The Anti-Foundation Axiom in Constructive Set Theories

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The paper investigates the strength of the anti-foundation axiom on the basis of various systems of constructive set theories.

1 Introduction

Intrinsically circular phenomena have come to the attention of researchers in differing fields such as mathematical logic, computer science, artificial intelligence, linguistics, cognitive science, and philosophy. Logicians first explored set theories whose universe contains what are called non-wellfounded sets, or hypersets (cf. [17], [5]). But the area was considered rather exotic until these theories were put to use in developing rigorous accounts of circular notions in computer science (cf. [7]). Instead of the Foundation Axiom these set theories adopt the so-called Anti-Foundation Axiom, $\text{AFA}$, which gives rise to a rich universe of sets. $\text{AFA}$ provides an elegant tool for modeling all sorts of circular phenomena. The application areas range from knowledge representation and theoretical economics to the semantics of natural language and programming languages.

The subject of hypersets and their applications is thoroughly and timely developed in the book [7] by J. Barwise and L. Moss. While reading [7], I asked myself whether most of the material could be developed on the basis of a constructive universe of hypersets rather than a classical one. My tentative answer is that large chunks of [7] only require constructive set theory. Research in this direction is under way. In the meantime the present paper investigates the strength of $\text{AFA}$

*Supported by the Volkswagen-Stiftung (RiP program Oberwolfach). Most of the results reported in this paper where first presented in a talk given in the Stockholm-Uppsala Logic Seminar in January 1999.

Submitted for Proceedings of LLC9
Ingrid van Loon, Grigori Mints, Reinhard Muskens (eds.)
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on the basis of various systems of constructive set theories, including ones with large set axioms.

Constructive set theory grew out of Myhill’s endeavours (cf. [21]) to discover a simple formalism that relates to Bishop’s constructive mathematics as $\text{ZFC}$ relates to classical Cantorian mathematics. Later on Aczel modified Myhill’s set theory to a system which he called Constructive Zermelo-Fraenkel set theory, $\text{CZF}$, and corroborated its constructiveness by interpreting it in Martin-Löf type theory ($\text{MLTT}$) (cf. [2]). The interpretation was in many ways canonical and can be seen as providing $\text{CZF}$ with a standard model in type theory.

Let $\text{CZF}^-$ be $\text{CZF}$ without $\in$-induction and let $\text{CZFA}$ be $\text{CZF}^-$ plus $\text{AFA}$. I. Lindström (cf. [19]) showed that $\text{CZFA}$ can be interpreted in $\text{MLTT}$ as well. Among other sources, the work of [19] will be utilized in calibrating the exact strength of various extensions of $\text{CZFA}$, in particular ones with inaccessible set axioms. The upshot is that $\text{AFA}$ does not yield any extra proof-theoretic strength on the basis of constructive set theory and is indeed much weaker in proof strength than $\in$-Induction. This contrasts with Kripke-Platek set theory, $\text{KP}$. The theory $\text{KPA}$, which adopts $\text{AFA}$ in place of the Foundation Axiom scheme, is proof-theoretically considerably stronger than $\text{KP}$ as was shown in [28].

This paper also contains several other new results.

2 The anti-foundation axiom

Definition 2.1 A graph will consist of a set of nodes and a set of edges, each edge being an ordered pair $(x, y)$ of nodes. If $(x, y)$ is an edge then we’ll write $x \rightarrow y$ and say that $y$ is a child of $x$.

A path is a finite or infinite sequence $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \ldots$ of nodes $x_0, x_1, x_2, \ldots$ linked by edges $(x_0, x_1), (x_1, x_2), \ldots$.

A pointed graph is a graph together with a distinguished node $x_0$ called its point. A pointed graph is accessible if for every node $x$ there is a path $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \ldots \rightarrow x$ from the point $x_0$ to $x$.

A decoration of a graph is an assignment $d$ of a set to each node of the graph in such a way that the elements of the set assigned to a node are the sets assigned to the children of that node, i.e.

$$d(a) = \{d(x) : a \rightarrow x\}.$$  

Definition 2.2 The Anti-Foundation Axiom, $\text{AFA}$, is the statement that every graph has a unique decoration.
3 AFA in constructive set theory

In this section we will present some results about the proof-theoretic strength of systems of constructive set theory with AFA instead of $\in$-Induction.

3.1 The theory CZFA

The language of CZF is the first order language of Zermelo-Fraenkel set theory, LST, with the non logical primitive symbol $\in$. We assume that LST has also a constant, $\omega$, for the set of the natural numbers.

Definition 3.1 (Axioms of CZF) CZF is based on intuitionistic predicate logic with equality. The set theoretic axioms of CZF are the following:

1. **Extensionality** $\forall a \forall b \left( \forall y \left( y \in a \leftrightarrow y \in b \right) \rightarrow a = b \right)$.
2. **Pair** $\forall a \forall b \exists x \forall y \left( y \in x \leftrightarrow y = a \lor y = b \right)$.
3. **Union** $\forall a \exists x \forall y \left( y \in x \leftrightarrow \exists z \in a \ y \in z \right)$.
4. **$\Delta_0$ - Separation scheme** $\forall a \exists x \forall y \left( y \in x \leftrightarrow y \in a \land \varphi(y) \right)$, for every bounded formula $\varphi(y)$, where a formula $\varphi(x)$ is bounded, or $\Delta_0$, if all the quantifiers occurring in it are bounded, i.e. of the form $\forall x \in b$ or $\exists x \in b$.
5. **Subset Collection scheme** $\forall a \forall b \exists c \forall u \left( \forall x \in a \exists y \varphi(x, y, u) \rightarrow \exists d \in c \left( \forall x \in a \exists y \in d \varphi(x, y, u) \land \forall y \in d \exists x \in a \varphi(x, y, u) \right) \right)$ for every formula $\varphi(x, y, u)$.
6. **Strong Collection scheme** $\forall a \left( \forall x \in a \exists y \varphi(x, y) \rightarrow \exists b \left( \forall x \in a \exists y \in b \varphi(x, y) \land \forall y \in b \exists x \in a \varphi(x, y) \right) \right)$ for every formula $\varphi(x, y)$.
7. **Infinity**
   
   \begin{align*}
   \omega 1 & \quad 0 \in \omega \land \forall y \left( y \in \omega \rightarrow y + 1 \in \omega \right) \\
   \omega 2 & \quad \forall x \left( 0 \in x \land \forall y \left( y \in x \rightarrow y + 1 \in x \right) \rightarrow \omega \subseteq x \right),
   \end{align*}

where $y + 1$ is $y \cup \{y\}$, and 0 is the empty set, defined in the obvious way.

Draft Manuscript (September 15, 2005)
8. $\in$ - Induction scheme

\[(IND_\in) \quad \forall a \ (\forall x \in a \ \varphi(x) \rightarrow \varphi(a)) \rightarrow \forall a \ \varphi(a),\]

for every formula $\varphi(a)$.

**Definition 3.2** A mathematically very useful axiom to have in set theory is the *Dependent Choices Axiom*, $\text{DC}$, i.e., for all formulae $\psi$, whenever

\[(\forall x \in a) (\exists y \in a) \psi(x, y)\]

and $b_0 \in a$, then there exists a function $f : \omega \rightarrow a$ such that $f(0) = b_0$ and

\[(\forall n \in \omega) \ \psi(f(n), f(n + 1)).\]

Even more useful in constructive set theory is the *Relativized Dependent Choices Axiom*, $\text{RDC}$.\(^1\) It asserts that for arbitrary formulae $\phi$ and $\psi$, whenever

\[\forall x [\phi(x) \rightarrow \exists y (\phi(y) \land \psi(x, y))]\]

and $\phi(b_0)$, then there exists a function $f$ with domain $\omega$ such that $f(0) = b_0$ and

\[\forall n \in \omega [\phi(f(n)) \land \psi(f(n), f(n + 1))].\]

A restricted form of $\text{RDC}$ is $\Delta_0\text{-RDC}$: For all $\Delta_0$-formulae $\theta$ and $\psi$, whenever

\[\forall x [\theta(x) \rightarrow (\exists y \in a) (\theta(y) \land \psi(x, y))]\]

and $b_0 \in a \land \phi(b_0)$, then there exists a function $f : \omega \rightarrow a$ such that $f(0) = b_0$ and

\[\forall n \in \omega [\theta(f(n)) \land \psi(f(n), f(n + 1))].\]

Letting $\phi(x)$ stand for $x \in a \land \theta(x)$, one sees that $\Delta_0\text{-RDC}$ is a consequence of $\text{RDC}$.

**Definition 3.3** Let $\text{CZF}^-$ be the system $\text{CZF}$ without the $\in$ - Induction scheme and let $\text{CZFA}$ be the theory $\text{CZF}^- + \text{AFA}$.

$\text{CZF}^-$ has certain defects from a mathematical point of view in that this theory appears to be too limited for proving the existence of the functions (as sets of ordered pairs) of addition and multiplication on $\omega$. Likewise, there seems to be no way of proving the existence of

\(^1\)In Aczel [3], $\text{RDC}$ is called the dependent choices axiom and $\text{DC}$ is dubbed the axiom of limited dependent choices. We deviate from the notation in [3] as it deviates from the usage in classical set theory texts.
the transitive closure of an arbitrary set from the axioms of \( \text{CZF}^- \). The first defect could be cured by just adding axioms which assert the existence of these functions, and this augmentation would then enable one to prove the existence of every primitive recursive functions on \( \omega \). However, in this paper I prefer to remedy these defects by slightly strengthening induction on \( \omega \) to
\[
\Sigma_1\text{-IND}_\omega \quad \phi(0) \land (\forall n \in \omega)(\phi(n) \to \phi(n+1)) \to (\forall n \in \omega)\phi(n)
\]
for all \( \Sigma_1 \) formulae \( \phi \). It is worth noting that \( \Sigma_1\text{-IND}_\omega \) actually implies \( \Sigma\text{-IND}_\omega \). It is worth noting that \( \Sigma_1\text{-IND}_\omega \) actually implies \( \Sigma\text{-IND}_\omega \).

The existence of the functions of addition and multiplication on \( \omega \) can also be proved in \( \text{CZF}^- + \Delta_0\text{-RDC} \).

\textbf{Draft Manuscript (September 15, 2005)}
3.2 Interpreting AFA in Martin-Löf type theory

The constructiveness of CZF was shown by Aczel by giving it an interpretation in Martin-Löf’s intuitionistic theory of types (cf. [2, 3, 4]). I. Lindström [19] and L. Haulås [16] have shown that CZFA can be interpreted in Martin-Löf type theory as well.

In this subsection we shall recall the interpretation of CZFA in Martin-Löf type theory, MLTT, as presented in [19]. In the following we work in MLTT with a universe \( U \) closed under the usual type constructors \( \Pi, \Sigma, +, I, N, N_0, N_1 \) (for details see [20]). We will denote the projection functions of the \( \Sigma \)-type by \( ()_0 \) and \( ()_1 \), respectively.

Definition 3.5 (System) A system over \( U \) consists of a type \( S \) together with an assignment of \( \bar{a} \in U \) and \( \tilde{a} \in \bar{a} \to S \) to each \( a \in S \).

A system \( S \) over \( U \) together with an additional assignment \( \sup_S(A, f) \in U \) to \( A \in U \) and \( f \in A \to S \) such that

\[
\sup_S(A, f) = A \in U \quad \text{and} \quad \sup_S(A, f) = f \in A \to S
\]

will be called a strong system over \( U \).

The primordial example of a strong system is Aczel’s type of iterative sets \( V \) which he used to interpret CZF in MLTT. [19] shows that Martin-Löf type theory allows one to prove that for every system \( S \) there exists a maximum bisimulation, \( \equiv_S \), on \( S \), and that \( S \) equipped with \( \equiv_S \) gives rise to an interpretation of CZFA.

Definition 3.6 (Bisimulation) A binary relation \( R \) on a system \( S \) over \( U \) is a bisimulation on \( S \) if, given \( a, b \in S \),

\[
R(a, b) \rightarrow \forall x \in \bar{a} \exists y \in \bar{b} R(\bar{a}x, \bar{b}y) \land \forall y \in \bar{b} \exists x \in \bar{a} R(\bar{a}x, \bar{b}y).
\]

The intuitive idea behind the definition of the relation \( \equiv_S \) is to define inductively certain approximations \( \equiv^n_S \) for each natural number \( n \), and to let \( (a \equiv_S b) \) hold if and only if \( (a \equiv^n_S b) \) holds for each \( n \in N \) and for \( m > n \) the proof of \( (a \equiv^m_S b) \) is an extension of the proof of \( (a \equiv^n_S b) \).

Definition 3.7 For \( a, b \in S \), define \( a \equiv^0_S b \) by recursion as follows:

\[
(a \equiv^0_S b) := N_1;
\]

\[
(a \equiv^{n+1}_S b) := (\forall x \in \bar{a})(\exists y \in \bar{b})(\bar{a}x \equiv^n_S \bar{b}y) \land (\forall y \in \bar{b})(\exists x \in \bar{a})(\bar{a}x \equiv^n_S \bar{b}y).
\]
For each natural number \( n \), define projection functions \( h_n \) as follows
\[
\begin{align*}
&h_0(a, b)(f, g) = 0_1, \\
&h_{n+1}(a, b)(f, g) = (f', g'), \quad \text{for } (f, g) \in (a \equiv_S b),
\end{align*}
\]
where \( 0_1 \) is the canonical element of \( \mathbb{N}_1 \) (the canonical one element set), and
\[
f' = (x) ((fx)_0, h_n(\bar{a}x, \bar{b}(fx)_0)(fx)_1),
g' = (y) ((gy)_0, h_n(\bar{a}(gy)_0, \bar{b}y)(gy)_1).
\]

Further, let
\[
(a \equiv_S b) = (\Pi n \in \mathbb{N}) (a \equiv_n b).
\]

Finally, let \( a \equiv_S b \) stand for
\[
(\Sigma \chi \in (a \equiv_S b)) (\Pi \in \mathbb{N}) h_n(\bar{a}, \bar{b}) \chi(n+1), \chi(n)).
\]

**Proposition 3.8** The relation \( \equiv_S \) is the maximum bisimulation on \( S \). In particular, for \( a, b \in S \),

(i) \( (a \equiv_S b) \in U \).

(ii) \( (a \equiv_S b) \rightarrow \forall x \in \bar{a} \exists y \in \bar{b} (\bar{a}x \equiv_S \bar{b}y) \land \forall y \in \bar{b} \exists x \in \bar{a} (\bar{a}x \equiv_S \bar{b}y) \).

(iii) If \( R \) is a relation on \( S \) such that:
\[
R(a, b) \rightarrow \forall x \in \bar{a} \exists y \in \bar{b} R(\bar{a}x, \bar{b}y) \land \forall y \in \bar{b} \exists x \in \bar{a} R(\bar{a}x, \bar{b}y),
\]
then \( R(a, b) \rightarrow (a \equiv_S b) \).

(iv) \( \equiv_S \) is an equivalence relation on \( S \) satisfying
\[
a \equiv_S b \iff \forall x \in \bar{a} \exists y \in \bar{b} (\bar{a}x \equiv_S \bar{b}y) \land \forall y \in \bar{b} \exists x \in \bar{a} (\bar{a}x \equiv_S \bar{b}y).
\]

**Proof:** It can be easily seen that \( (a \equiv_S b) \in U \). See [19] for a proof of (ii), (iii), (iv). \( \square \)

**Proposition 3.9** For every strong system \( S \) over a universe \( U \), define the relation \( \in_S \) on \( S \) by
\[
a \in_S b := (\exists y \in \bar{b})(a \equiv_S \bar{b}y).
\]

Then, as in Aczel’s interpretation [2], we get an interpretation of the language of set theory in which all theorems of \( \text{CZF}^- + \text{IND}_{\omega} + \text{RDC} \) are valid.
Proof: For CZF$^-$ see [19]. For RDC we shall draw on the proof of [3], Theorem 5.6. $\alpha \in S$ is said to be injectively presented if for all $x,y \in \bar{\alpha}$, 
\[
\bar{\alpha}(x) \equiv_S \bar{\alpha}(y) \rightarrow x = y \in \bar{\alpha}.
\]

The empty set in the interpretation is witnessed by $\emptyset_S := \sup_S(\mathbb{N}_0,(z)R_0(z))$. If $\alpha,\beta \in S$ let $\alpha \cup_S \beta \in S$ be defined by $\sup_S(\bar{\alpha} + \beta,g)$, where $g$ is defined by $g(i(a)) = \bar{\alpha}(a)$ for $a \in \bar{\alpha}$ and $g(j(a)) = \bar{\beta}(a)$ for $\bar{a} \in \bar{\beta}$. For $\alpha \in S$ let $\{\alpha\}_S \in S$ be defined by $\sup_S(\mathbb{N}_1,(z)R_1(z,\alpha))$. For $\alpha,\beta \in S$ the ordered pair of $\alpha$ and $\beta$ in the sense of $S$ is defined by $\langle \alpha,\beta \rangle_S := \{\{\alpha\}_S,\{\alpha\}_S \cup_S \{\beta\}_S \}$. 

The element of $S$ which plays the role of $\omega$ in the interpretation is 
\[
\omega_S = \sup_S(\mathbb{N}_1(v)\Delta(v))
\]
where $\Delta(0) = \emptyset_S$ and $\Delta(n+1) := \sup_S(\overline{\Delta(n)} + \mathbb{N}_1,h)$ with $h(i(a)) = \overline{\Delta(n)}(a)$ for $a \in \overline{\Delta(n)}$ and $h(j(a)) = R_1(a,\Delta(n))$ for $a \in \mathbb{N}_1$.

As $(\forall z \in \mathbb{N}_0)\bot$, it is obvious that $\emptyset_S$ is injectively presented. To show that $\omega_S$ is injectively presented we must show that for $n,m \in \mathbb{N}$,
\[
\Delta(n) \equiv_S \Delta(m) \rightarrow n = m \in \mathbb{N}.
\]

This can be shown by a routine double $\mathbb{N}$-induction, first on $n \in \mathbb{N}$ and within that on $m \in \mathbb{N}$.

We are now in a position to prove that RDC is valid under the interpretation in $S$. Let $B$ be an $\equiv_S$-extensional species over $S$, i.e., for all $\alpha,\beta \in S$, whenever $\alpha \equiv_S \beta$, then $B(\alpha) = B(\beta)$. Moreover, let $F$ be an $\equiv_S$-extensional species over $S \times S$ (i.e. $\equiv_S$-extensional in each argument) such that 
\[
(\forall x \in S)(B(x) \rightarrow (\exists y \in S)(B(y) \land F(x,y))).
\]

We claim that for each $\alpha \in S$ such that $B(\alpha)$ there is a $\delta \in S$ such that $\delta$ is a function with domain $\omega_S$, $\langle \emptyset_S,\alpha \rangle_S \in S \delta$ and for every $x \in S \omega_S$ and all $\beta,\gamma \in S$
\[
(\langle x,\beta \rangle_S \in S \delta \land \langle x',\gamma \rangle_S \in S \delta) \rightarrow (B(\beta) \land F(\beta,\gamma)).
\]

Proof of the claim: Let $\alpha \in S$ such that $B(\alpha)$. Then by the type theoretical version of RDC in [3], 1.15 there is $c \in \mathbb{N} \rightarrow S$ such that $c(0) = \alpha$ and for $n \in \mathbb{N}$ 
\[
B(c(n)) \land F(c(n),c(n+1)). \tag{1.2}
\]
Let $\eta = \sup_S (\mathbb{N}, c)$. Then $\eta \in S$ with $\bar{\eta} = \bar{\omega}_S$. Let
\[ \delta := \sup_S (\bar{\omega}_S(x), \bar{\omega}_S(x), \bar{\eta}(x))_S. \]

Then $\delta$ is a function with domain $\omega_S$ as $\omega_S$ is injectively presented. Since $\langle \emptyset_S, \alpha \rangle_S \equiv \langle \bar{\omega}_S(0), \bar{\eta}(0) \rangle_S \equiv \delta(0)$ it follows that $\langle \emptyset_S, \alpha \rangle_S \in S$. Finally, let $x \in S \omega_S$ and $\beta, \gamma \in S$ such that $\langle x, \beta \rangle_S \in S \delta$ and $\langle x', \gamma \rangle_S \in S \delta$. Then for some $n \in \mathbb{N}$ we have $x \equiv S \Delta(n)$, so that $x' \equiv S \Delta(n + 1)$ and hence $\beta \equiv S c(n)$ and $\gamma \equiv S c(n + 1)$. Hence, by (1.2) and the assumption that $B$ and $F$ are extensional, we get $B(\beta) \land F(\beta, \gamma)$.

If we apply the above result to the extensional predicates defined by formulae in the language of set theory we get the interpretation of RDC. □

Systems that satisfy a further completeness property allow for an interpretation of AFA as well.

**Proposition 3.10** Given a system $S$ define the type $S^*$ by
\[ S^* := (\Sigma f \in \mathbb{N} \to S) (\Pi n \in \mathbb{N}) (f(n) \equiv_{n+1} f(n+1)). \]

Then $S^*$ is a system under the following assignment of $(f, g) \in U$ and $(\bar{f}, \bar{g}) \in (f, g) \to S^*$,
for any $(f, g) \in S^*$:
\[
\begin{align*}
(f, g) & := \bar{f}(0), \\
(\bar{f}, \bar{g}) & := (a) (n) f(n+1) (t_{n+1} a).(n)((g(n+1))_01(t_{n+1} a)),
\end{align*}
\]
where $t_n \in \bar{f}(0) \to \bar{f}(n)$ for $n \in \mathbb{N}$ is given by
\[ t_0 := \text{id}, \quad t_{n+1} := (a)((g_0)0(t_{n+1} a))_0. \]

**Proof:** See [19], Proposition 3.1. □

**Proposition 3.11** Let $S$ be a strong system. Then the system $S^*$ as defined in Proposition 3.10 can be rendered a strong system under the following assignment $\sup_{S^*} (A, f) \in U$ to $A \in U$ and $f \in A \to S^*$:
\[ \sup_{S^*} (A, f) = (g, h), \]

Draft Manuscript (September 15, 2005)
where \( g_n \in S \) for \( n \in \mathbb{N} \) is given by
\[
\begin{align*}
f_0 &= \sup_{S} (A, (x)((bx)_{00})), \\
f(n + 1) &= \sup_{S} (A, (x)((bx)_{0n})),
\end{align*}
\]
and \( h_n \in \{ f_n \equiv_{n+1} f(n + 1) \} \) for \( n \in \mathbb{N} \) is defined by \( h_n = (\tau_n, \tau_n) \), where \( \tau_0 = (x)(x,0_1) \) and \( \tau_{n+1} = (x)(x,(bx)_1 n) \).

**Proof:** See [19].

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**Theorem 3.12** Let \( S \) be a strong system. Then the interpretation of \( \text{CZF}^- \) in the strong system \( S^* \) (as defined in Proposition 3.10) also validates AFA.

**Proof:** See [19], Proposition 3.5.

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### 3.3 Upper bounds

The results of the previous section can be utilized to read off upper bounds for the strengths of the systems \( \text{CZFA} + \Sigma_1\text{-IND}_\omega \) and \( \text{CZFA} + \text{IND}_\omega \) (also in combination with \( \text{RDC} \) and \( \Delta_0\text{-RDC} \)). As to the strength of type theory required for the results of the previous subsection, it is pivotal to observe that, unlike Aczel’s type of iterative sets, a strong system type is not required to have an inductive structure, i.e. there need not be an elimination rule for it. The strength of a type theory \( \text{ML}_1 \) with the type constructors \( \Pi, \Sigma, +, I, N, N_0, N_1 \) and one universe \( U \) closed under these same constructors has been determined by Aczel in [1]. \( \text{ML}_1 \) has the same strength as the theory \( \Sigma^1_1\text{-AC} \), a subsystem of second order arithmetic with the \( \Sigma^1_1 \) axiom of choice. Its proof theoretic ordinal is \( \varphi_{\varepsilon_0}0 \), where \( \varphi \) denotes the Veblen function (see [29]). If one adds a strong system type \( S \) to \( \text{ML}_1 \) the proof theoretic strength does not increase. This can be seen by emulating \( \text{ML}_1 + S \) in the theory \( \hat{\text{ID}}_1 \) of positive arithmetic fixed points similarly as in [14].

**Definition 3.13** Let \( L^+ \) be the language of Peano arithmetic augmented by a unary predicate symbol \( Q \). For each formula \( \phi(Q^+, u) \) of \( L^+ \) in which only the variable \( u \) occurs free and \( Q \) occurs only positively let \( I_\phi \) be a new unary predicate symbol. The *language of \( \hat{\text{ID}}_1 \), \( \hat{L} \), is the language of Peano arithmetic plus the predicate symbols \( I_\phi \) for each formula \( \phi(Q^+, u) \) of \( L^+ \).*
The axioms of $\hat{\text{ID}}_1$ comprise the axioms of Peano Arithmetic, with the induction scheme extended to all formulas of $\mathcal{L}$. In addition, $\hat{\text{ID}}_1$ has the fixed point axioms

$$(FP)\quad \forall x \left( \phi(I_\phi, x) \leftrightarrow I_\phi(x) \right)$$

for all formulae $\phi(Q^+, u)$ of $L^+$.}

**Definition 3.14** Occasionally the theory $\hat{\text{ID}}_1$ is too ‘coarse’ to obtain exact proof theoretic results. In those situations theories of natural numbers and ordinals which have been introduced by Jäger [18] provide a versatile tool. To give an example, the notion of being (a code for) a small type is simulated in $\hat{\text{ID}}_1$ via a formula which is no longer positive in the fixed point predicates (Aczel’s ‘trick’ in [1]) and on account of that the fragment of $\hat{\text{ID}}_1$ used for the interpretation of CZFA + $\Sigma_1$-IND$_\omega$ is proof theoretically too strong.

Let $\text{PA}_\Omega$ be the theory defined in [18]. Its main features are that the induction principles on the natural numbers and on the ordinals are restricted to so-called $\Delta^0_\Omega$ formulae; i.e. formulae in which all the ordinal quantifiers are restricted.

Let $\Sigma^\Omega$-IND$_\Omega$ be the scheme of induction on the natural numbers for $\Sigma^\Omega$ formulae, i.e. the smallest class of formulae which contains the $\Delta^0_\Omega$ formulae and is closed under $\land, \lor$, quantification over numbers, bounded quantification over ordinals, and (unbounded) existential quantification.

**Theorem 3.15**  
(i) The type theory used for interpreting the theory CZFA + IND$_\omega$ + RDC can be interpreted in $\hat{\text{ID}}_1$.

(ii) The type theory used for interpreting the theory CZFA + $\Sigma_1$-IND$_\omega$ + $\Delta_0$-RDC can be interpreted in $\text{PA}_\Omega$ + $\Sigma^\Omega$-IND$_\Omega$.

**Proof:** As to (i) one can extend the technique of [1] to simultaneously emulate a universe of sets together with a set of iterative sets over it via a positive fixed point definition. All the constructions of section can then be carried out in $\hat{\text{ID}}_1$.

Ad (ii): In $\text{PA}_\Omega$ one can define a universe of codes for types together with a type of iterative sets over it via a positive arithmetical inductive definitions. Thereby the notions of being a code for a type in the universe (a small type) and being a code for an iterative set become $\Sigma^\Omega$. The notion of being an element of a code for a small type is both $\Sigma^\Omega$ and $\Pi^\Omega$. Details of the above constructions can be found in [11].
One then has to scrutinize all the type theoretic constructions of section and [19] to determine the kind of structural induction and recursion on \( \mathbb{N} \) that is required for them. Closer inspection reveals that the strongest form of recursion on \( \mathbb{N} \) being used all come in the guise of defining functions from \( \mathbb{N} \) to \( \mathcal{U} \). Due to the previous observations the interpretation of these constructions can be justified via \( \Sigma^0\text{-IND}_{\mathbb{N}} \).

**Corollary 3.16** CZFA + IND\( _\omega \) + RDC can be interpreted in \( \Sigma^1_1\text{-AC} \).

**Proof:** This follows from the fact that the fixed point axioms of \( \hat{\text{ID}}_1 \) can be emulated in \( \Sigma^1_1\text{-AC} \) by interpreting the fixed points as \( \Sigma^1_1 \) sets (cf. [1] and [8]). \( \square \)

### 3.4 Lower bounds

Lower bounds for the theories CZFA + \( \Sigma_1\text{-IND}_\omega \) and CZFA + IND\( _\omega \) can be established by interpreting suitable intuitionistic theories \( \mathcal{R}A^*_\alpha \) of the ramified hierarchy up to level \( \alpha \) in them. For the definition of the theories \( \mathcal{R}A^*_\alpha \) see [15], chapter II, 1.2. We assume that an ordinal representation system for \( \varepsilon_0 \) has been formalized in CZF\( ^- \) + \( \Sigma_1\text{-IND}_\omega \) as a decidable subset of \( \omega \) (where \( A \subseteq \omega \) is said to be decidable if \( (\forall n \in \omega)(n \in A \lor n \notin A) \)). By Gentzen’s proof it follows that CZF\( ^- \) + IND\( _\omega \) proves transfinite induction up to \( \alpha \) for each (meta) \( \alpha < \varepsilon_0 \). Using Strong Collection combined with transfinite induction up to \( \alpha \) one readily shows the existence of the ramified hierarchy of length \( \alpha \) in CZF\( ^- \) + IND\( _\omega \). As a result we get that the theory \( \mathcal{R}A^*_\leq \varepsilon_0 := \bigcup_{\alpha < \varepsilon_0} \mathcal{R}A^*_\alpha \) can be interpreted in CZF\( ^- \) + IND\( _\omega \). To show that \( \mathcal{R}A^*_\omega \) can be interpreted in CZF\( ^- \) + \( \Sigma_1\text{-IND}_\omega \) we need a preparatory lemma.

**Lemma 3.17** For each (meta) \( n \), CZF\( ^- \) + \( \Sigma_1\text{-IND}_\omega \) proves

\[
(\forall \alpha)\left[ (\forall \beta < \alpha) \phi(\beta) \rightarrow \phi(\alpha) \right] \rightarrow (\forall \alpha < \omega^n) \phi(\alpha)
\]

for every \( \Sigma \) formula \( \phi \). Here we assume that the variables \( \alpha, \beta, \ldots \) range over the elements of the ordinal representation system and that \( < \) denotes the less-relation on them.

**Proof:** Recall that CZF\( ^- \) + \( \Sigma_1\text{-IND}_\omega \) proves \( \Sigma\text{-IND}_\omega \) as was explained in Definition 3.3, hence we have \( \Sigma\text{-IND}_\omega \) at our disposal. We
proceed by meta-induction on $n$. The case $n = 0$ is trivial. So assume $n = m + 1$ and that the assertion has been shown for $m$. Suppose
\[(\forall \alpha)\left[ (\forall \beta < \alpha) \phi(\beta) \rightarrow \phi(\alpha) \right],\]
where $\phi$ is $\Sigma$. Using the induction hypothesis for $m$ we get
\[\forall \gamma \left[ (\forall \beta < \gamma) \phi(\beta) \rightarrow (\forall \beta < \gamma + \omega^m) \phi(\beta) \right].\]
Using $\Sigma\text{-IND}^\omega$ with the formula $\psi(x) := (\forall \beta < \alpha + \omega^m \cdot x) \phi(\beta)$ we obtain
\[(\forall \beta < \alpha) \phi(\beta) \rightarrow (\forall x \in \omega)(\forall \beta < \alpha + \omega^m \cdot x) \phi(\beta).\]
The latter yields $\forall \alpha \left[ (\forall \beta < \alpha) \phi(\beta) \rightarrow (\forall \beta < \alpha + \omega^{m+1}) \phi(\beta) \right]$ and hence $(\forall \gamma < \omega^n) \phi(\gamma)$. $\square$

**Proposition 3.18**  
(i) $\text{RA}_{<\varepsilon_0}^i$ can be interpreted in $\text{CZF}^- + \text{IND}_\omega$.  
(ii) $\text{RA}_{<\omega^\omega}^i$ can be interpreted in $\text{CZF}^- + \Sigma_1\text{-IND}_\omega$.

**Proof:** (i) has been shown. (ii) Using the induction principle from Lemma 3.17 one can show the existence of the ramified hierarchy of sets up to level $\alpha$ for every (meta) $\alpha < \omega^\omega$, and hence one can interpret $\text{RA}_{<\alpha}^i$ in $\text{CZF}^- + \Sigma_1\text{-IND}_\omega$. $\square$

**Theorem 3.19**  
(i) The theories $\text{CZF}^- + \Sigma_1\text{-IND}_\omega$, $\text{CZFA} + \Sigma_1\text{-IND}_\omega + \Delta_0\text{-RDC}$, $\text{CZF}A + \Sigma_1\text{-IND}_\omega + \text{DC}$, and $\Sigma_1\text{-DC}_0$, are proof-theoretically equivalent. Their proof-theoretic ordinal is $\varphi_{\omega^0}$.

(ii) The theories $\text{CZF}^- + \text{IND}_\omega$, $\text{CZF}A + \text{IND}_\omega + \text{RDC}$, $\text{ID}_1$, and $\Sigma_1\text{-AC}$ are proof-theoretically equivalent. Their proof-theoretic ordinal is $\varphi_{\varepsilon_0^0}$.

**Proof:** It is known that the proof-theoretic ordinal of $\text{RA}_{<\omega^\omega}^i$ is $\varphi_{\omega^0}$; this follows from Theorem 3.2.13 and Theorem 3.1.11 in [15], chap. I. $\varphi_{\omega^0}$ is the proof-theoretic ordinal of $\Sigma_1\text{-DC}_0$, too, by [9]. As it can be shown that $\varphi_{\omega^0}$ is also the proof-theoretic ordinal of $\text{PA}_{\Omega}^\omega + \Sigma^\Omega_1\text{-IND}$, (i) follows from Corollary 3.15 and Proposition 3.18.

The proof that $\varphi_{\omega^0}$ is the proof-theoretic ordinal of $\text{PA}_{\Omega}^\omega + \Sigma^\Omega_1\text{-IND}$ proceeds as follows. A Gentzen-style (sequent calculus) version of $\text{PA}_{\Omega}^\omega + \Sigma^\Omega_1\text{-IND}$ allows one to carry out a partial cut elimination in that all cuts with formulae which are neither $\Sigma^\Omega$ nor $\Pi^\Omega$ can be removed.

*Draft Manuscript (September 15, 2005)*
In a next step one shows that such partially normalized derivation of sequents consisting of $\Delta^0_0(\Sigma^3_1)$ formulae (i.e. the smallest class of formulae containing the $\Sigma^3_1$ formulae which is closed under $\neg, \land, \lor, \text{number quantification}$, and bounded ordinal quantification) can be interpreted (asymmetrically) in the formal system $\text{RA}_{<\omega}$ by a method very similar to the one used in [22], Theorem 5.2.

It is also known that the proof-theoretic ordinal of $\text{RA}_{<\varepsilon_0}$ is $\varphi_{\varepsilon_0}0$ (cf. [15], chap. I., Theorem 3.2.13 and Theorem 3.1.11).

As the latter ordinal is the proof-theoretic ordinal of $\Sigma^1_1\text{-AC}$ as well, assertion (ii) follows from Corollary 3.16 and Proposition 3.18. \qed

4 Anti-foundation with inaccessible sets

It has been noted that an important aspect of the contrast between set theory and category theory is that, in the set-theoretic picture, well orderings and the axiom of foundation play a key role, while in the category-theoretic or structuralist picture, well foundedness almost completely disappears. In view of the proof-theoretic results of the previous sections, it is natural to conjecture that constructive category theory can be developed in weak theories.

The main focus of this section will be a constructive set theory in which a great deal of category theory can be formalized à la Grothendieck via universes. The theory is $\text{CZF}^-$ plus large set axioms which classically are equivalent to large cardinal axioms. Due to the lack of $\varepsilon$-induction this system is proof-theoretically weak. On the other hand it is a mathematically rich theory in which one can easily formalize Bishop style constructive mathematics. [12] investigated the strength of $\text{CZF}^-$ plus the statement that every set is contained in an inaccessible set, $\text{INAC}$. [12] showed that $\text{CZF}^- + \text{INAC}$ has a realizability interpretation in type theory, thereby establishing that the proof theoretic ordinal of $\text{CZF}^- + \text{INAC}$ is a surprisingly small ordinal known as the Feferman-Schütte ordinal $\Gamma_0$. The objective of this section is to indicate how the machinery of [12] can be utilized to show that the addition of $\text{AFA}$ to $\text{CZF}^- + \text{INAC}$ does not increase the proof theoretic strength.

The first large set axiom proposed in the context of constructive set theory was the Regular Extension Axiom, $\text{REA}$, which Aczel introduced to accommodate inductive definitions in $\text{CZF}$ (cf. [2], [4]).

**Definition 4.1** A set $c$ is said to be regular if it is transitive, inhabited (i.e. $\exists u \ u \in c$) and for any $u \in c$ and set $R \subseteq u \times c$ if $\forall x \in u \ \exists y (x, y) \in R$
then there is a set \( v \in c \) such that
\[
\forall x \in u \; \exists y \in v \; (x, y) \in R \land \forall y \in v \; \exists x \in u \; (x, y) \in R.
\]
We write \( \text{Reg}(a) \) for ‘\( a \) is regular’.

**RE** is the principle
\[
\forall x \exists y \; (x \in y \land \text{Reg}(y)).
\]

**Definition 4.2** Let \( \text{INAC} \) be the principle
\[
\forall x \exists y \; (x \in y \land \text{Reg}(y) \text{ and } y \text{ is a model of } \text{CZF}^-),
\]
i.e. the structure \( \langle y, \in \rangle \) is a model of \( \text{CZF}^- \).

We say that a set is **inaccessible** if it is regular and a model of \( \text{CZF}^- \) and write \( \text{INAC}(y) \) for ‘\( y \) is inaccessible’.

The formalization of the notion of inaccessibility in Definition 4.2 is somewhat awkward as it is very syntactic in that it requires a satisfaction predicate for formulae interpreted over a set. An alternative and more ‘algebraic’ characterization will be given next.

**Definition 4.3** Let \( \Omega := \{ x : x \subseteq \{0\} \} \). \( \Omega \) is the class of truth values with 0 representing falsity and 1 = \( \{0\} \) representing truth. Classically one has \( \Omega = \{0, 1\} \) but intuitionistically one cannot conclude that those are the only truth values.

For \( a \subseteq \Omega \) define
\[
\bigwedge a = \{ x \in 1 : (\forall u \in a) x \in u \}.
\]
A class \( B \) is **\( \bigwedge \)-closed** if for all \( a \subseteq B \), whenever \( a \subseteq \Omega \), then \( \bigwedge a \in B \).

For sets \( a, b \) let \( ^a b \) be the class of all functions with domain \( a \) and with range contained in \( b \). Let \( \text{mv}(^a b) \) be the class of all sets \( r \subseteq a \times b \) satisfying \( \forall u \in a \; \exists v \in b \; \langle u, v \rangle \in r \). A set \( c \) is said to be **full in** \( \text{mv}(^a b) \) if \( c \subseteq \text{mv}(^a b) \) and
\[
\forall r \in \text{mv}(^a b) \; \exists s \in c \; s \subseteq r.
\]

The expression \( \text{mv}(^a b) \) should be read as the collection of multi-valued functions from \( a \) to \( b \).

**Proposition 4.4 (CZF^-)** A set \( I \) is inaccessible if and only if the following are satisfied:
\begin{enumerate}
  \item \( I \) is a regular set,
  \item \( \omega \in I \),
\end{enumerate}
3. \((\forall a \in I) \bigcup a \in I\),

4. \(I\) is \(\Delta\)-closed,

5. \((\forall a, b \in I) \left[ \{x \in 1 : a = b\} \in I \land \{x \in 1 : a \in b\} \in I \right] \tag{3}\)

6. \((\forall a, b \in I)(\exists c \in I) \left[ c \text{ is full in } \text{mv}(a b) \right].

**Proof:** See [26], Proposition 3.4. \(\Box\)

Viewed classically inaccessible sets are closely related to inaccessible cardinals. Let \(V_\alpha\) denote the \(\alpha\)th level of the von Neumann hierarchy.

**Proposition 4.5 (ZFC)** A set \(I\) is inaccessible if and only if \(I = V_\kappa\) for some strongly inaccessible cardinal \(\kappa\).

**Proof:** This is a consequence of the proof of [24], Corollary 2.7. \(\Box\)

**Proposition 4.6** Let \(\text{EM}\) denote the principle of excluded middle. The theories \(\text{CZF}^- + \text{INAC} + \text{EM}\) and

\[
\text{ZFC} + \forall \alpha \exists \kappa (\alpha < \kappa \land \kappa \text{ is a strongly inaccessible cardinal})
\]

have the same proof theoretic strength.

**Proof:** [12], Lemma 2.10. \(\Box\)

The next two results show that the strength of \(\text{INAC}\) is quite modest when based on constructive set theory.

**Theorem 4.7** \(\text{CZF} + \text{REA}\) and \(\text{CZF} + \text{INAC}\) have the same proof-theoretic strength as the subsystem of second order arithmetic with \(\Delta^1_1\)-comprehension and bar induction.

**Proof:** \(\text{CZF} + \text{INAC}\) has a realizability in a type theory \(\text{MLS}^*\) with the following ingredients:\(^4\)

- \(\text{MLS}^*\) demands closure under the usual type constructors \(\Pi, \Sigma, +, I, N, N_0, N_1\) (but not the \(W\)-type).

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\(^3\)This clause may be omitted in the presence of \(\epsilon\)-induction.

\(^4\)Regarding the exact formalization of a superuniverse \(S\) and the universe operator \(U\) see [26].
• MLS* has a superuniverse $S$ which is closed under $\Pi, \Sigma, +, I, N, N_0, N_1$ and the $W$-type and the universe operator $U$.

• MLS* has a type $V$ of iterative sets over $S$.

• The universe operator $U$ takes a type $A$ in $S$ and a family of types $B : A \to S$ and produces a universe $U(A, B)$ in $S$ which contains $A$ and $B(x)$ for all $x : A$, and is closed under $\Pi, \Sigma, +, I, N, N_0, N_1$ (but not the $W$-type).

The type $V$ then yields a realizability interpretation of $\text{CZF} + \text{INAC}$ à la Aczel, using the techniques of [24]. On the other hand, MLS* can be interpreted in the classical set theory $\text{KPi}$ by the methods of [23], section 5. As $\text{CZF} + \text{REA}$ has the same strength as $\text{KPi}$ by [23], $\text{CZF} + \text{REA}$ and $\text{CZF} + \text{INAC}$ also have the same strength.

**Theorem 4.8** The proof theoretic ordinal of $\text{CZF}^- + \text{INAC}$ is the Feferman-Schütte ordinal $\Gamma_0$. $\text{CZF}^- + \text{INAC}$ has the same proof theoretic strength as the classical theory $\text{ATR}_0$ with arithmetical transfinite recursion and induction on the natural numbers restricted to sets.

**Proof:** [12], Corollary 9.14. □

The remainder of this section will be devoted to sketching a proof of the following.

**Theorem 4.9** The Feferman-Schütte ordinal $\Gamma_0$ is the proof theoretic ordinal of $\text{CZFA} + \text{INAC} + \Delta_0\text{-RDC}$. $\text{CZFA} + \text{INAC} + \Delta_0\text{-RDC}$ has the same proof theoretic strength as the classical theory $\text{ATR}_0$. 4.8 was shown in [12] by giving $\text{CZF}^- + \text{INAC}$ a realizability interpretation in a theory $\hat{\text{ID}}_\omega^*$ of iterated fixed point definitions. The latter were used to simulate two hierarchies of types, $(U_\alpha)_\alpha$ and $(V_\alpha)_\alpha$. $(U_\alpha)_\alpha$ is a hierarchy of universes and each $V_\alpha$ is a type of iterative sets over the universes $(U_\beta)_{\beta < \alpha}$. In particular, $U_0$ is a universe of small types closed under the usual type constructors ($\Pi, \Sigma, +, I, N, N_k$) and $V_1$ is a type of iterative sets over $U_0$. New levels in the hierarchy are introduced by reflecting on previous ones, i.e. each new universe contains all the objects of any earlier universe plus a code for that preceding universe. The types of iterative sets $V_\alpha$ are then endowed with an equivalence relation, $\equiv_\alpha$, which is the maximum bisimulation on $V_\alpha$, and an elementhood relation, $\in_\alpha$, to allow for a realizability interpretation of $\text{CZF}^-$.  

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*Draft Manuscript (September 15, 2005)*
From now on we assume familiarity with the definitions of [12]. Each \( \mathcal{V}_\alpha \) gives rise to a strong system over \( \mathcal{U}_{<\alpha} \) in the sense of Definition 3.5: If \( x \in \mathcal{V}_\alpha \) then \( x = \sup(a, b) \) for uniquely determined \( a, b \). Moreover, \( a \in \mathcal{U}_{<\alpha} \) and \( b \) satisfies \( bj \in \mathcal{V}_\alpha \) for every \( j \) such that \( \mathcal{U}_{<\alpha} \models j \in a \). Therefore we may put \( \bar{x} := a \) and \( \tilde{x} := b \) to render \( \mathcal{V}_\alpha \) a system. Conversely, if \( a \in \mathcal{U}_{<\alpha} \) and \( b \) satisfies \( \mathcal{V}_\alpha \models bj = bi \) for all \( j, i \) such that \( \mathcal{U}_{<\alpha} \models j = i \in a \), then \( \sup(a, b) \in \mathcal{V}_\alpha \), \( \sup(a, b) = a \), and \( \sup(a, b) = b \).

**Definition 4.10** To indicate the dependence on \( \alpha \), we shall denote the relations \( \equiv \alpha \) of [12], Definition 6.3 by \( \equiv^\alpha_n \). Guided by Proposition 3.10, we derive a system \( \mathcal{V}^\alpha \) which enables us to interpret \( \text{AFA} \):

\[
\mathcal{V}^\alpha_x := \sigma(\pi(\bar{N}, \Delta x. \varphi), \Delta f. \pi(\bar{N}, \Lambda n. f(n) \equiv^\alpha_{n+1} f(n + 1))),
\]

\[x \in \mathcal{V}^\alpha_x \text{ iff } \mathcal{U}_{\alpha+1} \models x \in \mathcal{V}^\alpha_x.\]

As shown in Proposition 3.10, \( \mathcal{V}^\alpha \) can be rendered a strong system. Let \( \equiv^\alpha_n \) be the maximum bisimulation on that system.

Note that \( \mathcal{U}_{\alpha+1} \models a \equiv^\alpha_n b \) set whenever \( b \in \mathcal{V}^\alpha_x \).

**Lemma 4.11** Let \( a, b \in \mathcal{V}^\alpha_x \) and let \( \beta \leq \alpha \). Then \( a, b \in \mathcal{V}^\beta_x \) and

\[\exists x \mathcal{U}_{\beta+1} \models x \in (a \equiv^\beta_n b) \text{ iff } \exists x \mathcal{U}_{\alpha+1} \models x \in (a \equiv^\alpha_n b).\]

**Proof:** The proof is similarly as in [12], Lemma 6.6.

**Definition 4.12** For \( a \in \mathcal{V}^\alpha_x \) let \( (b \equiv^\alpha_x a) := \sigma(\bar{a}, \Delta x. b \equiv^\alpha_x \bar{a} x) \).

**Definition 4.13** Let \( x \in \mathcal{V}^\alpha_x \) stand for \( \exists \alpha x \in \mathcal{V}^\alpha_x \).

For \( a, b \in \mathcal{V}^\alpha_x \) define

\[(a \equiv^\alpha_x b) := \exists \alpha (a \in \mathcal{V}^\alpha_x \land b \in \mathcal{V}^\alpha_x \land \exists x \mathcal{U}_{\alpha+1} \models x \in (a \equiv^\alpha_x b)).\]

In addition, for \( a \in \mathcal{V}^\alpha_x \) let \( b \in \mathcal{V}^\alpha_x \) set \( a := \exists \alpha (a \in \mathcal{V}^\alpha_x \land b \equiv^\alpha_x a) \).

**Definition 4.14 (Realizability in \( \mathcal{V}^\alpha_x \))** For each formula \( \varphi(x_1, \ldots, x_n) \) of \( \text{CZF}^- \) containing at most \( x_1, \ldots, x_n \), we define \( e \vdash^\alpha_{\alpha} \varphi(x_1, \ldots, x_n) \) as follows:

\[
\begin{align*}
& e \vdash^\alpha_{\alpha} \bot := \mathcal{U}_{\alpha+1} \models e \in \bar{N}_0; \\
& e \vdash^\alpha_{\alpha} (x = y) := \mathcal{U}_{\alpha+1} \models e \in (x \equiv^\alpha_x y) \land \mathcal{V}^\alpha_x \models x, y \text{ set;} \\
& e \vdash^\alpha_{\alpha} (x \in y) := \mathcal{U}_{\alpha+1} \models e \in (x \in^\alpha_x y) \land \mathcal{V}^\alpha_x \models x, y \text{ set;} \\
& e \vdash^\alpha_{\alpha} \psi \land \chi := e_0 \vdash^\alpha_{\alpha} \psi \land e_1 \vdash^\alpha_{\alpha} \chi.
\end{align*}
\]
\[ e \vDash^* \psi \lor \chi \quad \text{:=} \quad (e_0 = 0 \rightarrow e_1 \vDash^* \psi) \lor (e_0 \neq 0 \rightarrow e_1 \vDash^* \chi); \]
\[ e \vDash^* \psi \rightarrow \chi \quad \text{:=} \quad \forall q (q \vDash^* \psi \rightarrow eq \vDash^* \chi); \]
\[ e \vDash^* \exists x \in a \psi(x) \quad \text{:=} \quad \bigcup_{\alpha+1} \models e_0 \in \bar{a} \land e_1 \vDash^* \psi(\bar{a}(e_0)); \]
\[ e \vDash^* \forall x \in a \psi(x) \quad \text{:=} \quad \forall i (\bigcup_{\alpha+1} \models i \notin \bar{a} \lor ei \vDash^* \psi(\bar{a}i)); \]
\[ e \vDash^* \exists x \forall x \psi(x) \quad \text{:=} \quad e_1 \vDash^* \psi(e_0) \land V^*_\alpha \models e_0 \text{ set}; \]
\[ e \vDash^* \forall x \exists y \psi(x) \quad \text{:=} \quad \forall u \in V^*_\alpha (eu \vDash^* \psi(u)). \]

We say that a formula \( \varphi \) of \( \text{CZF}^- \) is realizable in \( V^* \) if there is an \( e \) such that \( e \vDash^{*+} \varphi \).

**Definition 4.15 (Realizability in \( V^* \))** For each formula \( \varphi(x_1, \ldots, x_n) \) of \( \text{CZF} \) containing at most \( x_1, \ldots, x_n \) free, we define \( e \vDash^{*+} \varphi(x_1, \ldots, x_n) \), by replacing in Definition 4.14 \( e \vDash^{*+} \ldots \) by \( e \vDash^{*+} \ldots \), and \( U_{\alpha+1} \), \( V^*_\alpha \) by \( U, \ V^* \), respectively, and \( \equiv^*_\alpha \), \( \in^*_\alpha \), by \( \equiv^*_\alpha \) and \( \in^*_\alpha \), respectively.

We say that a formula \( \varphi \) of \( \text{CZF} \) is realizable in \( V^* \), or simply realizable, if there is an \( e \) such that \( e \vDash^{*+} \varphi \).

**Theorem 4.16** For every \( \alpha \), \( V^*_\alpha \) is a realizability model of \( \text{CZFA} + \text{RDC} \).

**Proof:** This follows from Proposition 3.9. \( \square \)

**Theorem 4.17** \( V^* \) is a realizability model of \( \text{CZF} + \text{INAC} \) as well as \( \Delta_0 \)-RDC. Moreover, \( V^* \) is also a realizability model of the assertion \( \forall x \exists y [x \in y \land \text{Reg}(y) \land y \text{ is a model of } \text{CZF}] \).

**Proof:** That \( V^* \) realizes AFA and \( \Delta_0 \)-RDC follows from the fact that all \( V^*_\alpha \) realize AFA and RDC. That \( V^* \) realizes \( \text{CZF}^- + \text{INAC} \) is proved in the same way as [12], Theorem 8.2 and Theorem 8.4. \( \square \)

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Draft Manuscript (September 15, 2005)


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