of our investigation they could just as well be given in

Name	Axiom Schemata
T-In	$\forall x.\ A(x) \rightarrow T\ulcorner A(\dot{x})\urcorner$
T-Out	$\forall x.\ T\ulcorner A(\dot{x})\urcorner \rightarrow A(x)$

Name	Axioms
T-Rep(etition)	$\forall x.\ T(x) \rightarrow T\ulcorner T(\dot{x})\urcorner$
T-Del(etion)	$\forall x.\ T\ulcorner T(\dot{x})\urcorner \rightarrow T(x)$
T-Cons(istency)	$\forall x \neg(T(x) \wedge T(\mathbf{neg}(x)))$
T-Comp(leteness)	$\forall\ulcorner A\urcorner.\ T\ulcorner A\urcorner \vee T\ulcorner\neg A\urcorner$
$\forall$-Inf(erence)	$\forall\ulcorner A(x)\urcorner.\ \forall n T\ulcorner A(\dot{n})\urcorner \rightarrow T\ulcorner\forall x\, A(x)\urcorner$
$\exists$-Inf(erence)	$\forall\ulcorner A(x)\urcorner.\ T\ulcorner\exists x\, A(x)\urcorner \rightarrow \exists n\, T\ulcorner A(\dot{n})\urcorner$

Name	Rules of inference
T-Intro(duction)	From $A(x)$ infer $T\ulcorner A(\dot{x})\urcorner$
T-Elim(ination)	From $T\ulcorner A(\dot{x})\urcorner$ infer $A(x)$
$\neg$T-Intro(duction)	From $\neg A(x)$ infer $\neg T\ulcorner A(\dot{x})\urcorner$
$\neg$T-Elim(ination)	From $\neg T\ulcorner A(\dot{x})\urcorner$ infer $\neg A(x)$.

Table 1: List of Optional Axioms

closed form (e.g. from A infer $T\ulcorner A\urcorner$). For T-Introduction, $\neg$T-Introduction and T-Elimination this is a given by the converses to $\forall$-Inference and $\exists$-Inference being provable in $Base_T$ (Lemma 2.24) and (in the case of T-Elimination) since all the theories we consider that extend $Base_T$ will comprise $\forall$-Inference. $\neg$T-Elimination is derivable from its nonparametric form and $\exists$-Inference, but not all the theories we consider closed under $\neg$T-Elimination will contain $\exists$-Inference. However we find that those problematic theories are, in fact, vacuously closed under $\neg$T-Elimination, and thus the closed and parametric forms are equivalent.

2.5 Notation *If Γ is a set of formulae and $R_0, \ldots, R_n$ a collection of Optional Axioms the theory $\Gamma + R_0 + \cdots + R_n$ is defined as theory containing $\Gamma \cup \{R_i : R_i$ is an axiom$\}$ and closed under all rules R_j and natural deduction simultaneously.*

It should be noted that under this definition the theory $\Gamma + R + S$ need not coincide with $(\Gamma + R) + S$. This may occur if R is a rule of inference and S is an axiom. A set of Optional Axioms $\Delta = \{D_0, \ldots, D_n\}$ is said to be *consistent* over the theory S if the theory $S + D_0 + \cdots + D_n$ is consistent and *inconsistent* otherwise.

In [11] all subsets of the Optional Axioms were classified as either consistent or inconsistent over the base theory $Base_T$:

2.6 Theorem *(Friedman and Sheard, [11]) The following are the only maximal consistent subsets of the Optional Axioms, over $Base_T$.*

$\mathcal{A}$. *T-In, T-Del, $\forall$-Inf, $\exists$-Inf, T-Intro, T-Rep, $\neg$T-Elim, T-Comp.*

$\mathcal{B}$. *T-Rep, T-Cons, T-Comp, $\forall$-Inf, $\exists$-Inf.*

$\mathcal{C}$. *T-Del, T-Cons, T-Comp, $\forall$-Inf, $\exists$-Inf.*

$\mathcal{D}$. *T-Intro, T-Elim, T-Cons, T-Comp, $\forall$-Inf, $\exists$-Inf, $\neg$T-Elim, $\neg$T-Intro.*

$\mathcal{E}$. *T-Intro, T-Elim, T-Del, $\forall$-Inf, T-Cons, $\neg$T-Intro.*

$\mathcal{F}$. *T-Intro, T-Elim, T-Del, $\forall$-Inf, $\neg$T-Elim.*

$\mathcal{G}$. *T-Intro, T-Elim, T-Rep, $\forall$-Inf, $\neg$T-Elim.*

$\mathcal{H}$. *T-Out, T-Rep, $\forall$-Inf, T-Elim, T-Del, $\neg$T-Intro, T-Cons.*

$\mathcal{I}$. *T-Rep, T-Del, T-Elim, $\forall$-Inf, $\neg$T-Elim.*

Although in [11] the question of consistency was answered, the proof-theoretic strength of these theories remained largely unknown. Friedman and Sheard showed that the theory $\mathcal{H}$ without T-Repetition has the strength of one inductive definition and later Cantini [6] has shown that the inclusion of this axiom provides no additional strength. In addition, the theory $\mathcal{D}$ is known to have the same arithmetical theorems as the system $\mathrm{RA}_{<\omega}$ [14]. We will now consider each of the remaining theories in turn and analyse them for their proof-theoretic strength.

2.3 $\mathcal{A}$: T-In, T-Del, $\forall$-Inf, $\exists$-Inf, T-Intro, T-Rep, $\neg$T-Elim, T-Comp

Define an interpretation $^*\colon \mathcal{L}_T \to \mathcal{L}$ as follows

$$\begin{aligned} p^* &= p \quad \text{for } p \text{ an atomic formula.} \\ (\neg A)^* &= \neg(A^*) \\ (A \square B)^* &= A^* \square B^* \quad \text{for } \square \in \{\vee, \wedge\} \\ (QxA(x))^* &= Qx(A(x))^* \quad \text{for } Q \in \{\forall, \exists\} \\ (T(s))^* &= (s = s). \end{aligned}$$

2.7 Proposition *For any formula C if $\mathcal{A} \vdash C$ then* $\mathrm{PA} \vdash C^*$.

Proof By induction on the length of proof of C. If C is an axiom of the system $\mathcal{A}$, then C is either an axiom of $Base_T$, one of T-In, T-Deletion, T-Completeness $\forall$-Inference, $\exists$-Inference, or already an axiom of PA. In each of these cases it is clear that $\mathrm{PA} \vdash C^*$. If C is not an axiom then there must be a formula B such that $\mathcal{A} \vdash B$ and $\mathcal{A} \vdash B \to C$. Then by the induction hypothesis (and definition of *) $\mathrm{PA} \vdash B^* \wedge (B^* \to C^*)$, and hence by modus ponens $\mathrm{PA} \vdash C^*$, as required. ∎

2.8 Corollary $\mathcal{A}$ *is a conservative extension of* PA. *Moreover this can be established in* PA.

Proof If B is a formula in the language of PA then $B^* = B$. Thus by Proposition 2.7 any arithmetical theorem of $\mathcal{A}$ is a theorem of PA. The fact that this can be carried out within PA is a consequence of the fact that the interpretation * is primitive recursive allowing the argument in Proposition 2.7 to be carried out within PA. ∎

2.4 $\mathcal{B}$: T-Rep, T-Cons, T-Comp, $\forall$-Inf, $\exists$-Inf and $\mathcal{C}$: T-Del, T-Cons, T-Comp, $\forall$-Inf, $\exists$-Inf

2.9 NOTATION *Let* S_0 *denote the theory* $Base_T + \forall$-Inf. *Also let* $\mathsf{Con}(S)$ *denote the sentence* $\neg\mathsf{Prov}_S(\ulcorner\bar{0} = \bar{1}\urcorner)$.

2.10 PROPOSITION $Base_T + \text{T-Cons} \vdash \forall\ulcorner A\urcorner\, \neg T\ulcorner A \wedge \neg A\urcorner$.

PROOF As a consequence of Proposition 2.2 $Base_T \vdash \forall\ulcorner A\urcorner\forall\ulcorner B\urcorner.\, T\ulcorner A \wedge B\urcorner \leftrightarrow T\ulcorner A\urcorner \wedge T\ulcorner B\urcorner$. We now argue informally within $Base_T$. Suppose otherwise, so $T\ulcorner A \wedge \neg A\urcorner$ for some sentence A. Then we have $T\ulcorner A\urcorner \wedge T\ulcorner\neg A\urcorner$, contradicting T-Consistency. ■

2.11 PROPOSITION *(i). For all arithmetical formulae* $A(x_1, \ldots, x_r)$ *whose free variables are among those exhibited,*

$$S_0 \vdash A(x_1, \ldots, x_r) \rightarrow T\ulcorner A(\dot{x_1} \ldots, \dot{x_r})\urcorner.$$

(ii). $S_0 \vdash \forall x\, [\mathsf{Prov}_{\mathrm{PA}_T}(x) \wedge \mathsf{Sent}_{\mathcal{L}_T}(x) \rightarrow T(x)]$.

PROOF (i) is shown by meta induction on the complexity of $A(x_1, \ldots, x_r)$ and (ii) is shown by induction within S_0. See, for example, [25, Proposition 2.3]. ■

If we add T-Cons to S_0, we obtain equivalence in Proposition 2.11 (ii).

2.12 COROLLARY *For all arithmetical formulae* $A(x_1, \ldots, x_r)$ *whose free variables are among those exhibited,*

$$S_0 + \text{T-Cons} \vdash A(x_1, \ldots, x_r) \leftrightarrow T\ulcorner A(\dot{x_1} \ldots, \dot{x_r})\urcorner.$$

PROOF Suppose $T\ulcorner A(\dot{x_1} \ldots, \dot{x_r})\urcorner$. As $S_0 \vdash \neg A(x_1, \ldots, x_r) \rightarrow T\ulcorner\neg A(\dot{x_1} \ldots, \dot{x_r})\urcorner$, T-Cons yields $A(x_1, \ldots, x_r)$. ■

2.13 COROLLARY $\mathsf{Con}(\mathrm{PA})$ *is a theorem of* $\mathcal{B}$ *and* $\mathcal{C}$. ■

The theories $\mathcal{B}$ and $\mathcal{C}$, though, are much stronger than $\mathrm{PA} + \mathsf{Con}(\mathrm{PA})$. They prove the same arithmetical statements as the subsystem ACA of second order arithmetic which has arithmetical comprehension and the full induction scheme for all formulae of second order arithmetic. The proof-theoretic ordinal of ACA is $\varepsilon_{\varepsilon_0}$ (see [22], Theorem 23.4). Our first step will be to show that the model construction for both $\mathcal{B}$ and $\mathcal{C}$ provided in [11] can be carried out inside ACA. This is achieved by formalizing arithmetical truth in ACA.

2.14 PROPOSITION *One can formalise a truth predicate* T_{arith} *of complexity* Π^1_1 *in* ACA *such that for all arithmetical formulae* $A(x_1, \ldots, x_r)$,

$$\mathrm{ACA} \vdash A(x_1, \ldots, x_r) \leftrightarrow T_{arith}\ulcorner A(\dot{x_1} \ldots, \dot{x_r})\urcorner.$$

PROOF This is shown in [28], CH.3, §18, in particular Theorem 18.13. The theory S^1 used there is a subsystems of ACA wherein induction is restricted to Π^1_1 formulae. ■

2.15 PROPOSITION *Every arithmetical statement provable in* $\mathcal{B}$ *or* $\mathcal{C}$ *is provable in* ACA.

Proof For $\mathcal{B}$, we extend the predicate T_{arith} to a truth predicate for sentences C of $\mathcal{L}_T$ by letting $T^{\mathcal{B}}_{arith}(\ulcorner C \urcorner) := T_{arith}(\ulcorner C^b \urcorner)$ where C^b arises from C by replacing every subformula $T(s)$ of C by $s = s$. Define an interpretation $^*: \mathcal{L}_T \to \mathcal{L}_2$ as follows

$$\begin{aligned} p^* &= p \quad \text{for } p \text{ arithmetical.} \\ (\neg A)^* &= \neg A^* \\ (A \,\square\, B)^* &= A^* \,\square\, B^* \quad \text{for } \square \in \{\vee, \wedge\} \\ (QxA(x))^* &= Qx(A(x))^* \quad \text{for } Q \in \{\forall, \exists\} \\ (T(s))^* &= T^{\mathcal{B}}_{arith}(s). \end{aligned}$$

One then shows as in [11] that $\mathcal{B} \vdash C$ implies ACA $\vdash C^*$.

Similarly, for $\mathcal{C}$ one uses the extension $T^{\mathcal{C}}_{arith}(\ulcorner B \urcorner) := T_{arith}(\ulcorner B^c \urcorner)$ of the truth predicate where B^c arises from a formula B by replacing every subformula $T(s)$ of B by $s \neq s$, and defines an interpretation $^{**}: \mathcal{L}_T \to \mathcal{L}_2$ in the same vein as *, using $T^{\mathcal{C}}_{arith}$ instead of $T^{\mathcal{B}}_{arith}$. ■

2.16 Theorem *$\mathcal{B}$ and $\mathcal{C}$ prove the same arithmetical theorems as* ACA.

Proof In view of the previous result it suffices to give an interpretation of ACA in both $\mathcal{B}$ and $\mathcal{C}$, which preserves arithmetical formulae. We will use sets of the form $\{x \mid T(\ulcorner A(\dot{x}) \urcorner)\}$ to interpret the second order quantifiers. To avoid clashes of variables we will assume that $X_0, X_2, X_4, \ldots$ is the list of set variables and $x_1, x_3, x_5, \ldots$ is the list of numerical variables of $\mathcal{L}_2$. Let $\mathbf{subn}(n, m)$ be a primitive recursive function such that $\mathbf{subn}(\ulcorner A(x) \urcorner, m) = \ulcorner A(\bar{m}) \urcorner$ if x is the only variable which occurs free in the formula $A(x)$. If n is not the Gödel number of a formula with exactly one free variable let $\mathbf{subn}(n, m) = \ulcorner \bar{0} = \bar{1} \urcorner$. The translation $^\ddagger : \mathcal{L}_2 \to \mathcal{L}_T$ leaves first order atomic formulae untouched and commutes with propositional connectives and numerical quantifiers. $(t \in X_i)^\ddagger := T(\mathbf{subn}(x_i, t))$ and $(\forall X_i\, B(X_i))^\ddagger := \forall x_i\, (B(X_i))^\ddagger$. Note that $(t \in X_i \vee \neg t \in X_i)^\ddagger$ is $T(\mathbf{subn}(x_i, t)) \vee \neg T(\mathbf{subn}(x_i, t))$, which is provable in both target theories owing to T-Comp. ‡ translates instances of induction axioms into (other) instances of induction axioms and arithmetical statements without set variables remain unchanged. So only the translations of the arithmetical comprehension axioms need verifying. Let $A(u, \vec{v}, \vec{X})$ be without set quantifiers and with all free variables exhibited. By induction on the length of A one produces a term $s_A(\vec{v}, \vec{x})$ such that

$$\forall u\, [T(\mathbf{subn}(s_A(\vec{v}, \vec{x}), u) \leftrightarrow (A(u, \vec{v}, \vec{X}))^\ddagger] \tag{1}$$

holds provably in $\mathcal{B}$ and $\mathcal{C}$. This is obvious for atomic formulae either by design in the case of a formula $t \in X_i$ or by Corollary 2.12 for arithmetical atoms. In the case of complex formulae it follows easily from the following equivalences which are provable with the help of E-Inf, T-Cons and T-Comp:

$$\begin{gathered} \forall \ulcorner A \urcorner, \ulcorner B \urcorner [T(\ulcorner \neg A \urcorner) \leftrightarrow \neg T(\ulcorner A \urcorner)], \\ \forall \ulcorner A \urcorner, \ulcorner B \urcorner [T(\ulcorner A \wedge B \urcorner) \leftrightarrow T(\ulcorner A \urcorner) \wedge T(\ulcorner B \urcorner)], \\ \forall \ulcorner A(y) \urcorner [T(\ulcorner \exists y\, A(y) \urcorner) \leftrightarrow \exists y T(\ulcorner A(\dot{y}) \urcorner)]. \end{gathered}$$

(1) yields $\exists x_j \forall u[(u \in X_j)^\ddagger \leftrightarrow (A(u, \vec{v}, \vec{X}))^\ddagger]$, thus verifies the translation of arithmetical comprehension. ■

2.5 $\mathcal{D}$: T-Intro T-Elim, T-Cons, T-Comp, ∀-Inf, ∃-Inf, ¬T-Elim, ¬T-intro

Of the nine theories, $\mathcal{D}$ appears to have received the most attention. Friedman and Sheard proved that the theory $\mathcal{D}$ without ∀-Inference is a conservative extension of Peano Arithmetic and later, in [14], Halbach showed $\mathcal{D}$ is equi-consistent to the theory $\mathrm{RT}_{<\omega}$, the system of ω-times iterated truth. Sheard [25] showed that $\mathcal{D}$ proves the same arithmetical theorems as its sub-theory $Base_T +$ ∀-Inf + T-Elim. We shall provide a direct proof that $\mathcal{D}$ is equi-consistent with the second-order theory ACA_0^+, the theory ACA_0 together with the assertion that for every set X the ω-th Turing jump of X exists.

2.17 Remark ACA_0^+ *is sufficiently strong for the development of some interesting countable combinatorics. The Auslander/Ellis theorem of topological dynamics is provable in* ACA_0^+ *as is Hindman's Theorem (cf. [26, X.3]). The latter says that if the natural numbers are coloured with finitely many colours then there exists an infinite set* X *such that all finite sums of elements of* X *have the same colour.*

We also note that ACA_0^+ *is a theory that bears interesting connections to the bar rule and parameter free bar induction [21].*

2.18 Definition *Let* $\mathfrak{A}$ *be an* $\mathcal{L}$*-structure and* $X \subseteq \mathrm{dom}\,\mathfrak{A}$. *The pair* $\langle\mathfrak{A}, X\rangle$ *can be viewed as an extension of* $\mathfrak{A}$ *to the language* $\mathcal{L}_T$ *in the following way*

- *If* A *is arithmetical then* $\langle\mathfrak{A}, X\rangle \models A$ *if and only if* $\mathfrak{A} \models A$.
- $\langle\mathfrak{A}, X\rangle \models T(x)$ *if and only if* $x \in X$.
- *In all other cases follow the standard inductive definition of satisfaction for* $\langle\mathfrak{A}, X\rangle$.

Define by recursion a sequence of $\mathcal{L}_T$-structures

$$\mathfrak{M}_0 = \langle\mathbb{N}, \emptyset\rangle\,,$$
$$\mathfrak{M}_{n+1} = \langle\mathbb{N}, \{\ulcorner A\urcorner : \mathfrak{M}_n \models A\}\rangle$$

and define $\mathrm{Th}_\infty = \{A : \exists k \forall n > k.\ \mathfrak{M}_n \models A\}$. In [11] it was shown that Th_∞ is then a consistent theory containing $\mathcal{D}$.

2.19 Notation *We denote by* $j(X)$ *the* Turing jump *of the set* X, *that is,*

$$j(X) := \{\langle e, x\rangle : \{e\}^X(x)\downarrow\}.$$

where $\{e\}^X(x)\downarrow$ *denotes the formula stating "the eth recursive function with oracle* X *is defined on input* x*". Define the* n*-th jump of* X *as follows* $j_0(X) = X$, $j_{n+1} = j(j_n(X))$ *and the* ω*-jump of* X, $j_\omega(X)$, *to be* $\{\langle n, x\rangle : x \in j_n(X)\}$. *If* Y *is a set then* $(Y)_n := \{x : \langle n, x\rangle \in Y\}$ *is the* n*-*th section *of* Y. *If* A *and* B *are sets then* $A \leq_T B$ *denotes that* A *is Turing reducible to* B *i.e. the characteristic function of* A *is computable with the oracle* B.

2.20 Definition ACA_0^+ *is the theory of* ACA_0 *together with the axiom*

$$\forall X \exists Y\, ((Y)_0 = X \wedge \forall n\, (Y)_{n+1} = j((Y)_n)).$$

2.21 PROPOSITION ACA_0^+ *proves, given a set* X*, the existence of the set*

$$S^X := \{\ulcorner A \urcorner : A \textit{ is an } \mathcal{L}_T \textit{ sentence which is true when `}T\textit{' is interpreted by } X\}$$

PROOF ACA_0^+ is equivalent to the schema of ω-times iterated arithmetical comprehension (follows from [1, Lemma 3.4]), i.e. for every arithmetical formula $A(u, U)$ and set X there exists Y such that $(Y)_0 = X$ and

$$\forall n \forall x\, [x \in (Y)_{n+1} \leftrightarrow A(x, (Y)^n)]$$

where $(Y)^n$ denotes the set $\{\langle k, x\rangle : k \leq n \wedge x \in (Y)_k\}$. Define $Z^X = \{x : \mathsf{Ax}_{\mathrm{PRA}}(x) \vee \exists z \in X\, (x = \ulcorner T(\dot{z})\urcorner)\}$ and pick $A^X(x, U)$ to be the formula

$$\begin{aligned}&\exists \ulcorner B \urcorner, \ulcorner C \urcorner < x\, (x = \ulcorner B \wedge C \urcorner \wedge \ulcorner B \urcorner \in (U)_{|B|} \wedge \ulcorner C \urcorner \in (U)_{|C|}) \vee \\ &\quad \exists \ulcorner B \urcorner < x\, (x = \ulcorner \neg B \urcorner \wedge \ulcorner B \urcorner \notin (U)_{|B|}) \vee \\ &\exists \ulcorner B(x) \urcorner < x\, (x = \ulcorner \forall y B(y) \urcorner \wedge \forall n \ulcorner B(\dot{n}) \urcorner \in (U)_{|B(\bar{0})|}).\end{aligned}$$

In the above $|A|$ represents the complexity of A (with $|T(s)| = 0$) which is primitive recursive in $\ulcorner A \urcorner$. We now argue informally within ACA_0^+. By the observation above there then exists a set Y^X such that $(Y)_0 = Z^X$ and $(Y)_{n+1} = \{x : A^X(x, (Y)^n)\}$. By induction we verify that $\langle n, \ulcorner A \urcorner\rangle \in Y$ if and only if A is an $\mathcal{L}_T$ sentence with complexity n and which is true when 'T' is interpreted by X. Finally, pick $S^X = \{x : \exists n\, x \in (Y)_n\}$. Then since Y is known to exist S^X also exists and we are done. ■

2.22 THEOREM *Every arithmetical theorem of* $\mathcal{D}$ *is a theorem of* ACA_0^+.

PROOF We shall use the model construction for $\mathcal{D}$ in [11] to show that $\mathcal{D}$ is reducible to ACA_0^+. Let $\mathfrak{M}$ be an arbitrary model of ACA_0^+. Then $\mathfrak{M}$ is isomorphic to a model $\langle \mathfrak{A}, \mathfrak{X}\rangle \models \mathrm{ACA}_0^+$ where $\mathfrak{A} \models \mathrm{PA}$ and $\mathfrak{X} \subseteq \mathrm{Pow}(\mathrm{dom}\,\mathfrak{A})$ (the power set of the domain of $\mathfrak{A}$). We argue informally within $\mathfrak{M}$. We begin by constructing an hierarchy of models within ACA_0^+.

$$\mathfrak{A}_n = \langle \mathfrak{A}, S_n\rangle$$

where $S_0 = \emptyset$ and $S_{n+1} = S^{S_n}$ as defined in Proposition 2.21. There was also demonstrated that each step in the hierarchy is definable within $\mathfrak{M}$. Now let $\mathrm{Th}_\infty = \{A : \exists n \forall k > n\ \mathfrak{A}_k \models A\}$. We shall show that $A \in \mathrm{Th}_\infty$ whenever $\mathcal{D} \vdash A$. If $Base_T \vdash A$ or A is one of T-Consistency, T-Completeness, $\forall$-Inference or $\exists$-Inference then $\mathfrak{A}_k \models A$ for all $k > 0$ and so $A \in \mathrm{Th}_\infty$. If $A \in \mathrm{Th}_\infty$ then there is an n such that for all $k > n$, $\mathfrak{A}_k \models A$. Thus $\ulcorner A \urcorner \in S_{k+1}$ for all $k > n$, i.e. $\mathfrak{A}_k \models T(\ulcorner A \urcorner)$ for all $k > n+1$, and hence $T(\ulcorner A \urcorner) \in \mathrm{Th}_\infty$. Likewise the converse also holds and so Th_∞ is closed under T-Elimination.

Now we observe that if A is an arithmetical formula such that $\mathcal{D} \vdash A$ then there is an n such that for all $k > n$, $\mathfrak{A}_k \models A$. However, A is arithmetical, so we must have $\mathfrak{A} \models A$ and hence $\mathfrak{M} \models A$. But the choice of $\mathfrak{M}$ was arbitrary thus, it follows that every arithmetical theorem of $\mathcal{D}$ is a theorem of ACA_0^+. ■

We now attempt to show that ACA_0^+ can be reduced to $\mathcal{D}$. To do this we need to take a detour into an intermediary theory. Let $\mathbb{E}_n$ be new set constants and let $\mathbf{D}_n$ be the formula $j_\omega(\mathbb{E}_n) = \mathbb{E}_{n+1}$. The system ACA_0^ω is defined to be ACA_0 together with the axioms $\mathbf{D}_i$ for each $i \in \mathbb{N}$.

2.23 Lemma (Rathjen [21, 3.2]) *If $\forall X \exists Y F(X,Y)$ is a sentence of second-order arithmetic such that $F(X,Y)$ arithmetical then*

$$\mathrm{ACA}_0^+ \vdash \forall X \exists Y\, F(X,Y) \Longleftrightarrow \mathrm{ACA}_0^\omega \vdash \exists Y F(\mathbb{E}_0, Y)$$

Proof See [21]. ■

We will show that any model of $\mathcal{D}$ can be extended to a model of ACA_0^ω. To do this we will need to prove three auxiliary lemmata establishing Tarskian biconditionals for subsets of $\mathcal{L}_T$.

2.24 Lemma *For all $A \in \mathcal{L}_{\mathrm{PA}}$, $\mathcal{D} \vdash \forall x.\, T\ulcorner A(\dot{x})\urcorner \leftrightarrow A(x)$.*

Proof Axiom 3 ensures the above holds for all atomic formulae. The remaining axioms allow T to commute with all connectives, except for the case $\exists n\, T\ulcorner A(\dot{n})\urcorner \to T\ulcorner \exists x A(x)\urcorner$ an argument for which follows.

$$\mathcal{D} \vdash \forall n.\ \mathsf{Prov}_{\mathrm{PRA}}\ulcorner A(\dot{n}) \to \exists y A(y)\urcorner$$
$$\mathcal{D} \vdash \forall n.\ T\ulcorner A(\dot{n}) \to \exists y A(y)\urcorner$$
$$\mathcal{D} \vdash \exists n\, T\ulcorner A(\dot{n})\urcorner \to T\ulcorner \exists y A(y)\urcorner$$

■

Halbach showed (in [14]) that we can generalise the above lemma to a subclass of $\mathcal{L}_T$ given by the following stratification of $\mathcal{L}_T$.

(i). $\mathcal{L}_0 = \mathcal{L}$;

(ii). $\mathcal{L}_{n+1}$ is the language $\mathcal{L}_n$ expanded by all the formulae of the form

$$T(t) \wedge \mathsf{Sent}_{\mathcal{L}_n}(t)$$

for t an arbitrary term, where $\mathsf{Sent}_{\mathcal{L}_n}(x)$ is a predicate in $\mathcal{L}$ expressing that x is the Gödel number of a sentence of the language $\mathcal{L}_n$. $\mathcal{L}_{n+1}$ is the closed under the usual rules for formulation of formulae.

2.25 Lemma $\mathcal{D} \vdash \forall x.\ \mathsf{Sent}_{\mathcal{L}_n}(x) \to (T(x) \leftrightarrow T\ulcorner T(x)\urcorner)$.

Proof See [14, Lemma 5.2]. ■

We require one further lemma before we can begin the reduction. Denote by A^- the formula obtained by replacing (in A) each subformula of A of the form $T\ulcorner B(\dot{x})\urcorner$ where $B \in \mathcal{L}_T$ has at most x free with $B(x)$.

2.26 Lemma *Let A be an $\mathcal{L}_T$ formula in which all occurrences of $T(s)$ as subformulae are found amongst the list $T\ulcorner B_0(\dot{x}_0)\urcorner$, $T\ulcorner B_1(\dot{x}_1)\urcorner$, …, $T\ulcorner B_n(\dot{x}_n)\urcorner$ with $B_i \in \mathcal{L}_T$ with at most x_i free ($i \leq n$). Then*

$$\mathcal{D} \vdash \forall x.\ A(x) \leftrightarrow T\ulcorner A^-(\dot{x})\urcorner.$$

Proof By induction on the complexity of A. If A is atomic then either $A \in \mathcal{L}$, in which case $A^- = A$ and $A \leftrightarrow T\ulcorner A^-\urcorner$ by Lemma 2.24 or A is $T\ulcorner B_i(\dot{x}_i)\urcorner$ for some i in which case the result is immediate.

The case where A is a composition follows by induction since T commutes with all connectives and quantifiers. ■

2.27 THEOREM *Every arithmetical theorem of* ACA_0^+ *is a theorem of* $\mathcal{D}$.

PROOF Let $\mathfrak{M}$ be a model of $\mathcal{D}$. Our aim is to construct, within $\mathfrak{M}$, a model for ACA_0^ω and then invoke Lemma 2.23. $\mathfrak{M}$ is isomorphic to a model $\langle \mathfrak{A}, \mathfrak{T} \rangle$ of $\mathcal{D}$ where $\mathfrak{A} \models \mathrm{PA}$ and $\mathfrak{T} \in \mathrm{Pow}(\mathrm{dom}\, \mathfrak{A})$. Let $\mathcal{L}_\omega = \bigcup_n \mathcal{L}_n$. The universe of sets $\mathfrak{X}$ is given by all sets of the form

$$X_A := \{n : T\ulcorner A(\dot{n})\urcorner\}.$$

for $A \in \mathcal{L}_\omega$ with at most one free variable. Induction for arbitrary $\mathcal{L}_T$ formulae in $\mathcal{D}$ provides us with the induction axiom for sets in ACA_0. Arithmetical comprehension provides no problem because Lemma 2.24 shows that $\forall x.\, A(x) \leftrightarrow x \in X_A$ for all arithmetic A.

It remains to define the set constants $\mathbb{E}_n$. Define by double recursion on n and k the following formulae.

$$\begin{aligned} C_n^0(x) &\leftrightarrow \bot \\ C_0^{k+1}(x) &\leftrightarrow T\ulcorner C_{\dot{x}_1}^k(\dot{x}_1)\urcorner \wedge \mathsf{Sent}_{\mathcal{L}_k}\ulcorner C_{\dot{x}_1}^k(\dot{x}_2)\urcorner \\ C_{n+1}^{k+1}(x) &\leftrightarrow \{x_1\}^{C_n^{k+1}}(x_2)\downarrow \end{aligned}$$

Let f be a primitive recursive function such that $f(k,n,x) = \ulcorner C_n^k(\dot{x})\urcorner$. Now for each k define $\mathbb{E}_k = \{\langle n, x\rangle : \mathfrak{M} \models T(f(k,n,x))\}$.

2.28 PROPOSITION $\mathbb{E}_0 = \emptyset$ *and for each* $k \in \mathbb{N}$, $\langle n, x\rangle \in \mathbb{E}_{k+1}$ *if and only if* $x \in j_n(\mathbb{E}_k)$.

PROOF First we should observe that we have $\forall n \forall x\, \mathsf{Sent}_{\mathcal{L}_k}(f(\bar{k},n,x))$ for each k so by Lemma 2.25

$$\mathcal{D} \vdash \forall n \forall x.\, T(f(\bar{k},n,x)) \leftrightarrow T\ulcorner T(f(\bar{k},n,x))\urcorner.$$

We can now prove the claim by induction on k.

The base case is trivial with Lemma 2.24 and we prove the induction step as follows by arguing within the model $\mathfrak{M}$.

$$\begin{aligned} \langle 0, x\rangle \in \mathbb{E}_{k+1} &\iff T(f(k+1,0,x)) \\ &\iff T\ulcorner T(f(k,x_1,x_2))\urcorner \wedge \mathsf{Sent}_{\mathcal{L}_k}(f(k,x_1,x_2)) \\ &\qquad \text{by definition of } C_{n+1}^{k+1}(x) \text{ and Lemma 2.24} \\ &\iff T(f(k,x_1,x_2)) \\ &\iff x \in \mathbb{E}_k. \\ \langle n+1, x\rangle \in \mathbb{E}_{k+1} &\iff T(f(k+1,n+1,x)) \\ &\iff \{x_1\}^{T(f(k+1,n+1,x))}(x_2)\downarrow \\ &\qquad \text{by Lemma 2.26,} \\ &\iff x \in j\{y : \langle n+1, y\rangle \in \mathbb{E}_{k+1}\} \\ &\iff x \in j_{n+1}(\mathbb{E}_k). \end{aligned}$$

■

Finally, $\langle \mathfrak{A}, \mathfrak{X}, (\mathbb{E}_n)_{n\in\mathbb{N}}\rangle$ forms our model of ACA_0^ω and any arithmetical theorem of ACA_0^+ is an arithmetical theorem of ACA_0^ω by Lemma 2.23 and hence is a theorem of $\mathcal{D}$, as required. ■

2.29 COROLLARY ACA_0^+ *and* $\mathcal{D}$ *have the same arithmetical consequences.*

2.6 Lower bounds for $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{I}$

In order to analyse the stronger theories of $\mathcal{E}$ to $\mathcal{I}$ we will need to develop an ordinal notation system so that we may make sense of principles such as transfinite induction within these theories.

2.6.1 Ordinals

Let ON be the class of all ordinals. Define a class of functions $\varphi_\alpha : \text{ON} \to \text{ON}$ for $\alpha \in \text{ON}$ indexed by ordinals by transfinite induction as follows.

(i). φ_0 is the function $\alpha \mapsto \omega^\alpha$.

(ii). Define $\varphi_{\alpha+1}$ to be the enumerating function of the class $\{\xi : \varphi_\alpha(\xi) = \xi\}$, that is, $\varphi_{\alpha+1}$ enumerates the fix-points of the previously defined function.

(iii). For limit ordinals λ, φ_λ enumerates $\{\xi : \varphi_\beta(\xi) = \xi \text{ for all } \beta < \lambda\}$.

It is easy to show that, for each α, φ_α is continuous and strictly increasing, which means the class $\{\xi : \varphi_\alpha(\xi) = \xi\}$ is unbounded in ON ([22, V. Theorem 13.8]). This enables us to deduce that φ_α is well-defined for every $\alpha \in \text{ON}$. We will write $\varphi\alpha\beta$ in place of $\varphi_\alpha(\beta)$. We will denote by ε_α the ordinal $\varphi 1\alpha$ and say an ordinal γ is an ε-ordinal if $\gamma = \varepsilon_\alpha$ for some α. It should be remarked that since, by definition, an ordinal $\varphi\alpha\beta$ is a fix-point for every function φ_γ with $\gamma < \alpha$ it is the case that $\varphi\gamma\varphi\alpha\beta = \varphi\alpha\beta$ whenever $\gamma < \alpha$.

The Feferman-Schütte ordinal, Γ_0, is then the least ordinal closed under the two-place function $\alpha, \beta \mapsto \varphi\alpha\beta$, that is, the least ordinal greater than 0 such that $\varphi\alpha\beta < \Gamma_0$ whenever $\alpha, \beta < \Gamma_0$. Γ_0 is countable and hence there is a bijection between Γ_0 and $\mathbb{N}$, however for our theories to make sense of such an embedding we require the following result.

2.30 PROPOSITION *There are primitive recursive sets* $\text{OT} \subseteq \mathbb{N}$ *and* $\prec \subseteq \mathbb{N} \times \mathbf{N}$ *and a bijection* $\tau : \langle \Gamma_0, < \rangle \to \langle \text{OT}, \prec \rangle$ *such that* $\tau(0) = 0$ *and the functions* $\langle \tau(\alpha), \tau(\beta) \rangle \mapsto \tau(\alpha + \beta)$ *and* $\langle \tau(\alpha), \tau(\beta) \rangle \mapsto \tau(\varphi\alpha\beta)$ *are primitive recursive.*

2.6.2 Well-ordering proofs

Let OT, $\prec$ and τ be as given in Proposition 2.30. We now restrict ourselves to looking only at the initial segment of the ordinals given by Γ_0 and so, all ordinals are, unless otherwise specified, smaller than Γ_0. Furthermore we identify ordinals α with their representatives, $\tau(\alpha)$. Lower case Greek letters α, β, etc. are taken to range over ordinals (elements of OT) and lower case Roman letters x, y, m, n, etc. range over natural numbers. That is, $\forall\alpha\, A(\alpha)$ will abbreviate $\forall x.\, x \in \text{OT} \to A(x)$ and $\exists\alpha\, A(\alpha)$ denotes $\exists x.\, x \in \text{OT} \wedge A(x)$. When there is no case for confusion we write $\alpha < \beta$ in place of $\alpha \prec \beta$. Recall $S_0 = Base_T + \forall\text{-Inf}$.

2.31 DEFINITION *Let* $S_1 = S_0 + \text{T-Elim}$*,* $S_2 = S_0 + \text{T-Del} + \text{T-Intro} + \text{T-Elim}$.

2.32 NOTATION Transfinite induction *up to* α *for a formula* $A(x)$*,* $\mathsf{TI}(\alpha, A)$*, is given by*

$$\mathsf{Prog}\, A \to \forall\xi \prec \alpha\, A(\xi)$$

where $\mathsf{Prog}\, A$ *states the progressiveness of* $A(x)$ *in* x *along* $\prec$, *i.e.*

$$\forall\beta.\ (\forall\xi \prec \beta\, A(\xi)) \to A(\beta).$$

We denote by $I(y)$ *the formula* $\forall\ulcorner A(x)\urcorner\, T\ulcorner\mathsf{TI}(\dot{y}, A)\urcorner$.

2.33 Definition *The* proof-theoretic ordinal *of a theory* S *(written* $|S|$*) is the least ordinal* α *such that* S *and* PA $+ \{\mathsf{TI}(\bar{\beta}, A) : \beta < \alpha$ *and* $A \in \mathcal{L}\}$ *prove the same arithmetical statements and, moreover, that this can be verified within* PA.

Sheard [25] proved the following.

2.34 Lemma *([25, Lemma 3.6]) Let* $\alpha(n)$ *be an ordinal term (with variable* n*) and* λ *an ordinal such that* PA $\vdash \sup_n \alpha(n) = \bar{\lambda}$. *Then* $S_0 \vdash \forall n.\ I(\alpha(\dot{n})) \to I(\bar{\lambda})$. ■

2.35 Lemma *([25, Lemma 3.7])Let* $\delta(\alpha)$ *be an ordinal term (with* α *an ordinal variable) such that* PA $\vdash \forall\lambda.\ \mathsf{limit}(\lambda) \to \delta(\lambda) = \sup_{\alpha<\lambda}\delta(\alpha)$. *Then* $S_0 \vdash \forall\lambda.\ \mathsf{limit}(\lambda) \to (\forall\alpha < \lambda) I(\delta(\alpha)) \to I(\delta(\lambda))$. ■

We now use these results to establish some properties of the formula $I(x)$.

2.36 Lemma *For every ordinal* α, *if* $S_1 \vdash I(\bar{\alpha})$ *then* $S_1 \vdash \mathsf{TI}(\bar{\alpha}, A)$ *for all formulae* $A \in \mathcal{L}_T$. ■

2.37 Lemma $S_0 + T\text{-}Del \vdash \forall\alpha.\ T\ulcorner I(\dot{\alpha})\urcorner \to I(\alpha)$

Proof Notice that $Base_T \vdash \forall\ulcorner B(x)\urcorner.\ T\ulcorner\forall x B(x)\urcorner \to \forall x T\ulcorner B(\dot{x})\urcorner$. By instantiating $I(x)$ for $B(x)$ above and using T-Deletion we obtain the result. ■

2.38 Lemma $S_0 \vdash \forall\alpha.\ I(\alpha) \to I(\omega^\alpha)$.

Proof Since for each formula $A \in \mathcal{L}_T$ there is a formula $A' \in \mathcal{L}_T$ constructed primitive recursively from A such that $\mathrm{PA}_T \vdash \forall\beta.\ \mathsf{TI}(\beta, A') \to \mathsf{TI}(\omega^\beta, A)$. Proposition 2.11 implies $S_0 \vdash \forall\ulcorner A(x)\urcorner\exists\ulcorner A'(x)\urcorner\, T(\ulcorner\forall\beta.\ \mathsf{TI}(\beta, A') \to \mathsf{TI}(\omega^\beta, A)\urcorner)$. Thus $\forall$-Inference and $Base_T$ yield $S_0 \vdash \forall\beta.\ I(\beta) \to I(\omega^\beta)$. ■

2.39 Lemma $S_0 \vdash \forall\alpha\forall\beta.\ I(\alpha) \wedge I(\beta) \to I(\alpha + \beta)$.

Proof The proof is essentially the same as that of Lemma 2.38. ■

2.40 Lemma $S_0 \vdash \forall\alpha.\ I(\alpha) \to I(\alpha^{\varepsilon+})$ *where* $\alpha^{\varepsilon+}$ *denotes the least* ε*-number larger than* α.

Proof By Lemma 2.38 we may deduce, by induction, $S_0 \vdash \forall\alpha.\ I(\alpha) \to \forall n I(\omega_n(\alpha + 1))$, where $\omega_0(\alpha) = \alpha$ and $\omega_{n+1}(\alpha) = \omega^{\omega_n(\alpha)}$. Then, since $\alpha^{\varepsilon+} = \sup_n \omega_n(\alpha + 1)$, Lemma 2.34 yields the result. ■

Lemma 2.40 amounts to showing $\mathsf{Prog}\ A$ where $A(x)$ is $I(\varepsilon_x)$ (the limit case being provided by Lemma 2.35) and thus we may show

2.41 Theorem $S_1 \vdash \mathsf{TI}(\bar{\beta}, A)$ *for every* $\beta < \varphi 20$ *and every* $A \in \mathcal{L}_T$.

Proof Observe that $\varphi 20$ is the least ordinal $\alpha > 0$ such that $\varepsilon_\delta < \alpha$ whenever $\delta < \alpha$. Now if $S_1 \vdash \mathsf{TI}(\bar{\beta}, A)$ for every $A \in \mathcal{L}_T$ then $S_1 \vdash I(\varepsilon_{\bar{\beta}})$ and so $S_1 \vdash \mathsf{TI}(\varepsilon_{\bar{\beta}}, A)$ for every $A \in \mathcal{L}_T$. ■

2.42 Corollary $\varphi 20 \leq |S_1|$. ■

S_1 is a sub-theory of the theories $\mathcal{D}$ to $\mathcal{I}$. The above theorem shows that S_1 proves the same arithmetical theorems as $\mathcal{D}$ (since $|\mathrm{ACA}^+| = \varphi 20$). However, as we shall see, the theories $\mathcal{E}$ and $\mathcal{F}$ are strictly stronger than $\mathcal{D}$; both prove a reflection principle over S_1.

2.43 Proposition $S_2 \vdash \forall x.\ \mathsf{Prov}_{S_1}(x) \wedge \mathsf{Sent}_{\mathcal{L}_T}(x) \to T(x)$.

Proof By T-Introduction we have $S_2 \vdash T\ulcorner A \urcorner$ for each axiom, A, of S_1. From this together with Proposition 2.11 (i) (and since the property of being the Gödel number of an axiom of S_1 is primitive recursively decidable) we deduce that $S_2 \vdash \forall x.\ \mathsf{Ax}_{S_1}(x) \to T(x)$. We now argue informally within S_2 by induction on the length of proof. Since the predicate T is closed under logical deductions the only thing to check is the use of T-Elimination. Suppose B was deduced within S_1 via an application of T-Elimination. Then $T\ulcorner B \urcorner$ is provable in S_1 and by the induction hypothesis we have $T\ulcorner T\ulcorner B \urcorner\urcorner$. So, by T-Deletion, we have $T\ulcorner B \urcorner$ as required. ■

As opposed to working with reflection principles and attempting to extend the result of the previous Lemma, it is easier to obtain a lower bound for S_2 by extending the ordinal analysis we established for S_1 as follows.

2.44 Lemma *For each $n < \omega$, $S_2 \vdash \forall \alpha.\ I(\alpha) \to I(\varphi \bar{n} \alpha)$.*

Proof By induction on n. The case $n = 0$ is given by Lemma 2.38 so suppose $n = m + 1$ and (1) $S_2 \vdash \forall \alpha.\ I(\alpha) \to I(\varphi \bar{m} \alpha)$. We argue informally within S_2. By induction on k from (1) we deduce

$$(2) \qquad \forall \alpha.\ I(\alpha) \to \forall k\ I(\varphi^k \bar{m} \alpha)$$

where $\varphi^0 \alpha \beta = \beta$ and $\varphi^{k+1} \alpha \beta = \varphi \alpha (\varphi^k \alpha \beta)$. $\sup_k \varphi^k \alpha \beta = \varphi(\alpha + 1)\beta$ so from (2) we deduce (3) $I(\varphi \bar{n} 0)$ and (4) $\forall \alpha.\ I(\varphi \bar{n} \alpha) \to I(\varphi \bar{n}(\alpha + 1))$. Now (3), (4) and Lemma 2.35 lead us to $\mathsf{Prog}\ B$ where $B(x)$ is $I(\varphi \bar{n} x)$ so we deduce

$$(5) \qquad \forall \alpha.\ \mathsf{TI}(\alpha, B) \to I(\varphi \bar{n} \alpha).$$

An application of T-Introduction and Lemma 2.37 to (5) means $S_2 \vdash \forall \alpha.\ I(\alpha) \to I(\varphi \bar{n} \alpha)$ as required. ■

2.45 Corollary $|\mathcal{E}|, |\mathcal{F}| \geq \varphi \omega 0$.

Proof Fix some $\beta < \varphi \omega 0$. Then there is an n such that $\beta < \varphi n 0$. By Lemma 2.44 we deduce $S_2 \vdash I(\varphi \bar{n} 0)$. T-Elimination shows $S_2 \vdash \mathsf{TI}(\varphi \bar{n} 0, A)$ from which we may deduce $S_2 \vdash \mathsf{TI}(\bar{\beta}, A)$. However S_2 is a sub-theory of both $\mathcal{E}$ and $\mathcal{F}$. ■

3 A Proof Theory for truth theories: Upper bounds for $\mathcal{F}$, $\mathcal{G}$, $\mathcal{I}$ and $\mathcal{E}$.

We now take a detour into infinitary logic to establish upper-bounds on the proof-theoretical strength of the systems $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{I}$. See Cantini [6] for a detailed investigation of $\mathcal{H}$. We will begin with $\mathcal{F}$.

3.1 Definition *An $\mathcal{L}_T$-formula is in* negation normal form *if it is generated from atomic and negated atomic formula (called literals) via the usual connectives* $\wedge$, $\vee$, $\forall x$ *and* $\exists x$.

The rank *of an $\mathcal{L}_T$-sentence A, $|A|$, is defined in the usual sense as the complexity of A, but with all literals in $\mathcal{L}_T$ being assigned rank* 0 *(including $T(t)$ and $\neg T(t)$ for closed terms t).*

We now assume that all $\mathcal{L}_T$-sentences are presented in negation normal form and we only deal with sentences (closed formulae) unless otherwise stated. A *sequent* is a finite set of $\mathcal{L}_T$-sentences.

3.2 Notation *If Γ and Δ are sequents and A is an $\mathcal{L}_T$-sentence then by Γ, Δ and Γ, A we denote the sequents $\Gamma \cup \Delta$ and $\Gamma \cup \{A\}$ respectively.*

If s is a closed term then $s^{\mathbb{N}}$ denotes the interpretation of s in the standard model $\mathbb{N}$.

3.1 An infinitary system for $\mathcal{F}$

We are now ready to define an infinitary system for $\mathcal{F}$. We formulate it in the usual Tait-style sequent calculus as follows.

3.3 Definition (Inductive Definition of F_∞) *Let Γ be a finite set of sentences of $\mathcal{L}_T$. Define $\mathrm{F}_\infty \vdash^{\alpha}_{n,k} \Gamma$ for $\alpha \in \mathrm{OT}$ and $n, k < \omega$ by transfinite induction on α as follows ('F_∞' will be omitted for notational convenience).*

(Ax.1.) $\vdash^{\alpha}_{n,k} \Gamma, A$ *if A is a true arithmetical literal.*

(Ax.2.) $\vdash^{\alpha}_{n,k} \Gamma, \neg T(t), T(s)$ *if* $t^{\mathbf{N}} = s^{\mathbb{N}}$.

(Ax.3.) $\vdash^{\alpha}_{n,k} \Gamma, \neg T(t)$ *if $t^{\mathbb{N}}$ is not the Gödel number of an $\mathcal{L}_T$-sentence.*

($\wedge$) *If* $\vdash^{\alpha}_{n,k} \Gamma, A_0$, $\vdash^{\beta}_{n,k} \Gamma, A_1$ *and* $\alpha, \beta < \delta$ *then* $\vdash^{\delta}_{n,k} \Gamma, A_0 \wedge A_1$.

($\vee_i$) *If* $\vdash^{\alpha}_{n,k} \Gamma, A_i$ *and* $\alpha < \beta$ *then* $\vdash^{\beta}_{n,k} \Gamma, A_0 \vee A_1$. *($i \leq 1$)*

($\exists$) *If* $\vdash^{\alpha}_{n,k} \Gamma, A(t)$ *for some term t and* $\alpha < \beta$ *then* $\vdash^{\beta}_{n,k} \Gamma, \exists x A(x)$.

(ω) *If* $\vdash^{\alpha}_{n,k} \Gamma, A(\bar{m})$ *for each* $m < \omega$ *and* $\alpha < \beta$ *then* $\vdash^{\beta}_{n,k} \Gamma, \forall x A(x)$.

(Cut) *If* $\vdash^{\alpha}_{n,k} \Gamma, A$, $\vdash^{\beta}_{n,k} \Gamma, \neg A$, $|A| < k$ *and* $\alpha, \beta < \delta$ *then* $\vdash^{\delta}_{n,k} \Gamma$.

(T-Intro) *If* $\vdash^{\alpha}_{n,k} A$, $\alpha < \beta$ *and* $n < m$ *then* $\vdash^{\beta}_{m,k} \Gamma, T(t)$ *whenever* $t^{\mathbb{N}} = \ulcorner A \urcorner$.

(T-Imp) *If* $\vdash^{\alpha}_{n,k} \Gamma, T(t)$, $\vdash^{\beta}_{n,k} \Gamma, T(\mathbf{imp}(t,s))$ *and* $\alpha, \beta < \delta$ *then* $\vdash^{\delta}_{n,k} \Gamma, T(s)$.

(T-Del) *If* $\vdash^{\alpha}_{n,k} \Gamma, T\ulcorner T(t) \urcorner$, *where $t^{\mathbb{N}}$ is the Gödel number of an $\mathcal{L}_T$-sentence,* $s^{\mathbb{N}} = t^{\mathbb{N}}$ *and* $\alpha < \beta$ *then* $\vdash^{\beta}_{n,k} \Gamma, T(s)$.

(T-U-Inf) *If* $\vdash^{\alpha}_{n,\,k} \Gamma, T\ulcorner A(\bar{m})\urcorner$ *holds for all* m, $\alpha < \beta$ *then* $\vdash^{\beta}_{n,\,k} \Gamma, T(t)$ *whenever* $t^{\mathbf{N}} = \ulcorner\forall x A(x)\urcorner$.

We refer to the rules (T-Intro), (T-Imp), (T-Del), (T-U-Inf) as *T-rules*. In $\vdash^{\alpha}_{n,\,k} \Gamma$, α denotes the length of the derivation, k the cut-rank, and n the T-Intro rank of the derivation. As above we will omit explicit mention of the infinitary system under consideration whenever there is no room for confusion.

In each of the above rules the distinguished formulae that occur in the conclusion are called *active* and the distinguished formulae in the premise(s) are called *minor formulae*. Formulae that appear and are neither active nor minor are called *side formulae*.

3.4 Proposition *The following hold for all* α, n *and* k

(i). Monotonicity: $\vdash^{\alpha}_{n,\,k} \Gamma$, $\alpha \leq \beta$, $k \leq l$ *and* $n \leq m$ *implies* $\vdash^{\beta}_{m,\,l} \Gamma$.

(ii). Weakening: $\vdash^{\alpha}_{n,\,k} \Gamma$ *implies* $\vdash^{\alpha}_{n,\,k} \Gamma, \Delta$.

(iii). If $\vdash^{\alpha}_{n,\,k} \Gamma, A$ *and* A *is a false arithmetical literal then* $\vdash^{\alpha}_{n,\,k} \Gamma$.

(iv). If $\vdash^{\alpha}_{n,\,k} \Gamma, A(t)$ *and* $s^{\mathbb{N}} = t^{\mathbb{N}}$ *then* $\vdash^{\alpha}_{n,\,k} \Gamma, A(s)$.

(v). If $\vdash^{\alpha}_{n,\,k} \Gamma, T(t)$ *and* $t^{\mathbb{N}}$ *is not the Gödel number of an* $\mathcal{L}_T$*-sentence then* $\vdash^{\alpha}_{n,\,k} \Gamma$.

(vi). If $\vdash^{\alpha}_{n,\,k} \Gamma, (\neg)T(s), (\neg)T(t)$ *and* $s^{\mathbb{N}} = t^{\mathbb{N}}$ *then* $\vdash^{\alpha}_{n,\,k} \Gamma, (\neg)T(s)$.

Proof (i)–(vi) all follow easily by induction on α. For weakening it is important that (T-Intro) is formulated to allow side formulae in the conclusion. We shall prove (v) and (vi) as these differ most from the standard infinitary results.

(v). Suppose $\vdash^{\alpha}_{n,\,k} \Gamma, T(t)$ and $t^{\mathbb{N}}$ is not the Gödel number of an $\mathcal{L}_T$-sentence. If $\Gamma, T(t)$ is an axiom and $T(t)$ is not active then it follows that Γ is an axiom, thus we may suppose that $T(t)$ is active. In that case $\Gamma, T(t)$ must be an instance of (Ax.2) and so of the form $\Gamma', \neg T(s), T(t)$ with $s^{\mathbb{N}} = t^{\mathbb{N}}$ and $\Gamma' \subseteq \Gamma$. However, then $\Gamma', \neg T(s) (= \Gamma)$ is an axiom of the form (Ax.3) and the result holds. Now suppose the assumption arose from a rule of inference. If $T(t)$ is not active then $T(t)$ occurs in the premise of the rule used and so by the induction hypothesis and re-application of the rule in question $\vdash^{\alpha}_{n,\,k} \Gamma$ holds. Thus we may suppose $\alpha > 0$ and $T(t)$ is active. In this case the assumption was derived via a T-rule. This may not have been via an application of either (T-Intro), (T-Del) or (T-U-Inf) as these require $t^{\mathbb{N}}$ to be the Gödel number of an $\mathcal{L}_T$-sentence. For (T-Imp) we observe that if $t^{\mathbb{N}}$ is not the Gödel number of an $\mathcal{L}_T$-sentence then neither is $\mathbf{imp}(s,t)^{\mathbb{N}}$ for any term s and so the result follows by the induction hypothesis.

(vi). We need only consider the case where $(\neg)T(t)$ is active. If $\Gamma, (\neg)T(t)$ is an axiom then $\Gamma, (\neg)T(s)$ must be as well and we are done. The only other situation that may arise is that we are considering $\Gamma, T(s), T(t)$ and that this follows from an application of a T-rule. For (T-Imp) the argument is as follows.

Suppose we have

$$\vdash^{\beta_0}_{n,\,k} \Gamma,\ T(t'),\ T(s), T(t) \quad \text{and} \quad \vdash^{\beta_1}_{n,\,k} \Gamma,\ T(\mathbf{imp}(t',t)),\ T(s), T(t)$$

for some β_0, $\beta_1 < \alpha$. We have assumed $t^{\mathbb{N}} = s^{\mathbb{N}}$ and so also $\mathbf{imp}(t', s)^{\mathbb{N}} = \mathbf{imp}(t', t)^{\mathbb{N}}$, hence by applying the induction hypothesis twice we deduce

$$\mathrel{\big|}\!\frac{\beta_0}{n,\,k}\ \Gamma,\ T(t'),\ T(s) \qquad \text{and} \qquad \mathrel{\big|}\!\frac{\beta_1}{n,\,k}\ \Gamma,\ T(\mathbf{imp}(t', s)),\ T(s)$$

from which $\mathrel{|}\!\frac{\alpha}{n,\,k}\ \Gamma,\ T(s)$ follows by (T-Imp). The other cases involve essentially the same argument. ■

Henceforth we freely use (i)–(vi) above without mention.

3.5 Definition *Suppose $\alpha = \omega^{\alpha_0} + \cdots + \omega^{\alpha_m}$ and $\beta = \omega^{\beta_0} + \cdots + \omega^{\beta_n}$ with $\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_m$ and $\beta_0 \geq \cdots \geq \beta_n$. Let $\gamma_0, \ldots, \gamma_{m+n+1}$ be an enumeration of the ordinals $\alpha_0, \ldots \alpha_m$, $\beta_0, \ldots, \beta_n$ (with repetitions) such that $\gamma_k \geq \gamma_{k+1}$ for every $k < m+n+1$. Then define the natural sum of α and β, $\alpha \# \beta$, to be given by $\omega^{\gamma_0} + \cdots + \omega^{\gamma_{m+n+1}}$.*

3.6 Lemma (Reduction Lemma for F_∞) *Suppose $\mathrel{|}\!\frac{\alpha}{n,\,k}\ \Gamma, A$ and $\mathrel{|}\!\frac{\beta}{n,\,k}\ \Delta, \neg A$ with $|A| = k$. Then $\mathrel{|}\!\frac{\alpha\#\beta}{n,\,k}\ \Gamma, \Delta$.*

Proof By induction on the natural sum of α and β. Due to our definition of the rank $|A| = |\neg A|$.

Case I. Either A or $\neg A$ is not active.

Without loss of generality we may assume A is not active, for which we have two scenarios: either Γ is an axiom, in which case Γ, Δ is also an axiom and we are done; or $\alpha > 0$ and $\mathrel{|}\!\frac{\alpha}{n,\,k}\ \Gamma, A$ follows from a rule. Provided the rule was not (T-Intro) A must have been present in the premise(s), in which case application of the induction hypothesis to each premise in turn together with the derivation $\mathrel{|}\!\frac{\beta}{n,\,k}\ \Delta, \neg A$ followed by re-application of the rule considered achieves the desired conclusion. This is fine since A is assumed not active and if $\alpha_0 < \alpha$ then $\alpha_0 \# \beta < \alpha \# \beta$ for all β. However, if the last rule was (T-Intro) then since A was not active it may be omitted in the conclusion and so the result holds by monotonicity.

Case II. Both A and $\neg A$ are active. Here we have two sub-cases depending on the value of $|A|$.

Case IIa. $|A| = k = 0$. Then one of Γ, A and $\Delta, \neg A$ must be an axiom, since if A is not arithmetical then one of A and $\neg A$ must be of the form $\neg T(t)$, which may only play an active role in axioms. Without loss of generality we may assume Γ, A is an axiom.

(Ax.1.) If A is a true arithmetical literal, then $\neg A$ is a false arithmetical literal, and so $\mathrel{|}\!\frac{\alpha\#\beta}{n,\,k}\ \Gamma, \Delta$ follows from Proposition 3.4(iii), weakening and monotonicity.

(Ax.2.) In this case Γ, A is $\Gamma', \neg T(t), T(s)$ with $\Gamma' \subseteq \Gamma$ and $s^{\mathbb{N}} = t^{\mathbb{N}}$. If A is $T(s)$ then by assumption we have $\mathrel{|}\!\frac{\beta}{n,\,k}\ \Delta, \neg T(s)$. However, $s^{\mathbb{N}} = t^{\mathbb{N}}$ so by Proposition 3.4(vi) $\mathrel{|}\!\frac{\beta}{n,\,k}\ \Delta, \neg T(t)$ holds. As $\neg T(t) \in \Gamma$ we obtain $\mathrel{|}\!\frac{\alpha\#\beta}{n,\,k}\ \Gamma, \Delta$ by weakening. Now suppose A is $\neg T(t)$. Then there are two cases for us to consider: if $t^{\mathbb{N}}$ is the Gödel number of an $\mathcal{L}_T$-sentence then as before we may deduce $\mathrel{\substack{\beta \\ \overline{n,k}}}\Delta, T(s)$ and hence $\mathrel{\substack{\alpha\#\beta \\ \overline{n,k}}}\Gamma, \Delta$; and if $t^{\mathbb{N}}$ is not the Gödel number of an $\mathcal{L}_T$-sentence then by Proposition 3.4(v) we may conclude $\mathrel{\substack{\alpha \\ \overline{n,k}}}\Gamma$ and the result follows by weakening and monotonicity.

(Ax.3.) Here A is $\neg T(t)$ and $t^{\mathbb{N}}$ is not the Gödel number of an $\mathcal{L}_T$-sentence, so by Proposition 3.4(v) $\mathrel{\substack{\beta \\ \overline{n,k}}}\Delta$ and so again $\mathrel{\substack{\alpha\#\beta \\ \overline{n,k}}}\Gamma, \Delta$.

Case IIb. $|A| = k > 0$. Now A cannot have the form $T(t)$ or $\neg T(t)$. Suppose $\mathrel{\substack{\alpha \\ \overline{n,k}}}\Gamma, A$ was derived via

($\wedge$). Then A is $A_0 \wedge A_1$ and we have (1) $\mathrel{\substack{\alpha \\ \overline{n,k}}}\Gamma, A_0 \wedge A_1$ and (2) $\mathrel{\substack{\beta \\ \overline{n,k}}}\Delta, \neg A_0 \vee \neg A_1$. Since both A and $\neg A$ are assumed active (3) $\mathrel{\substack{\alpha_i \\ \overline{n,k}}}\Gamma, A_i, A_0 \wedge A_1$ and (4) $\mathrel{\substack{\beta_0 \\ \overline{n,k}}}\Delta, \neg A_i, \neg A_0 \vee \neg A_1$ must also hold for some $\alpha_i < \alpha$ $(i = 0, 1)$ and $\beta_0 < \beta$. Firstly we apply the induction hypothesis to the pairs (1) and (4), (3) and (2) to obtain

$$\mathrel{\substack{\alpha\#\beta_0 \\ \overline{n,k}}}\Gamma, \Delta, \neg A_i$$

and

$$\mathrel{\substack{\alpha_i\#\beta \\ \overline{n,k}}}\Gamma, \Delta, A_i\ .$$

Now by (Cut) (as $|A_i| < |A| = k$) $\mathrel{\substack{\alpha\#\beta \\ \overline{n,k}}}\Gamma, \Delta$.

(ω). Then A is $\forall x B(x)$ and $\mathrel{\substack{\beta \\ \overline{n,k}}}\Delta, \neg A$ was derived via ($\exists$). Therefore we have (1) $\mathrel{\substack{\alpha \\ \overline{n,k}}}\Gamma, \forall x B(x)$, (2) $\mathrel{\substack{\beta \\ \overline{n,k}}}\Delta, \exists x \neg B(x)$, (3) $\mathrel{\substack{\alpha_0 \\ \overline{n,k}}}\Gamma, B(\bar{m}), \forall x B(x)$ for every $m < \omega$ and some $\alpha_0 < \alpha$, and

$$\text{(4)} \qquad \mathrel{\substack{\beta_0 \\ \overline{n,k}}}\Delta, \neg B(t), \exists x \neg B(x)$$

for some term t and $\beta_0 < \beta$. (3) and Proposition 3.4 (iv) imply

$$\text{(5)} \qquad \mathrel{\substack{\alpha_0 \\ \overline{n,k}}}\Gamma, B(\bar{m}_0), \forall x B(x)$$

holds where $m_0 = t^{\mathbb{N}}$. Now we may apply the induction hypothesis to (1) and (4), and (2) and (5) to obtain

$$\text{(6)} \ \mathrel{\substack{\alpha\#\beta_0 \\ \overline{n,k}}}\Gamma, \Delta, \neg B(t) \quad \text{and} \quad \text{(7)} \ \mathrel{\substack{\alpha_0\#\beta \\ \overline{n,k}}}\Gamma, B(\bar{m}_0)$$

Proposition 3.4(iv) shows that from (6) we can deduce

$$\mathrel{\substack{\alpha\#\beta_0 \\ \overline{n,k}}}\Gamma, \Delta, \neg B(\bar{m}_0)$$

which cut with (7) yields $\mathrel{\substack{\alpha\#\beta \\ \overline{n,k}}}\Gamma, \Delta$ as desired.

($\vee_i$) or ($\exists$). These are symmetric to the cases ($\wedge$) and (ω) above.

(Cut). Since A and $\neg A$ are assumed active the last rule applied could not have been (Cut).

A T-rule. Again, since A is assumed active, and of positive rank the last rule applied could not have been a T-rule. ■

3.7 Theorem (Cut-Elimination Theorem for F_∞)

$$\mathrm{F}_\infty \vdash^{\alpha}_{n,\,k+1} \Gamma \ \textit{ implies } \mathrm{F}_\infty \vdash^{\omega^\alpha}_{n,\,k} \Gamma\,.$$

Proof By induction on the length of the derivation. If Γ is an axiom the result is immediate, and if the last rule used was anything other than (Cut) the result follows by an application of the induction hypothesis. So suppose we have $\vdash^{\alpha_0}_{n,\,k+1} \Gamma, A$ and $\vdash^{\alpha_1}_{n,\,k+1} \Gamma, \neg A$ with $\alpha_0,\, \alpha_1 < \alpha$. Applying the induction hypothesis we deduce $\vdash^{\omega^{\alpha_0}}_{n,\,k} \Gamma, A$ and $\vdash^{\omega^{\alpha_1}}_{n,\,k} \Gamma, \neg A$. If $|A| < k$ then we may re-apply (Cut) and are done. However, if $|A| = k$ then by the Reduction Lemma $\vdash^{\omega^{\alpha_0}\#\omega^{\alpha_1}}_{n,\,k} \Gamma$ holds. But $\omega^{\alpha_0} \# \omega^{\alpha_1} < \omega^\alpha$ so the result holds by monotonicity. ■

The next step is to show that F_∞ is closed under T-Elimination. This is not immediate because of the (possible) presence of side formulae when analysing a specific inference rule but we may overcome the problem by defining an hierarchy of models indexed by natural numbers and ordinals which provide models of 'finite' subsystems of F_∞. We write $\mathrm{F}_\infty \vdash^{\alpha}_{n} \Gamma$ for $\mathrm{F}_\infty \vdash^{\alpha}_{n,\,0} \Gamma$.

3.8 Definition *For each $n \in \mathbb{N}$ define $f_n : \alpha \mapsto \varphi n(\varphi 1 \alpha)$. Now for every $n > 0$ and α define the $\mathcal{L}_T$-structure*

$$\mathfrak{M}^\alpha_n = \left\langle \mathbb{N}, \{\ulcorner B \urcorner : \mathrm{F}_\infty \vdash^{f_n(\alpha)}_{m} B \ \textit{ for some } m < n\} \right\rangle$$

and let $\mathfrak{M}^\alpha_0 = \langle \mathbb{N}, \emptyset \rangle$.

If Γ is a finite set of sentences, Γ is said to be *T-positive* if for each $A \in \Gamma$ the predicate T only occurs positively in A, and we write $\mathfrak{M} \models \Gamma$ to mean $\mathfrak{M}$ satisfies the disjunction of all members of Γ.

3.9 Proposition (Monotonicity of $\mathfrak{M}^\alpha_n$) *If $m \leq n$, $\alpha \leq \beta$ and A is T-positive. Then $\mathfrak{M}^\alpha_m \models A$ implies $\mathfrak{M}^\beta_n \models A$.*

Proof As $f_m(\alpha) \leq f_n(\beta)$ this is a consequence of monotonicity for F_∞. ■

3.10 Lemma (T-Elimination for F_∞) *For every T-positive sequent Γ and every $n < \omega$ we have*

(i). If $\mathrm{F}_\infty \vdash^{\alpha}_{n} \Gamma$ then $\mathfrak{M}^\alpha_n \models \Gamma$.

(ii). If $\mathrm{F}_\infty \vdash^{\alpha}_{n} T(s)$ and $s^{\mathbb{N}} = \ulcorner A \urcorner$ then $\mathrm{F}_\infty \vdash^{f_n(\alpha)}_{m} A$ for some $m < n$.

Proof We proceed by **main induction** on n and **subsidiary induction** on α.

Case $n = 0$**.** (i). This immediately holds for axioms (Ax.1) and (Ax.2) and the rules $(\wedge)$, $(\vee_i)$, $(\exists)$, and (ω) by the inductive definition of satisfaction and the choice of $\mathfrak{M}_0^\alpha$. The case of axiom (Ax.3) and the T-rules hold vacuously since the interpretation of T is empty. Furthermore, no applications of Cut have been assumed. (ii) holds because $\mathrm{F}_\infty \overset{\alpha}{\underset{0}{\vdash}} T(s)$ implies $\mathfrak{M}_0^\alpha \models T(s)$ by (i), and this is impossible.

Case $n = m + 1$**.** We now proceed by induction on α. (i) holds for axioms (Ax.1) and (Ax.2) and for the rules $(\vee_i)$, $(\wedge)$, $(\exists)$, (ω) as before. Due to our choice for $\mathfrak{M}_n^\alpha$ we observe that (Ax.3) is also satisfied. By assumption no applications of (Cut) have been used so this leaves only the T-rules to consider which will be dealt with individually as follows.

(T-Intro). If $\mathrm{F}_\infty \overset{\gamma}{\underset{m}{\vdash}} A$ for some $\gamma < \alpha$ then $\gamma < f_n(\alpha)$ and so $\mathfrak{M}_n^\alpha \models \Gamma \vee T(s)$ whenever $s^{\mathbb{N}} = \ulcorner A \urcorner$.

(T-Imp). If $\mathfrak{M}_n^\alpha \models \Gamma$ then we are done, so suppose otherwise. Then the induction hypothesis implies $\mathfrak{M}_n^{\gamma_0} \models T(t)$ and $\mathfrak{M}_n^{\gamma_1} \models T(\mathbf{imp}(t, s))$ for some γ_0, $\gamma_1 < \alpha$. If $\mathfrak{M}_n^\alpha \not\models \Gamma$ then $\mathrm{F}_\infty \overset{f_n(\gamma_0)}{\underset{m}{\vdash}} B$ and $\mathrm{F}_\infty \overset{f_n(\gamma_1)}{\underset{m}{\vdash}} \neg B, A$ for A and B such that $\ulcorner B \urcorner = t^{\mathbb{N}}$ and $\ulcorner A \urcorner = s^{\mathbb{N}}$. We may now apply (Cut) to deduce $\mathrm{F}_\infty \overset{f_n(\alpha)}{\underset{m,\,k}{\vdash}} A$ where $k = |B| + 1$ from which we obtain, by Theorem 3.7 applied k times and observing that $\omega^{f_n(\alpha)} = f_n(\alpha)$, $\mathrm{F}_\infty \overset{f_n(\alpha)}{\underset{m}{\vdash}} A$ and so $\mathfrak{M}_n^\alpha \models T(s)$.

(T-Del). Suppose $\gamma < \alpha$ and $\mathfrak{M}_n^\gamma \models T\ulcorner T \ulcorner A \urcorner \urcorner$ for some $\mathcal{L}_T$-sentence A. Thus $\mathrm{F}_\infty \overset{f_n(\gamma)}{\underset{m}{\vdash}} T\ulcorner A \urcorner$ holds. However by (i) we know $m \neq 0$ and so by the main induction hypothesis (ii) we obtain $\mathrm{F}_\infty \overset{f_m(f_n(\gamma))}{\underset{p}{\vdash}} A$ for some $p < m$. However since $n > 1$ we have $\varphi 1 f_n(\alpha) = f_n(\alpha)$, and so $f_m(f_n(\alpha)) = \varphi m(\varphi 1 f_n(\alpha)) = f_n(\alpha)$. Thus we may deduce $\mathfrak{M}_n^\alpha \models T\ulcorner A \urcorner$ by monotonicity and the definition of $\mathfrak{M}_n^\alpha$.

(T-U-Inf). If, for each k, $\gamma_k < \alpha$ and $\mathfrak{M}_n^{\gamma_k} \models T\ulcorner A(\bar{k})\urcorner$then $\mathrm{F}_\infty \overset{f_n(\gamma_k)}{\underset{m}{\vdash}} A(\bar{k})$ for each k. Thus $\mathrm{F}_\infty \overset{f_n(\alpha)}{\underset{m}{\vdash}} \forall x A(x)$ by (ω) (since $f_n(\alpha) > f_n(\gamma_k)$) and hence $\mathfrak{M}_n^\alpha \models T\ulcorner \forall x A(x) \urcorner$.

(ii) is now an immediate consequence of (i) and monotonicity. ■

As a consequence of Lemma 3.10 we obtain

3.11 Theorem (T-Elimination Theorem for F_∞) *For every* $\alpha < \varphi\omega 0$, $n \in \mathbb{N}$ *and* $\mathcal{L}_T$*-sentence* A

(i). $\mathrm{F}_\infty \overset{\alpha}{\underset{n}{\vdash}} T\ulcorner A \urcorner$ *implies* $\mathrm{F}_\infty \overset{f_n(\alpha)}{\underset{n}{\vdash}} A$.

(ii). $\mathrm{F}_\infty \overset{\alpha}{\underset{n}{\vdash}} \neg T\ulcorner A \urcorner$ *implies* $\mathrm{F}_\infty \overset{\alpha}{\underset{n}{\vdash}} \neg A$.

Proof (i) is now an immediate consequence of Lemma 3.10(ii) since $f_n(\alpha) < f_n(\varphi\omega 0) = \varphi\omega 0$.

(ii). As observed in the Reduction Lemma for F_∞ and since the derivation of $\neg T\ulcorner A \urcorner$ is assumed cut-free, $\overset{\alpha}{\underset{n}{\vdash}} \neg T\ulcorner A \urcorner$ must be an axiom. The only axiom available is (Ax.3) which assumes $\ulcorner A \urcorner$ is not the Gödel number of an $\mathcal{L}_T$-sentence. Contradiction; hence (ii) holds vacuously. ■

3.12 Notation *If A is an $\mathcal{L}_T$-formulae then denote by A^* the sentence obtained from A by replacing all free variables in A with arbitrary closed terms. This notation is extended to sets Γ of $\mathcal{L}_T$-formulae.*

3.13 Proposition *(i). If A is an axiom of $Base_T$ then* $\mathrm{F}_\infty \vdash^{\varepsilon_0}_{1,0} A^*$.

(ii). $\mathrm{F}_\infty \vdash^{p}_{0}$ T-Del $\wedge$ $\forall$-Inf *for some finite p.*

Proof (i). It is easily shown for each theorem A of PA_T (including each instance of $\mathcal{L}_T$ induction) that $\mathrm{F}_\infty \vdash^{\omega+p}_{0,k} A^*$ for some $p, k < \omega$ (cf. [23], 3.4.2). Hence $\mathrm{F}_\infty \vdash^{\alpha}_{0,0} A^*$ by Theorem 3.11 for some $\alpha < \varepsilon_0$.

All that remains is to show the truth axioms are also derivable. We will deal with axiom (3) of $Base_T$; (2) follows by the same argument. From what we have already established and an application of (T-Intro) $\mathrm{F}_\infty \vdash^{p}_{1,0} T\ulcorner A^* \urcorner$ holds for some finite p whenever A is an axiom of PRA. However, since the property of being the Gödel number an axiom of PRA is decidable we may conclude that for every term t there is a finite p such that $\mathrm{F}_\infty \vdash^{p}_{1,0} \neg \mathsf{Ax}_{\mathrm{PRA}}(t), T(t)$ hence $\mathrm{F}_\infty \vdash^{\omega}_{1,0} \forall x.\ \mathsf{Ax}_{\mathrm{PRA}}(x) \rightarrow T(x)$.

For axiom (1) we need to show $\forall x, y.\ T(x) \wedge T(\mathbf{imp}(x, y)) \rightarrow T(y)$ is derivable in F_∞. Suppose m and n are Gödel numbers for $\mathcal{L}_T$-sentences. Then (Ax.2) gives

$$\vdash^{0}_{0} \neg T(\bar{m}), \neg T(\mathbf{imp}(\bar{m}, \bar{n}), T(\bar{m})$$

and

$$\vdash^{0}_{0} \neg T(\bar{m}), \neg T(\mathbf{imp}(\bar{m}, \bar{n}), T(\mathbf{imp}(\bar{m}, \bar{n}).$$

By an application of (T-Imp) we can therefore deduce

(1) $$\vdash^{1}_{0} \neg T(\bar{m}), \neg T(\mathbf{imp}(\bar{m}, \bar{n}), T(\bar{n})$$

whenever m and n are Gödel numbers for $\mathcal{L}_T$-sentences. By (Ax.3) we also have (1) where one of m or n is not an $\mathcal{L}_T$-sentence, hence (1) may be deduced for any m and n. Applications of $(\vee_i)$ yields $\vdash^{5}_{0} \neg(T(\bar{m}) \wedge T(\mathbf{imp}(\bar{m}, \bar{n})) \vee T(\bar{n})$ and so (ω) completes the proof.

(ii). We will show T-Deletion is derivable, $\forall$-Inf is similar. Suppose m is an $\mathcal{L}_T$-sentence. By (Ax.2.) $\vdash^{0}_{0} T\ulcorner T(\bar{m})\urcorner, \neg T\ulcorner T(\bar{m})\urcorner$ holds and so by (T-Del) we can deduce

(2) $$\vdash^{1}_{0} T(\bar{m}), \neg T\ulcorner T(\bar{m})\urcorner.$$

However (Ax.3) also allows us to deduce (2) for any m not an $\mathcal{L}_T$-sentence. Hence, by an application of $(\vee_0)$ and $(\vee_1)$ we deduce

$$\vdash^{3}_{0} \neg T\ulcorner T(\bar{m})\urcorner \vee T(\bar{m})$$

holds for each m. (ω) finishes the derivation of T-Deletion. ∎

3.14 Theorem *Suppose $\mathcal{F} \vdash A$ and n is the length of this derivation, i.e. the total number of symbols occurring in the proof. Then* $\mathrm{F}_\infty \vdash^{f_n(0)}_{n,\,n} A^*$.

Proof Note that $f_m(0) < f_k(0)$ holds for all $m < k$.

We proceed by induction on n. Observe that all the axioms of $\mathcal{F}$ are provable in F_∞ with length $f_0(0) = \varepsilon_0$, cut-rank 0 and T-Intro rank 1 owing to Proposition 3.13.

Suppose the last inference was T-Elim. Then $\mathcal{F} \vdash T\ulcorner A \urcorner$ with total length $m < n$. Inductively we then have $\mathrm{F}_\infty \vdash^{f_m(0)}_{m,\,m} T\ulcorner A \urcorner$. Theorem 3.11 implies that $\mathrm{F}_\infty \vdash^{f_m(f_m(0))}_{m,\,m} A$. As $f_m(f_m(0)) \le f_n(0)$ we arrive at $\mathrm{F}_\infty \vdash^{f_n(0)}_{n,\,n} A$.

If the last inference was $\neg$T-Elimination a simplified argument employing Theorem 3.11 (ii) can be used. Furthermore, any application of T-Introduction follows through to F_∞. In the case of any other inference the assertion follows immediately from the induction hypothesis using the same inference or (ω) if the last inference was $(\forall)$. Additionally observe that in the case of (Cut) the total length of the proof exceeds the cut rank of any cut formula appearing in it. ■

3.15 Theorem *The theories $\mathcal{F}$ and* $\mathrm{PA}+\mathsf{TI}(<\varphi\omega 0)$ *prove the same arithmetical theorems.*

Proof The lower bound is provided by Corollary 2.45. For the upper bound suppose A is an arithmetical sentence, $\mathcal{F} \vdash A$ and the length of this proof is n. By Theorem 3.14 we deduce that $\mathrm{F}_\infty \vdash^{f_n(0)}_{n,\,n} A$. The cuts in this derivation may be eliminated and as $f_n(0) < \varphi(n+1)0$ we deduce $\mathrm{F}_\infty \vdash^{\varphi(n+1)0}_{n} A$. Moreover we deduce $\mathrm{F}_\infty \vdash^{\varphi(n+1)0}_{0} A$ as A is arithmetical and thus A is derivable in F_∞ directly from axiom (Ax.1) and rules $(\wedge)$, $(\vee)$, (ω), and $(\exists)$. Whence if A is of complexity Π^0_k for some (meta) k, then all formulae occurring in this cut-free derivation belong to the same complexity class. Thus by employing a partial truth predicate for formulae of complexity at most Π^0_k and transfinite induction up to $\varphi(n+1)0$ we would like to conclude that A holds. For this procedure to work, though, it is necessary to formalize infinite derivations in our background theory. To be able to do this we need to show that certain infinite derivations and operations like cut elimination we perform on them can be dealt with in the theory $\mathrm{PA} + \mathsf{TI}(<\varphi\omega 0)$.

The language of PA is not rich enough to allow for quantification over arbitrary infinite derivations. However, it is well-known that for proof-theoretic purposes it suffices to deal with recursive infinite derivations or codes for recursive derivations (even primitive recursive ones) where the nodes of the proof tree are given by a recursive function (cf. [23, 12]). A recursive F_∞-proof is a recursive tree, with each node labeled by: A sequent, a rule of inference or the designation "Axiom", two sets of formulas specifying the set of principal and minor formulas,respectively, of that inference, and three ordinals (length, T-rank, and cut-rank) such that the sequent is obtained from those immediately above it through application of the specified rule of inference of F_∞. The well-foundedness of a proof–tree is then witnessed by the (first) ordinal "tags" which are in reverse order of the tree order. We then have to show that none of our manipulations on recursive F_∞–derivations leads us beyond this class of recursive proof–trees. This is standard for the cut-elimination theorem and its pertaining lemmas (cf. [23], section 5). The proof of Lemma 3.10 requires transfinite induction only up to $f_n(0)$ for a sufficiently large (meta) n. The predicates $\mathfrak{M}^\beta_m \models \Gamma$ for $m < n$ occur only in case $\beta \leq f_n(0)$ and Γ is a sequent containing literals. But it is also necessary to couch the definition of the structures $\mathfrak{M}^\alpha_n$ for $n > 0$ in terms of recursive proofs, i.e.,

$$\mathfrak{M}^\alpha_n = \left\langle \mathbb{N}, \{\ulcorner B \urcorner : \mathrm{F}^{rec}_\infty \vdash^{f_n(\alpha)}_m B \text{ for some } m < n\} \right\rangle,$$

where $\mathrm{F}^{rec}_\infty \vdash^{\gamma}_m \Gamma$ signifies that Γ is derivable via a *recursive* cut-free proof in F_∞ with length $\leq \gamma$ and T-intro rank $\leq m$.

Moreover, by viewing instances of (T-Intro) as axioms a cut-free derivation also involves only formulae with complexity at most that of the conclusion. Hence, given $\mathrm{E}_\infty \vdash^\alpha_n A$ with A an arithmetical formula of complexity k we may replicate the proof of Lemma 3.10 within $\mathsf{PA} + \mathsf{TI}(f_n(0))$ by replacing $\mathfrak{M}^\beta_m$ for $m \leq n$ with partial truth predicates (indexed by ordinals) for formulae of complexity k. ∎

3.2 An infinitary system for $\mathcal{G}$

3.16 Definition ($\mathrm{G}_\infty \vdash^\alpha_k \Gamma$) *Define* $\mathrm{G}_\infty \vdash^\alpha_k \Gamma$ *by induction on* α *by the rules (Ax.1), (Ax.2), (Ax.3), ($\vee_i$), ($\wedge$), ($\forall$), ($\exists$), (Cut), (T-Imp), (T-U-Inf) (as defined in Definition 3.3 but without a bound on the T-Intro rank of the derivation) together with the rules*

(T-Intro) *If* $\vdash^\alpha_k A$ *and* $\alpha < \beta$ *then* $\vdash^\beta_k \Gamma, T(t)$ *whenever* $t^\mathbb{N} = \ulcorner A \urcorner$.

(T-Rep) *If* $\vdash^\alpha_k \Gamma, T(t)$ *and* $\alpha < \beta$ *then* $\vdash^\beta_k \Gamma, T(s)$ *whenever* $s^\mathbb{N} = \ulcorner T(t) \urcorner$.

Note, that the rule (T-Del) in F_∞ *has been dropped in favour of (T-Rep) above. (T-Rep) is also considered a T-rule.*

3.17 Remark *Unlike in* F_∞ *we are not interested in keeping a bound on the applications of (T-Intro) used in a derivation; therefore we drop it as a parameter in the deductions.*

3.18 Proposition *Cases (i)–(vi) of Proposition 3.4 are all provable for* G_∞.

Proof First we note that the existence of (T-Del) was completely independent for the results of Proposition 3.4 in that it played no role except when considering the case where (T-Del) was the last used inference. We provide the additional argument for (vi) (the other results follow by similar arguments).

Suppose, instead that

$$\vdash^\alpha_k \Gamma, T(t), T(s)$$

($t^\mathbb{N} = s^\mathbb{N}$) was derived via (T-Rep). Without loss of generality we may assume $T(s)$ is active. Then $\vdash^\beta_k \Gamma, T(t), T(s'), T(s)$ holds for some $\beta < \alpha$ and s such that $s^\mathbb{N} = \ulcorner T(s') \urcorner$. By (iv) of this proposition we deduce $\vdash^\beta_k \Gamma, T(s), T(s')$ holds, and hence we deduce $\vdash^\alpha_k \Gamma, T(t)$ by (T-Rep) as desired. ∎

The following now holds true for G_∞.

3.19 Lemma (Reduction Lemma for G_∞) *Suppose* $\frac{\alpha}{k}\, \Gamma, A$ *and* $\frac{\beta}{k}\, \Delta, \neg A$ *with* $|A| = k$. *Then* $\frac{\alpha \# \beta}{k}\, \Gamma, \Delta$.

Proof Precisely the same proof may be applied here as for the Reduction Lemma 3.6 for F_∞. The proof of Lemma 3.6 does not conflict with the removal of (T-Del) and addition of (T-Rep) since the only situation a problem may arise is in case IIa and there the proof only depends on those rules and axioms deriving negative formulae, that is, deductions of the form $\frac{\alpha}{k}\, \Gamma, \neg T(t)$ with $\neg T(t)$ active. The presence of (T-Rep) and the loss of (T-Del) have no effect on this. ■

3.20 Theorem (Cut-Elimination Theorem for G_∞)

$$\mathrm{G}_\infty \frac{\alpha}{k+1}\, \Gamma \ \textit{ implies } \mathrm{G}_\infty \frac{\omega^\alpha}{k}\, \Gamma \,.$$

Proof Immediate consequence of the Reduction Lemma. ■

The next step is to show that G_∞ is closed under T-Elimination. For this we employ a similar method as for F_∞ (Theorem 3.11).

3.21 Definition *For each ordinal* α *in* OT *define*

$$\mathfrak{M}^G_\alpha = \left\langle \mathbb{N}, \{\ulcorner B \urcorner : \mathrm{G}_\infty \frac{\varepsilon_\alpha}{0}\, B\,\} \right\rangle .$$

3.22 Proposition *Let* $\alpha \leq \beta$ *and* A *be a* T*-positive sentence. Then* $\mathfrak{M}^G_\alpha \models A$ *implies* $\mathfrak{M}^G_\beta \models A$.

Proof Immediate by the definition of $\mathfrak{M}^G_\alpha$. ■

3.23 Lemma *If* Γ *is* T*-positive and* $\mathrm{G}_\infty \frac{\alpha}{0}\, \Gamma$ *then* $\mathfrak{M}^G_\alpha \models \Gamma$.

Proof We proceed by induction on α. If Γ is an axiom then $\mathfrak{M}^G_\alpha \models \Gamma$ is immediate, so assume $\frac{\alpha}{0}\, \Gamma$ was derived via a rule. The cases of $(\vee_i)$, $(\wedge)$, $(\exists)$, (ω) follow by the inductive definition of satisfaction and as the derivation is assumed cut-free no application of (Cut) may have been used.

(T-Intro). In this case Γ is $\Gamma', T(t)$ where $t^{\mathbb{N}} = \ulcorner A \urcorner$ and $\frac{\gamma}{0}\, A$ holds for some $\gamma < \alpha$. Indeed by assumption we have $\mathfrak{M}^G_\alpha \models T(t)$.

(T-Imp). Now suppose $\frac{\alpha}{0}\, \Gamma, T(s)$, $\frac{\gamma}{0}\, \Gamma, T(t), T(s)$ and $\frac{\delta}{0}\, \Gamma, T(\mathbf{imp}(t, s)), T(s)$ hold where γ, $\delta < \alpha$. Seeking a contradiction suppose $\mathfrak{M}^G_\alpha \models \neg\Gamma \wedge \neg T(s)$. Then we have $\mathfrak{M}^G_\gamma \models T(t)$ and $\mathfrak{M}^G_\delta \models T(\mathbf{imp}(t, s))$ by the induction hypothesis and Proposition 3.22. Therefore $s^{\mathbb{N}}$ and $t^{\mathbb{N}}$ are sentences, say A and B respectively, and both $\mathrm{G}_\infty \frac{\varepsilon_\gamma}{0}\, B$ and $\mathrm{G}_\infty \frac{\varepsilon_\delta}{0}\, \neg B, A$ hold. By (Cut) we deduce $\mathrm{G}_\infty \frac{\varepsilon_\alpha}{k}\, A$, where $k = |B| + 1$, and so $\mathrm{G}_\infty \frac{\varepsilon_\alpha}{0}\, A$ follows from Theorem 3.20, since $\omega^{\varepsilon_\alpha} = \varepsilon_\alpha$. Thus $\mathfrak{M}^G_\alpha \models T(s)$, yielding a contradiction. Hence $\mathfrak{M}^G_\alpha \models \Gamma \vee T(s)$.

(T-Rep). Suppose $\mathfrak{M}^G_\gamma \models \Gamma \vee T(s)$ and $\gamma < \alpha$. We want to show $\mathfrak{M}^G_\alpha \models \Gamma \vee T(t)$ where $t^{\mathbf{N}} = \ulcorner T(s) \urcorner$. If $\mathfrak{M}^G_\gamma \models \Gamma$ we are done by Proposition 3.22, so suppose $\mathfrak{M}^G_\gamma \models T(s)$. In this case $s^{\mathbb{N}}$ is the Gödel number of a sentence, say A and $\mathrm{G}_\infty \overset{\varepsilon_\gamma}{\underset{0}{\vdash}} A$ holds. By an application of (T-Intro) we obtain $\mathrm{G}_\infty \overset{\varepsilon_\alpha}{\underset{0}{\vdash}} T(s)$ and hence $\mathfrak{M}^G_\alpha \models T(t)$.

(T-U-Inf). If, for each n, $\gamma_n < \alpha$ and $\mathfrak{M}^G_{\gamma_n} \models T\ulcorner A(\bar{n}) \urcorner$ then we deduce $\overset{\varepsilon_\alpha}{\underset{0}{\vdash}} \forall x A(x)$ and so $\mathfrak{M}^G_\alpha \models T\ulcorner \forall x A(x) \urcorner$. ■

3.24 Theorem (T-Elimination Theorem for G_∞) *For every α and every $\mathcal{L}_T$-sentence A*

(i). $\mathrm{G}_\infty \overset{\alpha}{\underset{0}{\vdash}} T\ulcorner A \urcorner$ *implies* $\mathrm{G}_\infty \overset{\varepsilon_\alpha}{\underset{0}{\vdash}} A$.

(ii). $\mathrm{G}_\infty \overset{\alpha}{\underset{0}{\vdash}} \neg T\ulcorner A \urcorner$ *implies* $\mathrm{G}_\infty \overset{\alpha}{\underset{0}{\vdash}} \neg A$.

Proof (i). If $\mathrm{G}_\infty \overset{\alpha}{\underset{0}{\vdash}} T\ulcorner A \urcorner$ holds then by Lemma 3.23 it follows that $\mathfrak{M}^G_\alpha \models T\ulcorner A \urcorner$, and so $\mathrm{G}_\infty \overset{\varepsilon_\alpha}{\underset{0}{\vdash}} A$ holds, as desired.

For part (ii), as for F_∞, we observe we cannot make a cut-free derivation of $\neg T(\ulcorner A \urcorner)$ for any sentence A and so (ii) holds vacuously. ■

3.25 Proposition *(i). If A is an axiom of $Base_T$ then* $\mathrm{G}_\infty \overset{\omega+n}{\underset{k}{\vdash}} A^*$ *for some finite n and k.*

(ii). $\mathrm{G}_\infty \overset{n}{\underset{0}{\vdash}}$ T-Rep $\wedge$ $\forall$-Inf *for some n.*

Proof The same argument as in Proposition 3.13. ■

3.26 Theorem (Embedding Theorem for G_∞) *Suppose $\mathcal{G} \vdash A$ and the length of this proof is n. Then* $\mathrm{G}_\infty \overset{g(n)}{\underset{0}{\vdash}} A^*$ *where $g(0) = \varepsilon_0$ and $g(n+1) = \varepsilon_{g(n)}$.*

Proof Proposition 3.25 and Theorem 3.24 allow us to deduce $\mathrm{G}_\infty \overset{g(n)}{\underset{n}{\vdash}} A^*$ in the same manner as Theorem 3.14. Eliminating the cuts completes the proof. ■

3.27 Corollary *$\mathcal{G}$ and* PA + $\mathsf{TI}(<\varphi 20)$ *prove the same arithmetical statements.*

Proof We proceed with essentially the same argument as for Theorem 3.15, however Theorem 2.41 provides the lower bound for $\mathcal{G}$ and the previous theorem provides the basis for the upper bound. ■

3.3 An infinitary system for $\mathcal{I}$

To establish an upper bound for the system $\mathcal{I}$ we have to refine the the model construction technique used for $\mathcal{G}$ and replace G_∞ in the definition of $\mathfrak{M}^G_\alpha$ by an infinitary version of PA_T.

3.28 Definition ($\mathrm{I}_\infty\overset{\alpha}{\underset{k}{\vert}}\Gamma$) *Let* PA^T_∞ *denote the infinitary system given by the arithmetical axioms and rules and (Ax.2), (Ax.3) and (T-Intro), i.e. (Ax.1), (Ax.2), (Ax.3), ($\wedge$), ($\vee$), (ω), ($\exists$), (Cut) and (T-Intro).*

Now we define the system I_∞ *by the rules of* G_∞ *excluding (T-Intro) with the addition of (T-Del) (defined in Definition 3.3, without the parameter 'n') and the axiom*

(Ax.4) $\mathrm{I}_\infty\overset{\alpha}{\underset{k}{\vert}}\Gamma, T(t)$ *whenever* $t^{\mathbb{N}}$ *is the Gödel number of an* $\mathcal{L}_T$*-sentence* A *and* $\mathrm{PA}^T_\infty\overset{\alpha}{\underset{0}{\vert}} A$ *.*

Explicitly, I_∞ *has the axioms (Ax.1), (Ax.2), (Ax.3), and (Ax.4) and the rules ($\vee_i$), ($\wedge$), (ω), ($\exists$), (Cut), (T-Imp), (T-U-Inf), (T-Rep) and (T-Del).*

In view of our work in previous sections the following are easily shown

3.29 Proposition *(i). Cases (i)–(vi) of Proposition 3.4 hold for both* PA^T_∞ *and* I_∞*;*

(ii). $\mathrm{PA}^T_\infty\overset{\alpha}{\underset{0}{\vert}}\Gamma$ *implies* $\mathrm{I}_\infty\overset{\alpha}{\underset{0}{\vert}}\Gamma$ *;*

(iii). If $\mathrm{PA}^T_\infty\overset{\alpha}{\underset{0}{\vert}} T\ulcorner A\urcorner$ *then* $\mathrm{PA}^T_\infty\overset{\alpha}{\underset{0}{\vert}} A$ *.*

Proof We prove (iii) as this is the only non-obvious case considering our previous work. $\mathrm{PA}^T_\infty\overset{\alpha}{\underset{0}{\vert}} T\ulcorner A\urcorner$ may only arise by (T-Intro), hence $\mathrm{PA}^T_\infty\overset{\beta}{\underset{0}{\vert}} A$ for some $\beta \leq \alpha$. ■

3.30 Theorem *Both* I_∞ *and* PA^T_∞ *enjoy cut elimination with the same bounds as for* F_∞*.*

Proof (Ax.4) produces only T-positive active formulae, thus it is clear that the same proof as Theorem 3.7 works for I_∞. ■

As before we still require to show I_∞ is closed under T-Elimination. This will be done with a similar method to before.

3.31 Definition *Define* $\mathcal{L}_T$*-structures* $\mathfrak{M}^I_\alpha$ *by*

$$\mathfrak{M}^I_\alpha = \left\langle \mathbb{N}, \{\ulcorner B\urcorner : \mathrm{PA}^T_\infty\overset{\varepsilon_\alpha}{\underset{0}{\vert}} B\,\}\right\rangle .$$

Then the following Lemma holds.

3.32 Lemma *If* $\mathrm{I}_\infty\overset{\alpha}{\underset{0}{\vert}}\Gamma$ *and* Γ *is a T-positive sequent then* $\mathfrak{M}^I_\alpha \models \Gamma$*.*

Proof The proof follows the same routine as for G_∞. If Γ is an instance of (Ax.4) then $\mathfrak{M}^I_\alpha \models \Gamma$ by definition and if the last inference was (T-Del) the result follows by Proposition 3.29(iii). The other cases are identical to before. ■

By combining this and Proposition 3.29 we can conclude that I_∞ is closed under T-Elimination.

3.33 Theorem $\mathrm{I}_\infty\overset{\alpha}{\underset{0}{\vert}} T\ulcorner A\urcorner$ *implies* $\mathrm{I}_\infty\overset{\varepsilon_\alpha}{\underset{0}{\vert}} A$ *.* ■

All that remains is to embed $\mathcal{I}$ into I_∞ and read off the ordinal bounds; the following are proved by essentially the same arguments as their counterparts in the last section.

3.34 THEOREM *Suppose $\mathcal{I} \vdash A$ and n is the total length of this derivation. Then* $\mathrm{I}_\infty \frac{\alpha}{0} A^*$ *for some $\alpha < g(n)$, where g is defined as in Theorem 3.26.*

3.35 COROLLARY *$\mathcal{I}$ is a conservative extension of* PA + $\mathsf{TI}(<\varphi 20)$ *and thus* $|\mathcal{I}| = \varphi 20$.

PROOF We proceed with essentially the same argument as for Theorem 3.15, however Theorem 2.41 provides the lower bound for $\mathcal{I}$ and the previous theorem provides the basis for the upper bound. ■

Our techniques also allow us the following observation.

3.36 THEOREM *The conjunction of the following two sentences is independent of $\mathcal{I}$.*

$$T(\ulcorner \forall x \forall y.\ (T(x) \wedge T(\mathbf{imp}(x,y))) \rightarrow T(y)\urcorner)$$
$$T(\ulcorner \forall \ulcorner A(x)\urcorner.\ \forall n\, T(\ulcorner A(\dot{n})\urcorner) \rightarrow T(\ulcorner \forall x A(x)\urcorner)\urcorner).$$

Moreover, the theory obtained by adding this as an axiom to $\mathcal{I}$ has proof-theoretic ordinal $\varphi 30$.

PROOF We will sketch the proof here; full details can be found in [18]. Denote by (†) the conjunction of the two sentences.

Consistency of the resulting theory is a consequence of Friedman and Sheard's model construction for $\mathcal{I}$, however, some work must be done in order to establish an upper bound on its strength. To that aim we observe that (†) is derivable in the system $\mathrm{I}^\dagger_\infty$ given as I_∞ with (Ax.4) replaced by

(Ax.4†) $\mathrm{I}^\dagger_\infty \frac{\alpha}{k} \Gamma, T(t)$ whenever $t^{\mathbb{N}}$ is the Gödel number of an $\mathcal{L}_T$-sentence A and $\mathrm{G}_\infty \frac{\alpha}{0} A$.

Closure under T-Elimination can be shown for this system in the same manner as before using the model

$$\mathfrak{M}^\dagger_\alpha := \left\langle \mathbb{N}, \{\ulcorner B\urcorner : \mathrm{G}_\infty \frac{\varphi 2\alpha}{0} B\,\}\right\rangle$$

but with necessarily larger bounds, namely $\mathrm{I}^\dagger_\infty \frac{\varphi 2\alpha}{0} A$ whenever $\mathrm{I}^\dagger_\infty \frac{\alpha}{0} T\ulcorner A\urcorner$. Thus $\mathcal{I}$ + (†) proves no more arithmetical statements than PA + $\mathsf{TI}(<\varphi 30)$.

The lower bound is the easier to establish as we simply observe that

$$\mathcal{I} + (\dagger) \vdash \forall x.\mathsf{Prov}_{S_1}(x) \wedge \mathsf{Sent}_{\mathcal{L}_T}(x) \rightarrow T(x)$$

from which we deduce $\mathcal{I} + (\dagger) \vdash I(\varphi\bar{2}\bar{\alpha})$, if $\mathcal{I} + (\dagger) \vdash I(\bar{\alpha})$. As the addition of (†) to $\mathcal{I}$ yields a stronger (but consistent) theory (†) must be independent from $\mathcal{I}$. ■

3.4 An infinitary system for $\mathcal{E}$

$\mathcal{E}$ is the most difficult theory to analyse. The presence of T-Consistency (along with T-Introduction and ∀-Inference) means $\mathcal{E}$ is ω-inconsistent (McGee [19]). Therefore the natural choice for E_∞ could not possibly support full cut elimination. In fact the situation is worse; due to ¬T-Introduction if we attempted to carry out the proof of Theorem 3.7 in this situation it would appear necessary to deal with negative occurrences of T in sequents. However, it was crucial to the proof that we only ever encountered T-positive sequents (Proposition 3.9 would fail otherwise).

Fortunately all is not lost. By formulating the rule (T-Cons) to represent T-Consistency immediately followed by a cut we can ensure T-positive premises yield only T-positive conclusions and reproduce Theorem 3.7 in the context of $\mathcal{E}$.

3.37 Definition (Inductive definition of E_∞) *Define* $\mathrm{E}_\infty \overset{\alpha}{\underset{n,k}{\vdash}} \Gamma$ *as* F_∞ *in Definition 3.3 with the addition of*

$$\text{(T-Cons)}\quad \textit{If}\ \overset{\alpha}{\underset{n,k}{\vdash}} \Gamma, T(s),\ \ \overset{\beta}{\underset{n,k}{\vdash}} \Gamma, T(t),\ s^{\mathbb{N}} = (\mathbf{neg}(t))^{\mathbb{N}}\ \textit{and}\ \alpha,\beta < \delta\ \textit{then}\ \overset{\delta}{\underset{n,k}{\vdash}} \Gamma.$$

That is, E_∞ *has the rule (T-Cons) and all the rules and axioms of* F_∞*, namely (Ax.1), (Ax.2), (Ax.3), (*$\wedge$*), (*$\vee_i$*), (*$\exists$*), (*$\forall$*), (Cut), (T-Intro), (T-Imp), (T-Del), (T-U-Inf).*

3.38 Proposition *Cases (i)–(vi) of Proposition 3.4 hold for* E_∞. ■

3.39 Notation *Let* $\mathrm{E}_\infty \overset{\alpha}{\underset{n}{\vdash}} \Gamma$ *denote* $\mathrm{E}_\infty \overset{\alpha}{\underset{n,0}{\vdash}} \Gamma$*, that is no applications of rule (Cut) have been used. (The rule (T-Cons) may still have been used.)*

3.40 Theorem (Cut Elimination for E_∞)

$$\mathrm{E}_\infty \overset{\alpha}{\underset{n,k}{\vdash}} \Gamma\ \textit{implies}\ \mathrm{E}_\infty \overset{\omega_k(\alpha)}{\underset{n}{\vdash}} \Gamma,$$

where $\omega_0(\alpha) = \alpha$ *and* $\omega_{k+1}(\alpha) = \omega^{\omega_k(\alpha)}$.

Proof Since the rule (T-Cons) has no active formulae the same argument as for Theorem 3.7 suffices here. ■

3.41 Remark *Although we are able to remove all applications of (Cut) in a derivation we are not, in general, able to remove applications of (T-Cons). Therefore, although it may still be possible to derive the empty sequent, any cut-free derivation of it must involve only literals.*

We will now establish that E_∞, like F_∞, is closed under T-Elimination. Define, for each $n < \omega$, the $\mathcal{L}_T$-structure

$$\mathfrak{M}_n^\alpha = \left\langle \mathbf{N}, \{\ulcorner B \urcorner : \mathrm{E}_\infty \overset{f_n(\alpha)}{\underset{m}{\vdash}} B\ \text{ for some } m < n\} \right\rangle$$

with $\mathfrak{M}_0^\alpha$ denoting $\langle \mathbf{N}, \emptyset \rangle$. Recall $f_n(\alpha) = \varphi n \varphi 1 \alpha$.

3.42 Theorem *Suppose* $\mathrm{E}_\infty \overset{\alpha}{\underset{n}{\vdash}} \Gamma$ *and* Γ *is* T*-positive. Then the following holds*

(i). $\mathfrak{M}_n^\alpha \models \Gamma$*;*

(ii). Γ *is not the empty sequent;*

(iii). If Γ *is* $T(t)$ *then* $t^{\mathbb{N}} = \ulcorner A \urcorner$ *for some* $\mathcal{L}_T$*-sentence* A *and* $\mathrm{E}_\infty \overset{f_n(\alpha)}{\underset{m}{\vdash}} A$ *for some* $m < n$*.*

Proof By main induction on n and subsidiary transfinite induction on α.

(iii) is an immediate consequence of (i) and (ii) holds if the last applied rule was not (T-Cons). Furthermore if $\mathrm{E}_\infty \vdash^{\alpha}_{n} \Gamma$ arose by an application of any rule other than (T-Cons) the argument in Lemma 3.10 would suffice to show (i).

Hence we may assume $\mathrm{E}_\infty \vdash^{\beta}_{n} \Gamma, T(s)$ and $\mathrm{E}_\infty \vdash^{\beta}_{n} \Gamma, T(\mathbf{neg}(s))$ for some $\beta < \alpha$ and term s and the last rule used was (T-Cons). By the induction hypothesis we deduce $\mathfrak{M}^{\beta}_{n} \models \Gamma, T(s)$ and $\mathfrak{M}^{\beta}_{n} \models \Gamma, T(\mathbf{neg}(s))$. If $n = 0$ we deduce Γ may not be empty and $\mathfrak{M}^{\alpha}_{n} \models \Gamma$. Thus we may assume $n = m + 1 > 0$. If either Γ is empty or $\mathfrak{M}^{\alpha}_{n} \not\models \Gamma$ then $\mathfrak{M}^{\beta}_{n} \models T(s) \wedge T(\mathbf{neg}(s))$ as Γ is T-positive. Therefore $s^{\mathbb{N}} = \ulcorner A \urcorner$, say, and both $\mathrm{E}_\infty \vdash^{f_n(\beta)}_{m} A$ and $\mathrm{E}_\infty \vdash^{f_n(\beta)}_{m} \neg A$ hold; hence we may deduce $\mathrm{E}_\infty \vdash^{f_n(\alpha)}_{m} \emptyset$, in contradiction with the induction hypothesis (ii). That is (i) and (ii) must hold. ∎

3.43 Remark At this point it might be instructive to see why E_∞, in contrast to F_∞, is not closed under $\neg T$-Elimination. The proof of Theorem 3.11 (ii) fails because even if the proof of $\neg T \ulcorner A \urcorner$ is assumed cut-free, $\mathrm{E}_\infty \vdash^{\alpha}_{n} \neg T \ulcorner A \urcorner$ need not be an axiom. Indeed the last inference could have been (T-Cons). From [11, p.14] we know that the addition of $\neg T$-Elimination would render $\mathcal{E}$ and E_∞ inconsistent. If we take B to be the usual Liar Sentence, that is, PA_T proves $B \leftrightarrow \neg T \ulcorner B \urcorner$, $Base_T$ then proves $T \ulcorner \neg B \urcorner \leftrightarrow T \ulcorner T \ulcorner B \urcorner \urcorner$ (see [11, p.14]), and hence $\mathrm{E}_\infty \vdash^{\alpha}_{n} \neg T \ulcorner T \ulcorner B \urcorner \urcorner, T \ulcorner \neg B \urcorner$ for some α, n. With the help of (T-Del) we also get $\mathrm{E}_\infty \vdash^{\alpha'}_{n'} \neg T \ulcorner T \ulcorner B \urcorner \urcorner, T \ulcorner B \urcorner$ for some α', n'. By means of (T-Cons) we can conclude that $\vdash^{\beta}_{m} \neg T \ulcorner T \ulcorner B \urcorner \urcorner$ for some β, m. But $\neg T \ulcorner B \urcorner$ cannot be derived since this would entail the derivability of B and consequently $T \ulcorner T \ulcorner B \urcorner \urcorner$ and therefore yield an inconsistency. Thus $\neg T \ulcorner T \ulcorner B \urcorner \urcorner$ provides an explicit example for the failure of $\neg T$-Elimination in E_∞.

3.44 Theorem *Suppose $\mathcal{E} \vdash A$ and n is the length of this derivation, i.e. the total number of symbols occurring in the proof. Then* $\mathrm{E}_\infty \vdash^{f_n(0)}_{n,\,n} A^*$.

Proof Suppose $\mathcal{E} \vdash A$ and the length of this derivation is n. We will show $\mathrm{E}_\infty \vdash^{\alpha}_{n,n} A^*$ for some $\alpha \leq f_n(0)$ by induction on n.

Since ¬T-Introduction is derivable (in $\mathcal{E}$) from T-Introduction and T-Consistency and applications of T-Elimination are covered by Theorem 3.42 all that remains is to show T-Consistency is derivable in E_∞. This can be obtained by simply applying the rule (T-Cons) to the axioms $\neg T(\mathbf{neg}(\bar{n})), \neg T(\bar{n}), T(\bar{n})$ and $\neg T(\mathbf{neg}(\bar{n})), \neg T(\bar{n}), T(\mathbf{neg}(\bar{n}))$. ∎

Finally we may characterize the proof strength of $\mathcal{E}$.

3.45 Theorem *If A is arithmetical then $\mathcal{E} \vdash A$ iff* $\mathrm{PA} + \mathsf{TI}({<}\varphi\omega 0) \vdash A$.

Proof The lower bound is provided by Corollary 2.45. For the upper bound we proceed with essentially the same argument as for Theorem 3.15, but using the previous Theorem 3.44 in place of Theorem 3.14. ∎

3.46 Corollary $|\mathcal{E}| = \varphi\omega 0$. ∎

4 Conclusion

We have now established all the technical results needed to locate the truth theories on the scale of standard theories. The next Definition will recall some of these standard systems.

4.1 Definition The following theories all share the language of second order arithmetic (see [26]) with lower case variables $x, y, z, \ldots$ intented to range over natural numbers and upper case variables $X, Y, Z, \ldots$ intended to range over sets of natural numbers.

The Σ^1_1-DC (Dependent Choice) scheme is the collection of formulae

$$\forall x \forall X \exists Y B(x, X, Y) \to \forall U \exists Z[(Z)_0 = U \wedge \forall x B(x, (Z)_x, (Z)_{x+1})]$$

where B is Σ^1_1. Here by $(Z)_x$ we refer to the x-th section of the set Z, i.e. $(Z)_x$ denotes the set $\{y \mid \langle x, y\rangle \in Z\}$ with $\langle ., . \rangle$ being a standard primitive recursive pairing function. The system $\Sigma^1_1\text{-DC}_0$ is defined to be $\text{ACA}_0 + \Sigma^1_1\text{-DC}$. The proof-theoretic ordinal of $\Sigma^1_1\text{-DC}_0$ was determined to be $\varphi\omega 0$ in [5].

The rule

$$\text{(HCR)} \quad \frac{\forall x[\exists Y\, A(x, Y) \leftrightarrow \forall Z\, B(x, Z)]}{\exists X \forall x\, [x \in X \leftrightarrow \exists Y\, A(x, Y)]}$$

with $A(x, Y)$ and $B(x, Z)$ arithmetical, known as the *hyperarithmetic* or Δ^1_1-*comprehension rule*, was introduced by Kreisel (cf. [17]). Let Δ^1_1-CR be the theory ACA + HCR. The strength of Δ^1_1-CR was investigated by Feferman and Schütte (cf. [7, 22]). Δ^1_1-CR has proof-theoretic ordinal $\varphi\omega 0$ according to [22, VIII,Theorem 23.5].

Another well-known rule is the *Bar Rule*

$$\text{(BR)} \quad \frac{\text{WO}(\prec)}{\text{TI}(\prec, F)}$$

where $\prec$ is any primitive recursive ordering, $\text{WO}(\prec)$ expresses that $\prec$ is a well-ordering, and $\text{TI}(\prec, F)$ asserts transfinite induction along $\prec$ for arbitrary formulae $F(x)$ of second order arithmetic.

The scheme of *parameter-free Bar induction* for primitive recursive orderings, $\text{BI}^{\square}_{\text{PR}}$, consists of all formulae

$$\text{WO}(\prec) \to \text{TI}(\prec, F)$$

where $F(x)$ may be any formula of second order arithmetic and $\prec$ is a primitive recursive ordering whose defining formula does not depend on parameters (for details see [21]).

The theory ID^*_1 is a restricted version of the ubiquitous theory of positive inductive definitions ID_1 (cf. [3]) in that the scheme for proof by induction on the inductively defined predicates is only permitted for formulae in which the predicates for the inductively defined sets occur positively (cf. [8, 4, 20, 2]).

4.2 Theorem *We shall write $T \equiv S$ to convey that the theories T and S prove the same arithmetical statements.*

(i). $\mathcal{A} \equiv \text{PA}$.

(ii). $\mathcal{B} \equiv \mathcal{C} \equiv \text{ACA}$.

(iii). $\mathcal{D} \equiv \mathcal{G} \equiv \mathcal{I} \equiv \mathrm{ACA}_0^+ \equiv \mathrm{ACA}_0 + \mathrm{BR} \equiv \mathrm{ACA}_0 + \mathrm{BI}_{\mathrm{PR}}^{\Box}$.

(iv). $\mathcal{E} \equiv \mathcal{F} \equiv \Sigma^1_1\text{-DC}_0 \equiv \Delta^1_1\text{-CR} \equiv \mathrm{ID}_1^*$.

Proof (i) follows from 2.8. (ii) is a consequence of 2.16. (iii) follows from 2.29, 3.27, and 3.35 in conjunction with [21, Theorem 3.5]. The first three equivalences of (iv) are due to 3.15, 3.45 and Theorem 4.3 of [5].

The equivalence with Δ^1_1-CR can be obtained from [22, VIII,Theorem 23.5] while the equivalence with the theory of positive induction ID_1^* follows from [20] as well as [2]. Many more examples of natural theories of ordinal strength $\varphi\omega 0$ can be found in [16]. ■

References

[1] B. Afshari and M. Rathjen: *Reverse Mathematics and Well-ordering Principles: A Pilot Study*, Annals of Pure and Applied Logic 160 (2009) 231-237.

[2] B. Afshari and M. Rathjen: *A note on* ID_1^*, Archive for Mathematical Logic, to appear.

[3] Wilfried Buchholz, Solomon Feferman, Wolfram Pohlers and Wilfried Sieg: *Iterated inductive definitions and subsystems of Analysis: Recent Proof-Theoretical Studies theories* (Springer-Verlag, Berlin, Heidelberg, 1981).

[4] Andrea Cantini: *A note on a predicatively reducible theory of elementary iterated induction*, Bollettino U.M.I. (1985) 413–430.

[5] A. Cantini: *On the relation between choice and comprehension principles in second order arithmetic*, Journal of Symbolic Logic 51 (1986) 360–373.

[6] A. Cantini: *A theory of formal truth arithmetically equivalent to* ID_1, Journal of Symbolic Logic 55 (1990) 244–259.

[7] S. Feferman: *Systems of predicative analysis*, Journal of Symbolic Logic 29 (1964) 1–30.

[8] Solomon Feferman: *Iterated inductive fixed-point theories: Application to Hancock's conjecture*, Patras Logic Symposion (North-Holland, Amsterdam, 1982) 171–196.

[9] S. Feferman: *Toward useful type-free theories, I*, Journal of Symbolic Logic 49 (1984) 75–111.

[10] S. Feferman: *Reflecting on Incompleteness*, Journal of Symbolic Logic 56 (1991) 1–49.

[11] H. Friedman and M. Sheard: *An Axiomatic Approach to Self-Referential Truth*, Annals of Pure and Applied Logic 33 (1987) 1–21.

[12] H. Friedman and M. Sheard: *Elementary descent recursion and proof theory*, Annals of Pure and Applied Logic 71 (1995) 1–45.

[13] Volker Halbach: *Truth and Reduction*, Erkenntnis 53 (2000) 97–126.

[14] V. Halbach: *A System of Complete and Consistent Truth*, Notre Dame Journal of Formal Logic 35 (1994).

[15] D. Hilbert, P. Bernays: *Grundagen der Mathematik I*, (Springer, Berlin,1939).

[16] G. Jäger, T. Strahm: *Some theories with positive induction of ordinal strength* $\varphi\omega 0$, Journal of Symbolic Logic 61 (1996) 818–842.

[17] G. Kreisel: *The axiom of choice and the class of hyperarithmetic functions*, Koninklijke Nederlands Akadenie van Wetenschappen, proceedings, ser. A, vol. 65 (1962) 307–319.

[18] G. Leigh: *Reflecting on Truth*, PhD thesis, University of Leeds, forthcoming.

[19] V. McGee: *How Truth-like can a predicate be? A negative result*, Journal of Philosophical Logic 14 (1985) 399–410.

[20] D. Probst: *The proof-theoretic analysis of transfinitely iterated quasi least fixed points*, Journal of Symbolic Logic 71 (2006) 721–726.

[21] M. Rathjen: *The Role of Parameters in Bar Rule and Bar Induction*, Journal of Symbolic Logic 56 (1991) 715–730.

[22] Kurt Schütte: *Proof Theory* (Springer-Verlag, Berlin, Heidelberg, 1977).

[23] H. Schwichtenberg: *Proof theory: Some applications of cut-elimination*, in: J. Barwise (ed.): *Handbook of Mathematical Logic*, (North-Holland 1977).

[24] M. Sheard: *A Guide to truth predicates in the Modern Era*, Journal of Symbolic Logic 59 (1994) 1032–1053.

[25] M. Sheard: *Weak and Strong Theories of Truth*, Studia Logica 68 (2001) 89–101.

[26] S. Simpson: *Subsystems of Second Order Arithmetic*, (Springer-Verlag, Berlin, Heidelberg, 1998).

[27] C. Smorynski: *The Incompleteness Theorems*, in: J. Barwise (ed.): *Handbook of Mathematical Logic*, (North-Holland 1977) 821–865.

[28] G. Takeuti: *Proof Theory*, second edition (North-Holland, Amsterdam, 1987).

[29] A. Tarski: *Der Wahrheitsbegriff in den formalisierten Sprachen*, Studia Philosophica 1 (1936) 261-404.