DMV Lectures
on
Representations of quivers, preprojective algebras and deformations of quotient singularities

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Introduction

In May 1999 there was a German Mathematical Society (DMV) Seminar on Quantizations of Kleinian singularities at the Mathematical Research Institute in Oberwolfach. The organizers were Ragnar Buchweitz, who spoke about deformation theory, Peter Slodowy, who spoke about Kleinian singularities, and myself. These are a slightly revised and expanded version of the notes that I prepared for the meeting.

I assume throughout that K is an algebraically closed field of characteristic zero and work with algebras and varieties over K. The aim is to study quotient singularities V/Γ, where V is a smooth affine variety and Γ is a finite group acting on V. In particular the Kleinian singularities K^2/Γ with $\Gamma \subset SL_2(K)$. For example one would like to construct and understand deformations, quantizations and desingularizations of these singularities.

The key idea is to try to realize V/Γ as a moduli space of modules for the skew group algebra $K[V]\#\Gamma$ formed by the action of Γ on the coordinate ring of V. This idea seems to be implicit in some of the recent work on the higher dimensional McKay correspondence, where it sometimes leads to a desingularization of V/Γ. To get deformations and quantizations of V/Γ, we look for deformations of the algebra $K[V]\#\Gamma$. It is easy to write some down in case V is a vector space and Γ preserves a symplectic form on V. To determine the properties of our deformations, we restrict to the Kleinian singularity case, when $K[V]\#\Gamma$ is Morita equivalent to a "preprojective algebra", and one can use representations of quivers. The lectures therefore begin with an introduction to these topics.

Much of this material comes from W.Crawley-Boevey and M.P.Holland, Noncommutative deformations of Kleinian singularities, Duke Math. J. 92 (1998), 605-635.

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Lecture 1. Representations of quivers

1.1. QUIVERS. Let Q be a quiver with vertex set I. Thus Q is a directed graph with a finite number of arrows and vertices. Each arrow a has its tail at a vertex t(a) and its head at a vertex h(a). We also write $a:i\longrightarrow j$ to indicate that $i = t(a)$ and $j = h(a)$.

A representation of Q consists of a vector space M_i for each vertex and a linear map $M_i\longrightarrow M_j$ for each arrow $a:i\longrightarrow j$. A homomorphism between two representations $M\longrightarrow N$ consists of a linear map $M_i\longrightarrow N_i$ for each vertex, such that for each arrow $a:i\longrightarrow j$ the square

$$\begin{array}{ccc} M_i & \longrightarrow & N_i \\ \downarrow & & \downarrow \\ M_j & \longrightarrow & N_j \end{array}$$

commutes. Clearly in this way one obtains a category of representations, and the isomorphisms turn out to be those homomorphisms in which all linear maps $M_i\longrightarrow N_i$ are invertible.

The dimension vector of a representation M is the vector $\alpha\in\mathbb{N}^I$ whose components are given by $\alpha_i = \dim M_i$. The notation $\alpha = \underline{\dim}\, M$ is often used. Choosing bases for the vector spaces, any representation of dimension α is given by an element of

$$\mathrm{Rep}(Q,\alpha) = \prod_{a:i\to j} \mathrm{Mat}(\alpha_j\times\alpha_i,K).$$

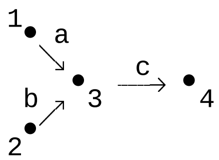
We write K_x for the representation corresponding to a point $x\in\mathrm{Rep}(Q,\alpha)$. Thus the vector space at a vertex i is K^{α_i} and the linear maps are given by the matrices. Another common notation is to use the corresponding capital letter X for the representation.

The group

$$GL(\alpha) = \prod_{i\in I} GL(\alpha_i,K)$$

acts on $\mathrm{Rep}(Q,\alpha)$ by conjugation. The group elements in which all matrices are the same nonzero multiple of the identity matrix act trivially, so the quotient group $G(\alpha) = GL(\alpha)/K^*$ acts. Clearly the orbits (for either group action) correspond to the isomorphism classes of representations of Q of dimension α. Moreover the stabilizer of x in $GL(\alpha)$ is evidently the automorphism group $\mathrm{Aut}(K_x)$, so the stabilizer of x in $G(\alpha)$ is $\mathrm{Aut}(K_x)/K^*$.

1.2. PATH ALGEBRAS. The *path algebra* KQ associated to a quiver Q is the associative algebra with basis the paths in Q. This includes a *trivial path* e_i for each vertex i. For example the path algebra of the quiver



has basis e_1, e_2, e_3, e_4, a, b, c, ca, cb. The multiplication in KQ is given by composition of paths if they are compatible, or zero if not. In the example we have a.b = 0, c.b = cb, e_4.c = c, e_3.c = 0, etc. Note that our convention for the order of arrows is to compose them as if they were functions. Clearly the e_i are orthogonal idempotents, and the sum of them is an identity element for KQ. The path algebra is finite-dimensional if and only if Q has no oriented cycles.

Studying representations of Q is essentially the same as studying KQ-modules. (By default this means *left* modules.) The connection is as follows.

- If M is a representation of Q, so given by vector spaces M_i for each vertex i, and linear maps, then $\mathbf{M} = \oplus_i M_i$ can be turned into a KQ-module as follows. If i is a vertex, then multiplication by e_i acts as the projection onto M_i. If $a:i \longrightarrow j$ is an arrow, then multiplication by a acts as the composition

$$M \twoheadrightarrow M_i \xrightarrow{a} M_j \hookrightarrow M.$$

- Conversely, if **M** is a KQ-module, there is a representation M with $M_i = e_i\mathbf{M}$, and with the linear map $M_i \longrightarrow M_j$ corresponding to an arrow $a:i \longrightarrow j$ given by left multiplication by a.

This defines an equivalence of categories, but perhaps it is not an isomorphism. Nevertheless, in future we shall blur the difference between representations and modules.

1.3. INDECOMPOSABLES. Recall that a module M is said to be indecomposable if it cannot be written as a direct sum of two proper submodules $M = X \oplus Y$. For finite-dimensional modules for an algebra, which is our interest in these notes, there are two key results:

- Fitting's Lemma says that a module is indecomposable if and only if every endomorphism is of the form $\lambda 1 + \theta$ where $\lambda \in K$ and θ is nilpotent.

- Any finite-dimensional module can clearly be written as a direct sum of indecomposable submodules, but such a decomposition is not unique. However, the Krull-Schmidt Theorem says that any two decompositions have the same number of indecomposable summands, and the summands can be paired off so that corresponding summands are isomorphic.

1.4. STANDARD RESOLUTION. If S is an algebra and V is an S-S-bimodule then the tensor algebra of V over S is

$$T_S V = S \oplus V \oplus (V \otimes_S V) \oplus (V \otimes_S V \otimes_S V) \oplus \dots$$

with the natural multiplication. If $A = T_S V$, there is a canonical exact sequence

$$0 \longrightarrow A \otimes_S V \otimes_S A \xrightarrow{f} A \otimes_S A \xrightarrow{m} A \longrightarrow 0$$

where $f(a \otimes v \otimes a') = av \otimes a' - a \otimes va'$ and $m(a \otimes a') = aa'$.

The path algebra KQ is a special cases of a tensor algebra, with S the commutative semisimple algebra $S = \prod_{i \in I} K$ and $V = \oplus_{a:i \longrightarrow j} K$, considered as an S-S-bimodule via $svs' = (s_j v_a s'_i)_{a:i \longrightarrow j}$. In fact any tensor algebra $T_S V$ with S,V finite dimensional and S commutative semisimple arises this way.

LEMMA (Standard resolution). Any KQ-module X has a projective resolution

$$0 \longrightarrow \oplus_{a:i\to j} KQe_j \otimes X_i \longrightarrow \oplus_i KQe_i \otimes X_i \longrightarrow X \longrightarrow 0$$

In particular gl.dim KQ ≤ 1.

PROOF. Here $X_i = e_i X$ is the vector space at vertex i in the corresponding representation, and the tensor products are over K. Thus $KQe_i \otimes X_i$ is isomorphic as a KQ-module to a direct sum of copies of KQe_i, indexed by a basis of X_i. Now since e_i is idempotent, KQe_i is a projective KQ-module, and hence so is the direct sum. Thus the terms are indeed projective modules.

The sequence is obtained by applying $-\otimes_{KQ} X$ to the canonical exact sequence for $KQ = A = T_S V$. The canonical exact sequence is a sequence of A-A-bimodules, and it is clearly split as a sequence of right A-modules, so it remains exact under the tensor product.

Since any left KQ-module has a projective resolution with two terms, we deduce that KQ has left global dimension at most 1. But the opposite algebra of KQ is also a path algebra, of the opposite algebra, so the same applies for right global dimension.

1.5. BILINEAR FORMS. The _Ringel form_ for Q is the bilinear form on $\mathbb{R}^I$ defined by

$$\langle\alpha,\beta\rangle = \sum_i \alpha_i\beta_i - \sum_{a:i\to j} \alpha_i\beta_j.$$

The _Tits form_ is the quadratic form $q(\alpha) = \langle\alpha,\alpha\rangle$. The corresponding _symmetric bilinear form_ is

$$(\alpha,\beta) = \langle\alpha,\beta\rangle + \langle\beta,\alpha\rangle.$$

If X,Y are (f.d.) representations of Q then there is _Ringel's formula_:

$$\dim \mathrm{Hom}(X,Y) - \dim \mathrm{Ext}^1(X,Y) = \langle\underline{\dim}\, X,\underline{\dim}\, Y\rangle.$$

which follows from applying the functor Hom(-,Y) to the standard resolution for X and using the fact that $\mathrm{Hom}(KQe_i,Y) \cong Y_i$ to compute dimensions.

1.6. ROOTS. Let $\varepsilon_i \in \mathbb{Z}^I$ denote the coordinate vector at vertex i. The matrix $A_{ij} = (\varepsilon_i,\varepsilon_j)$ is a Generalized Cartan Matrix (at least when Q has no loops), and so there is an associated Kac-Moody Lie algebra. This algebra has a root system associated to it. We need the same combinatorics.

If i is a loopfree vertex in Q (meaning that there is no arrow with head and tail at i), then there is a <u>reflection</u>

$$s_i : \mathbb{Z}^I \longrightarrow \mathbb{Z}^I,\ s_i(\alpha) = \alpha - (\alpha,\varepsilon_i)\varepsilon_i$$

The <u>Weyl</u> <u>group</u> is the subgroup $W \subseteq \mathrm{Aut}(\mathbb{Z}^I)$ generated by the s_i. The <u>fundamental</u> <u>region</u> is

$$F = \{\alpha\in\mathbb{N}^I : \alpha\neq 0,\ \alpha \text{ has connected support, and } (\alpha,\varepsilon_i)\leq 0 \text{ for all } i\}$$

By definition the <u>real</u> <u>roots</u> for Q are the orbits of coordinate vectors ε_i (for i loopfree) under W. The <u>imaginary</u> <u>roots</u> for Q are the orbits of $\pm\alpha$ (for $\alpha \in F$) under W.

If α is a root, then so is $-\alpha$. This is true by definition for imaginary roots. It holds for real roots since $s_i(\varepsilon_i) = -\varepsilon_i$ if i is a loopfree vertex. It can be shown that every root has all components ≥ 0 or ≤ 0. This can be deduced from Lie Theory, but one could also prove it using the methods of Lecture 2. Thus one can speak of <u>positive</u> and <u>negative</u> roots.

It is easy to check that $q(s_i(\alpha)) = q(\alpha)$, so that the Weyl group preserves the Tits form. It follows that the real roots have $q(\alpha)=1$, and the imaginary roots have $q(\alpha)\leq 0$. In general, however, not all vectors with these properties are roots. (But see §1.9.)

A nonzero element α of $\mathbb{Z}^I$ is said to be <u>indivisible</u> if $\gcd(\alpha_i) = 1$. Clearly any real root is indivisible, and if α is a real root, only $\pm\alpha$ are roots. On the other hand every imaginary root is a multiple of an indivisible root, and all other nonzero multiples are also roots.

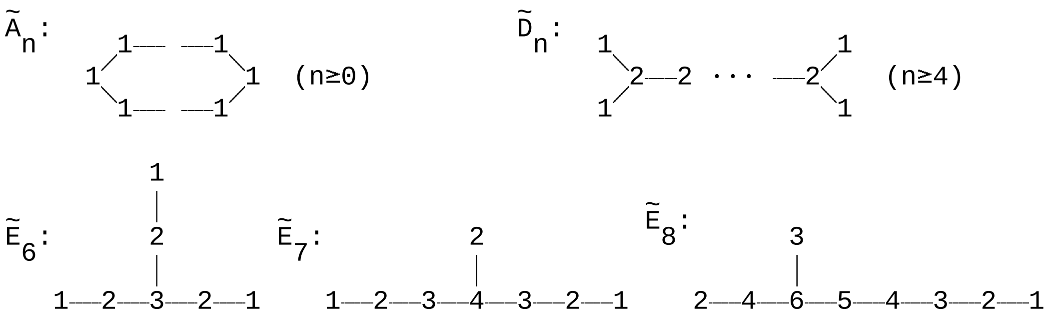
1.7. KAC'S THEOREM. (i) If there is an indecomposable representation of Q of dimension α, then α is a root.

(ii) If α is a positive real root there is a unique indecomposable of dimension α (up to isomorphism).

(iii) If α is a positive imaginary root then there are infinitely many indecomposables of dimension α (up to isomorphism).

In Lecture 2 we shall prove (i) and (ii). In the rest of this lecture we shall assume the truth of (i) and (ii), discuss Dynkin and extended Dynkin quivers, and prove a very special case of (iii). Thus, although we do not prove all of Kac's Theorem, we do prove everything we need for Kleinian singularities.

1.8. DYNKIN AND EXTENDED DYNKIN QUIVERS. The <u>extended</u> <u>Dynkin</u> <u>quivers</u> are those whose undelying graph is one of $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$ (with n+1 vertices). In each case we've indicated a special vector $\delta \in \mathbb{N}^I$ by marking each vertex i with the component δ_i.



Thus $\tilde{A}_0$ consists of one vertex and one loop. An <u>extending</u> <u>vertex</u> is one with $\delta_i = 1$. The <u>Dynkin</u> <u>quivers</u> A_n, D_n, E_6, E_7, E_8 are obtained by deleting an extending vertex. We have the following observations.

(1) Let Q be an arbitrary quiver. By definition the <u>radical</u> of the Tits form q is $\mathrm{Rad}(q) = \{\alpha \in \mathbb{R}^I : (\alpha,\varepsilon_i)=0 \text{ for all } i\}$. Writing n_{ij} for the number of edges i—j (loops count twice), we have

$$\alpha \in \mathrm{Rad}(q) \iff (2 - n_{ii})\alpha_i = \sum_{j\neq i} n_{ij}\alpha_j \quad \text{for all } i.$$

(2) If α is a radical vector with $\alpha_i > 0$ for all i, then by calculation

$$q(\beta) = \sum_{i<j} n_{ij} \frac{\alpha_i\alpha_j}{2} \left(\frac{\beta_i}{\alpha_i} - \frac{\beta_j}{\alpha_j}\right)^2$$

for any β. It follows that q is positive semidefinite (meaning that $q(\alpha) \geq 0$ for all $\alpha \in \mathbb{R}^I$). Assuming that Q is connected the only vectors on which q vanishes are the elements of $\mathbb{R}\alpha$, so this is also the radical of q.

(3) One can easily check that $\delta \in \mathrm{Rad}(q)$ for Q extended Dynkin. Thus q is positive semidefinite and $\mathrm{Rad}(q) = \mathbb{R}\delta$.

(4) It follows immediately that q is positive definite for Q Dynkin (meaning that $q(\alpha) > 0$ for all $0 \neq \alpha \in \mathbb{R}^I$).

(5) A case-by-case analysis shows that any connected quiver which is not Dynkin or extended Dynkin must properly contain an extended Dynkin quiver, and this implies that q is indefinite for such quivers (so takes both positive and negative values).

1.9. ROOTS FOR DYNKIN AND EXTENDED DYNKIN QUIVERS. Let Q be a Dynkin or extended Dynkin quiver.

(1) We show that if $\alpha \in \mathbb{Z}^I$ and $q(\alpha)\leq 1$, then α is either positive or negative. Write $\alpha = \alpha^+ - \alpha^-$ with $\alpha^+,\alpha^- \in \mathbb{N}^I$ having disjoint support. For a contradiction suppose that α^+ and α^- are both nonzero. Now

$$1 \geq q(\alpha) = q(\alpha^+) + q(\alpha^-) - (\alpha^+,\alpha^-) \geq q(\alpha^+) + q(\alpha^-)$$

but $q(\alpha^+)$ and $q(\alpha^-)$ are integers and q is positive semidefinite, so one term must vanish, say $q(\alpha^+) = 0$. This implies that Q is extended Dynkin and α^+ is a nonzero multiple of δ. But all components of δ are nonzero, so we must have $\alpha^- = 0$. Contradiction.

(2) The roots for Q are exactly the $0 \neq \alpha \in \mathbb{Z}^I$ with $q(\alpha)\leq 1$. Certainly any root has these properties. On the other hand, if α has these properties then we apply a sequence of reflections to minimize $|\sum\alpha_i|$. If α is now a multiple of a coordinate vector at a loopfree vertex, then since $q(\alpha) \leq 1$

we see that the multiple is ±1, so α is a real root. Otherwise, if i is any loopfree vertex then the reflection at i cannot change the sign of α_i, for otherwise it leads to a vector with both positive and negative components. By minimality this implies that $(\alpha,\varepsilon_i) \le 0$. Thus $q(\alpha) \le 0$, so α is a multiple of δ, so in the fundamental region, and hence an imaginary root.

(3) Clearly the imaginary roots for an extended Dynkin quiver are exactly the multiples of δ.

(4) Clearly a Dynkin quiver has only real roots. In fact it has only finitely many roots, for they form a discrete subset of the compact set $\{\alpha\in\mathbb{R}^I : q(\alpha)=1\}$. Thus Kac's Theorem implies Gabriel's Theorem, that the quivers with only finitely many indecomposables are the Dynkin quivers.

1.10. LEMMA (Ringel). An indecomposable f.d. KQ-module which is not a brick has a submodule which is a brick with self-extensions.

(By definition a brick is a module X with End(X) = K, and X has self-extensions if $\mathrm{Ext}^1(X,X) \neq 0$).

PROOF. By induction it suffices to prove that X has an indecomposable proper submodule with self-extensions. For a contradiction, suppose not. Let $\theta \in \mathrm{End}(X)$ be a nonzero endomorphism with $I = \mathrm{Im}(\theta)$ of minimal dimension. By hypothesis I is indecomposable, so has no self-extensions. Now $\theta^2 = 0$, for $\mathrm{Im}(\theta^2) \subseteq I$, and if they are equal then the composition

$$I \hookrightarrow X \xrightarrow{\theta} I$$

is an isomorphism, so I is a direct summand of X. Thus $I \subseteq \mathrm{Ker}(\theta)$. Write $\mathrm{Ker}(\theta)$ as a direct sum of indecomposables, say $\mathrm{Ker}(\theta) = \oplus K_i$, and let $\pi_i : \mathrm{Ker}(\theta) \twoheadrightarrow K_i$ be the projections. For some j we must have $\pi_j(I) \neq 0$. Suppose for a contradiction that $\mathrm{Ext}^1(K_j,K_j) = 0$.

Minimality implies that $\pi_j|_I$ is injective (considering the composition $X \twoheadrightarrow I \longrightarrow K_j \hookrightarrow X$). Applying $\mathrm{Hom}(-,K_j)$ to the short exact sequence

$$0 \longrightarrow I \longrightarrow K_j \longrightarrow K_j/I \longrightarrow 0$$

gives a long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^1(K_j,K_j) \xrightarrow{f} \mathrm{Ext}^1(I,K_j) \longrightarrow \mathrm{Ext}^2(K_j/I,K_j) \longrightarrow \cdots$$

and the Ext^2 term vanishes, so f is onto.

Now consider the pushout of the short exact sequence

$$0 \longrightarrow \mathrm{Ker}(\theta) \longrightarrow X \longrightarrow I \longrightarrow 0$$

along π_j, say

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(\theta) & \longrightarrow & X & \longrightarrow & I & \longrightarrow & 0 \\ & & \downarrow & & \downarrow g & & \| & & \\ 0 & \longrightarrow & K_j & \xrightarrow{h} & Y & \longrightarrow & I & \longrightarrow & 0 \end{array}$$

If it splits, then h has a retraction, and its composition with g is a retraction for the inclusion of K_j in X. But this implies that K_j is a direct summand of X, which is nonsense. Thus we must have $\mathrm{Ext}^1(I,K_j) \neq 0$.

It follows that $\mathrm{Ext}^1(K_j,K_j) \neq 0$. Contradiction.

1.11. LEMMA. For Q extended Dynkin, the general element of $\mathrm{Rep}(Q,\delta)$ is a brick, and there are only finitely many other orbits.

PROOF. There are only finitely many orbits of decomposable modules since there are only finitely many roots which are less than δ, and they are all real roots. Now using the fact that $q(\delta)=0$,

$$\dim \mathrm{Rep}(Q,\delta) = \sum_{a:i\to j} \delta_i\delta_j = \sum_i \delta_i^2, \quad \dim G(\delta) = \sum_i \delta_i^2 - 1,$$

so there must be infinitely many orbits. Thus the general element of $\mathrm{Rep}(Q,\delta)$ must be indecomposable. Now Ringel's Lemma says that each indecomposable is either a brick, or it has a proper submodule M which is a brick with self-extensions. But then

$$q(\underline{\dim}\ M) = \dim \operatorname{End}(M) - \dim \operatorname{Ext}^1(M,M) \le 0,$$

so $\underline{\dim}$ M is a multiple of δ. This is impossible.

1.12. FURTHER READING. The best reference for Kac's Theorem is his last paper on the topic, V.G.Kac, Root systems, representations of quivers and invariant theory, in: Invariant theory, Proc. Montecatini 1982, ed. F. Gherardelli, Lec. Notes in Math. 996, Springer, Berlin, 1983, 74-108.

Another useful reference is H.Kraft and Ch.Riedtmann, Geometry of representations of quivers, in: Representations of algebras, Proc. Durham 1985, ed. P. Webb, London Math. Soc. Lec. Note Series 116, Cambridge Univ. Press, 1986, 109-145.

The definitive reference for extended Dynkin quivers is Section 3.6 of C.M.Ringel, Tame algebras and integral quadratic forms, Lec. Notes in Math. 1099, Springer, Berlin, 1984.

Lecture 2. Preprojective algebras

Let Q be a quiver with vertex set I.

2.1. PREPROJECTIVE ALGEBRAS. The <u>double</u> of Q is the quiver obtained by adjoining an arrow $a^*:j\longrightarrow i$ for each arrow $a:i\longrightarrow j$ in Q.

The <u>preprojective</u> algebra is the associative algebra

$$\Pi(Q) = K\overline{Q} \,/\, (\textstyle\sum_{a\in Q} [a,a^*]).$$

More generally, the <u>deformed</u> <u>preprojective</u> <u>algebra</u> of <u>weight</u> $\lambda\in K^I$ is

$$\Pi^\lambda(Q) = K\overline{Q} \,/\, (\textstyle\sum_{a\in Q} [a,a^*] - \sum_{i\in I} \lambda_i e_i)$$

2.2. REMARKS. (1) The preprojective algebra first appeared with the relation $\sum_{a\in Q}(aa^*+a^*a) = 0$. It is easy to see that this gives an isomorphic algebra provided the quiver is bipartite, meaning that the vertices can be divided into two sets and no arrow has both head and tail in the same set.

(2) If Q has no oriented cycles then KQ is a finite-dimensional algebra. For such algebras there are Auslander-Reiten operators DTr and TrD, and it can be shown that

$$\Pi(Q) \cong \oplus_{n=0}^{\infty} (TrD)^n(KQ).$$

This means that $\Pi(Q)$ is the sum of all indecomposable preprojective KQ-modules.

(3) $\Pi^{\lambda}(Q)$ doesn't depend on the orientation of Q. Just reverse the role of a and a^*, and change the sign of one of them.

(4) If r is the defining relation for $\Pi^{\lambda}(Q)$, then $r = \sum e_i r e_i$, and

$$e_i r e_i = \sum_{h(a)=i} a a^* - \sum_{t(a)=i} a^* a - \lambda_i e_i.$$

Thus $\Pi^{\lambda}(Q)$-modules correspond to representations of $\overline{Q}$ in which the linear maps satisfy the relations

$$\sum_{h(a)=i} a a^* - \sum_{t(a)=i} a^* a = \lambda_i \, Id$$

for all i. With this identification we can speak of the dimension vector of a $\Pi^{\lambda}(Q)$-module.

(5) If there is a $\Pi^{\lambda}(Q)$-module of dimension α then $\lambda\cdot\alpha = \sum_i \lambda_i\alpha_i$ must be equal to zero. To see this, take the traces of all the relations, and sum. On the left hand side every term $tr(aa^*)$ is cancelled by a term $-tr(a^*a)$. On the right hand side the traces add up to $\lambda\cdot\alpha$.

2.3. MOMENT MAP. The relations for the deformed preprojective algebra arise from a moment map.

Let V be a vector space with a symplectic form ω, a skew symmetric bilinear form $V\times V \longrightarrow K$ which is non-degenerate in the sense that $\omega(u,v)=0$ for all v implies u=0. Let an algebraic group G act on V preserving ω. Differentiation gives an action of the Lie algebra $\mathfrak{g}\times V \longrightarrow V$. Since G preserves ω, it follows that

$$\omega(\theta v, v') = -\omega(v, \theta v')$$

for all $\theta \in \mathfrak{g}$ and $v, v' \in V$. By definition the <u>moment</u> <u>map</u> in this situation is the map $\mu: V \longrightarrow \mathfrak{g}^*$ defined by $\mu(v)(\theta) = \frac{1}{2}\,\omega(v, \theta v)$ for $v \in V$ and $\theta \in \mathfrak{g}$. It has the required property of moment maps in symplectic geometry: its derivative $d\mu_v: V \longrightarrow \mathfrak{g}^*$ at $v \in V$ satisfies

$$d\mu_v(v')(\theta) = \frac{1}{2}\,(\omega(v', \theta v) + \omega(v, \theta v')) = \omega(v, \theta v').$$

To apply this to quivers we equip $\mathrm{Rep}(\overline{Q}, \alpha)$ with the symplectic form coming from its identification with the cotangent bundle $T^*\mathrm{Rep}(Q, \alpha)$. Explicitly

$$\omega(x, y) = \sum_{a \in Q} \mathrm{tr}(x_{a^*}\, y_a) - \mathrm{tr}(x_a\, y_{a^*})$$

for $x, y \in \mathrm{Rep}(\overline{Q}, \alpha)$. The group $G(\alpha) = GL(\alpha)/K^*$ acts by conjugation and preserves ω.

$$(gx)_{a: i \to j} = g_i x_a g_j^{-1}$$

Its Lie algebra is identified with $\mathrm{End}(\alpha)/K$ where $\mathrm{End}(\alpha) = \prod_i \mathrm{Mat}(\alpha_i, K)$, and the action is given by

$$(\overline{\theta} x)_{a: i \to j} = \theta_i x_a - x_a \theta_j.$$

Let $\mathrm{End}(\alpha)_0 = \{\theta \in \mathrm{End}(\alpha) : \sum \mathrm{tr}(\theta_i) = 0\}$. The trace pairing gives an isomorphism

$$\mathrm{End}(\alpha)_0 \longrightarrow (\mathrm{End}(\alpha)/K)^*, \quad \theta \longmapsto (\overline{\phi} \longmapsto \sum_i \mathrm{tr}(\theta_i \phi_i)).$$

The moment map is thus $\mu_\alpha : \mathrm{Rep}(\overline{Q}, \alpha) \longrightarrow \mathrm{End}(\alpha)_0$ given by

$$x \longmapsto (\sum_{h(a)=i} x_a\, x_{a^*} - \sum_{t(a)=i} x_{a^*}\, x_a)_i$$

Now $G(\alpha)$ acts by conjugation on $\mathrm{End}(\alpha)_0$, and the invariant elements are those in which each component is a multiple of the identity matrix. We identify these with elements of $\{\lambda \in K^I : \lambda \cdot \alpha = 0\}$. Then $\mu_\alpha^{-1}(\lambda)$ is identified with the space of $\Pi^\lambda(Q)$-modules of dimension α.

2.4. LEMMA. If $x \in \mathrm{Rep}(Q,\alpha)$ and X is the corresponding KQ-module, then there is an exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(X,X)^* \longrightarrow \mathrm{Rep}(Q^{op},\alpha) \xrightarrow{f} \mathrm{End}(\alpha) \xrightarrow{t} \mathrm{End}(X)^* \longrightarrow 0$$

where $t(\theta)(\phi) = \sum_{i\in I} \mathrm{tr}(\theta_i\phi_j)$ comes from the trace pairing, and $f(y) = \sum_{a\in Q} [x_a, y_{a^*}]$. Thus the fibre of f over $\lambda\in K^I$ consists of the different ways of extending the action of KQ on X to an action of $\Pi^\lambda(Q)$.

PROOF. Apply $\mathrm{Hom}_{KQ}(-,X)$ to the standard resolution of X, dualize, and use trace pairings to identify terms.

2.5. THEOREM. If X is a KQ-module then the action of KQ on X can be extended to an action of $\Pi^\lambda(Q)$ if and only if $\lambda\cdot\underline{\dim}\, Y = 0$ for any KQ-module summand Y of X.

PROOF. Suppose that the action extends. Let X be given by $x \in \mathrm{Rep}(Q,\alpha)$. Then in the lemma we have $\lambda \in \mathrm{Im}(f)$, so $t(\lambda) = 0$, so $\sum \lambda_i \mathrm{tr}(\theta_i) = 0$ for any $\theta \in \mathrm{End}_{KQ}(X)$. If Y is a KQ-module summand of X, apply this with θ the projection onto Y to see that $\lambda\cdot\underline{\dim}\, Y = 0$.

For the converse, it suffices to prove that an indecomposable X with $\lambda\cdot\underline{\dim}\, X = 0$ lifts. By Fitting’s Lemma End(X) consists of multiples of the identity plus a nilpotent endomorphism, so it is easy to see that $t(\lambda) = 0$.

2.6. REMARK. Assuming Kac’s Theorem, it follows that the possible dimension vectors of $\Pi^\lambda(Q)$-modules are exactly the sums of roots α with $\lambda\cdot\alpha=0$. Note that although we don’t prove all of Kac’s Theorem, we prove enough to justify this claim for Q Dynkin or extended Dynkin, for when writing a vector as a sum of roots α with $\lambda\cdot\alpha=0$ you can take all these roots to be either real roots or δ, and what we prove is sufficient.

In fact one can prove that α is the dimension vector of a simple $\Pi^\lambda(Q)$-module if and only if α is a positive root, $\lambda\cdot\alpha = 0$, and

$$1-q(\alpha) > (1-q(\beta)) + (1-q(\gamma)) + \ldots$$

whenever $\alpha=\beta+\gamma+\ldots$ a sum of positive roots with $\lambda\cdot\beta=\lambda\cdot\gamma=\ldots=0$. See §2.14.

2.7. PROPOSITION. For an extended Dynkin quiver Q, all fibres of μ_δ are irreducible of dimension $1 + \sum \delta_i^2$. Thus μ_δ is flat.

PROOF. If $\theta\in\mathrm{End}(\alpha)_0$, let π be the composition

$$\mu_\delta^{-1}(\theta) \hookrightarrow \mathrm{Rep}(\bar{Q},\alpha) \twoheadrightarrow \mathrm{Rep}(Q,\alpha).$$

If $x\in\mathrm{Rep}(Q,\alpha)$ then $\pi^{-1}(x) \cong f^{-1}(\theta)$ in the sequence of Lemma 2.4, so it is either empty, or a coset of $\mathrm{Ext}^1(X,X)^*$. Thus it is either empty or irreducible of dimension dim End(X) + $q(\delta)$ = dim End(X) by Ringel's formula.

The bricks form a dense open set $B \subseteq \mathrm{Rep}(Q,\delta)$. They have nonempty fibres. Thus $\pi^{-1}(B)$ is irreducible of dimension dim B + 1 = $\sum \delta_i^2 + 1$.

Besides the bricks, there are only finitely many other orbits of $G(\delta)$ on $\mathrm{Rep}(Q,\delta)$. The stabilizer of x is identified with $\mathrm{Aut}(X)/K^*$, so the orbit of x has dimension

$$\dim G(\delta) - \dim \mathrm{Aut}(X)/K^*$$

and its inverse image under π (if non-empty) has dimension

$$\dim G(\delta) - \dim \mathrm{Aut}(X)/K^* + \dim \mathrm{End}(X) = \sum \delta_i^2.$$

Now any irreducible component of a fibre of μ_δ has dimension at least

$$\dim \mathrm{Rep}(\bar{Q},\delta) - \dim \mathrm{End}(\delta_0) = \sum \delta_i^2 + 1.$$

It follows that each fibre is irreducible of dimension $\sum \delta_i^2 + 1$. This implies flatness since μ_δ is a map between smooth irreducible varieties.

2.8. REFLECTION FUNCTORS. If i is a loopfree vertex, we have a reflection

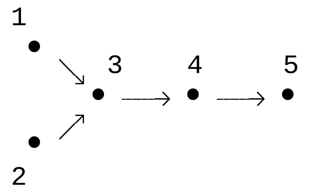
$$r_i : K^I \longrightarrow K^I, \quad r_i(\lambda)_j = \lambda_j - (\varepsilon_i,\varepsilon_j)\lambda_i.$$

dual to s_i. The duality means that $r_i\lambda \cdot \alpha = \lambda \cdot s_i\alpha$ for all λ, α. We say that the reflection is admissible for λ if $\lambda_i \neq 0$. In this case there is a Morita equivalence

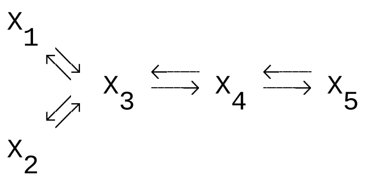
$$\Pi^\lambda(Q)\text{-modules} \longrightarrow \Pi^{r_i\lambda}(Q)\text{-modules}$$

which acts as s_i on dimension vectors. We call this a reflection functor. (Do not confuse this with a reflection functor in the sense of Bernstein, Gelfand and Ponomarev - they are for KQ-modules, and are not equivalences.)

EXAMPLE. If Q is the quiver



then a $\Pi^\lambda(Q)$-module X is given by vector spaces and linear maps



satisfying the deformed preprojective relations. For vertex i=3, the linear maps combine to give maps

$$X_3 \xrightarrow{\theta} X_1 \oplus X_2 \oplus X_4 \xrightarrow{\phi} X_3$$

and (inserting minus signs suitably) the relations ensure that $\phi\theta = \lambda_3$ Id. Now if $\lambda_3 \neq 0$ this implies that θ is the inclusion of a direct summand and

$$X_1 \oplus X_2 \oplus X_4 = \mathrm{Im}(\theta) \oplus \mathrm{Ker}(\phi).$$

The functor sends X to the $\Pi^\lambda(Q)$-module

$$\begin{array}{lll} X_1 & & \\ & \nwarrow\!\searrow & \\ & & \mathrm{Ker}(\phi) \overset{\longleftarrow}{\longrightarrow} X_4 \overset{\longleftarrow}{\longrightarrow} X_5 \\ & \swarrow\!\nearrow & \\ X_2 & & \end{array}$$

in which the linear maps to and from $\mathrm{Ker}(\phi)$ come from the two decompostions of $X_1 \oplus X_2 \oplus X_4$.

2.9. CONSEQUENCE. Let the Weyl group W act on K^I via $w\lambda \cdot \alpha = \lambda \cdot (w^{-1}\alpha)$ for all λ, α. We claim that if $\lambda' \in W\lambda$ then $\Pi^{\lambda}(Q)$ and $\Pi^{\lambda'}(Q)$ are Morita equivalent, that is, there is an equivalence

$$\Pi^{\lambda}(Q)\text{-modules} \longrightarrow \Pi^{\lambda'}(Q)\text{-modules}$$

Namely, write

$$\lambda' = r_{i_n} \dots r_{i_1} \lambda.$$

Doing this with n as small as possible, the reflections at each stage are admissible (for if $\lambda_i = 0$ then $r_i\lambda = \lambda$). The reflection functors then give the equivalence.

2.10. LEMMA. If there is a simple module for $\Pi^{\lambda}(Q)$ of dimension α, and i is a vertex, then $\alpha = \varepsilon_i$ or $(\alpha, \varepsilon_i) \le 0$ or $\lambda_i \ne 0$.

PROOF. Suppose otherwise. Since $(\alpha, \varepsilon_i) > 0$ there is no loop at i. If X is the simple module, since $\lambda_i = 0$, the linear maps combine to give maps

$$X_i \xrightarrow{\theta} \oplus_{j - i} X_j \xrightarrow{\phi} X_i$$

with composition zero. (The direct sum is over all arrows incident at i, and the corresponding term is the space X_j at the other end of the arrow.) Now θ is injective, for $\mathrm{Ker}(\theta)$ is a submodule of X, and if $X = \mathrm{Ker}(\theta)$ then X lives at i, and simplicity implies that $\alpha = \varepsilon_i$. Dually ϕ is surjective. But then

$$\dim \oplus_{j - i} X_j \ge 2 \dim X_i,$$

so $(\alpha,\varepsilon_i) \le 0$, a contradiction.

2.11. LEMMA. The dimension vector of any simple $\Pi^\lambda(Q)$-module is a root.

PROOF. Suppose that there is a simple of dimension α. Applying a sequence of admissible reflections, we follows the effect on λ and α:

$$\begin{matrix}\lambda \\ \alpha\end{matrix} \longrightarrow \begin{matrix}r_i\lambda \\ s_i\lambda\end{matrix} \longrightarrow \begin{matrix}r_jr_i\lambda \\ s_js_i\alpha\end{matrix} \longrightarrow \ldots \longrightarrow \begin{matrix}\lambda' \\ \alpha'\end{matrix}$$

Because of the reflection functors there is a simple $\Pi^{\lambda'}(Q)$-module of dimension α'. Thus α' is positive. We choose the sequence to make α' as small as possible. This implies that $(\alpha',\varepsilon_i) \le 0$ for any vertex i with $\lambda_i \neq 0$.

The previous lemma now implies that α' is either a coordinate vector at a loopfree vertex or has $(\alpha',\varepsilon_i) \le 0$ for all vertices i. Of course α' has connected support because of the existence of a simple $\Pi^{\lambda'}(Q)$-module of dimension α'. Thus in the latter case, α' is in the fundamental region. It follows that α is a root.

2.12. PROPOSITION. If there is an indecomposable for KQ of dimension α then α is a root.

PROOF. Write $\alpha = k\beta$ with β indivisible. Choose $\lambda \in K^I$ with $\lambda\cdot\alpha = 0$, but $\lambda\cdot\gamma \neq 0$ for any $0\le\gamma\le\alpha$ which is not a multiple of β. This is possible since K has characteristic zero.

The indecomposable KQ-module extends to an indecomposable $\Pi^\lambda(Q)$-module, and any composition factor of this must have dimension $m\beta$ for some m. Thus $m\beta$ is a root.

Now apply admissible reflections to λ and $m\beta$ as in the proof of Lemma 2.11. We pass to λ' and a vector which is easily seen to be of the form $m\beta'$ for some indivisible β'. The reflection functors can also be applied to the indecomposable $\Pi^\lambda(Q)$-module of dimension $\alpha = k\beta$ to give an indecomposable

$\Pi^{\lambda'}(Q)$-module of dimension $k\beta'$.

Now either $m\beta'$ is a coordinate vector at a loopfree vertex, or in the fundamental region.

In the first case m=1, but also because there is an indecomposable $\Pi^{\lambda'}(Q)$-module of dimension $k\beta'$, we must have k=1. Thus α is a root.

In the second case β' and $k\beta'$ are also in the fundamental region, so that α is again a root.

2.13. PROPOSITION. If α is a positive real root then (up to isomorphism) there is a unique indecomposable KQ-module of dimension α.

PROOF. We use the fact that every root is positive or negative. Write

$$\alpha = s_{i_n} \dots s_{i_1}(\varepsilon_j)$$

with j a loopfree vertex and n as small as possible. Then all intermediate terms $\alpha^k = s_{i_k} \dots s_{i_1}(\varepsilon_j)$ are positive roots. Define $\nu \in K^I$ by

$$\nu_i = \begin{cases} 0 & (i=j) \\ 1 & (\text{else})' \end{cases}$$

and let $\nu^k = r_{i_k} \dots r_{i_1}(\nu)$. Now $(\nu^k)_{i_{k+1}} = \nu^k \cdot \varepsilon_{i_{k+1}} = \nu \cdot \beta$ where

$$\beta = s_{i_1} \dots s_{i_k}(\varepsilon_{i_{k+1}}).$$

Now $\nu \cdot \beta \neq 0$, for since β is a real root it is positive or negative, so the condition $\nu \cdot \beta = 0$ implies that $\beta = \pm\, \varepsilon_j$, but then

$$\pm\, \varepsilon_{i_{k+1}} = s_{i_k} \dots s_{i_1}(\varepsilon_j) = \alpha^k$$

contradicting the minimality of n. Thus the reflection at i_{k+1} is admissible for ν^k. Thus there are reflection functors

$$\Pi^{\nu}(Q)\text{-modules} \longrightarrow \Pi^{\nu^1}(Q)\text{-modules} \longrightarrow \Pi^{\nu^2}(Q)\text{-modules} \longrightarrow \ldots$$

Clearly there is a unique $\Pi^{\nu}(Q)$-module of dimension ε_j, and it is simple. Thus, letting $\lambda = \nu^n$, there is a unique $\Pi^{\lambda}(Q)$-module M of dimension α and it is simple.

Now M is indecomposable as a KQ-module, for if it has an indecomposable summand of dimension β, then

$$0 = \lambda\cdot\beta = \nu^n\cdot\beta = \nu\cdot\gamma \quad \text{where} \quad \gamma = s_{i_1}\ldots s_{i_n}\,\beta.$$

But β is a root by Proposition 2.12, hence so is γ, and the condition $\nu\cdot\gamma = 0$ implies that $\gamma = \pm\,\varepsilon_j$. Thus $\beta = \pm\, s_{i_n}\ldots s_{i_1}(\varepsilon_j) = \pm\,\alpha$, so in fact $\beta = \alpha$.

Finally the indecomposable KQ-module of dimension α is unique since any such module can be extended to a $\Pi^{\lambda}(Q)$-module, but there is a unique $\Pi^{\lambda}(Q)$-module of dimension α.

2.14. FURTHER READING. The deformed preprojective algebra and the reflection functors were introduced in W.Crawley-Boevey, and M.P.Holland, Noncommutative deformations of Kleinian singularities, Duke Math. J. 92 (1998), 605-635.

The construction of the preprojective algebra using TrD is in D.Baer, W.Geigle and H.Lenzing, The preprojective algebra of a tame hereditary Artin algebra, Commun. Algebra 15 (1987), 425-457.

However, to see that these two descriptions of the preprojective algebra are the same, see C.M.Ringel, The preprojective algebra of a quiver, in: Algebras and modules, II (Geiranger, 1996), 467-480, CMS Conf. Proc., 24, Amer. Math. Soc., Providence, RI, 1998, or W.Crawley-Boevey, Preprojective algebras, differential operators and a Conze embedding for deformations of Kleinian singularities, Comment. Math. Helv. 74 (1999), 548-574. This latter paper contains much more about deformed preprojective algebras.

The paper W.Crawley-Boevey, Geometry of the moment map for representations of quivers, to appear in Composito Math., proves the characterization of the dimensions of simple modules for deformed preprojective algebras. In an appendix it also contains the elementary deduction of much of Kac’s Theorem given here.

Lecture 3. Module varieties and skew group algebras

We are interested in finite dimensional left modules for a finitely generated K-algebra A (associative, with 1).

3.1. MODULE VARIETIES. There is an affine variety

$$\begin{aligned} \mathrm{Mod}(A,n) &= \{A\text{-module structures on } K^n\} \\ &= \{K\text{-algebra maps } A \longrightarrow \mathrm{Mat}(n,K)\} \\ &= \{(\theta_1,\dots,\theta_m)\in \mathrm{Mat}(n,K)^m : r(\theta_1,\dots,\theta_m)=0 \text{ for all } r\in R\} \end{aligned}$$

on chosing generators of A, and hence writing $A = K\langle x_1,\dots,x_m\rangle/R$, where $K\langle x_1,\dots,x_m\rangle$ is the free associative algebra on generators $x_1,\dots,x_m$ and R is an ideal.

We need a variation on this, relative to a semisimple subalgebra. Let A be a f.g. K-algebra and $S \subseteq A$ a f.d. semisimple subalgebra (or more generally let $S \longrightarrow A$ be a homomorphism). If M is a f.d. S-module, let

$$\mathrm{Mod}_S(A,M) = \{A\text{-module structures on } M \text{ extending its } S\text{-module structure}\}$$

It is an affine variety. (Choosing a basis of M, it can be identified with a fibre of the map $\mathrm{Mod}(A,n) \longrightarrow \mathrm{Mod}(S,n)$.)

The group $G(M) = \mathrm{Aut}_S(M)/K^*$ acts on $\mathrm{Mod}_S(A,M)$. It orbits are in 1-1 correspondence with isomorphism classes of A-modules X with ${}_SX \cong M$. We write $\mathcal{O}(X)$ for the orbit corresponding to X. The stabilizer of a point $x \in \mathcal{O}(X)$ can be identified with $\mathrm{Aut}_A(X)/K^*$.

Special cases:

(1) Suppose $A = T_SV$ is the tensor algebra on an S-S-bimodule. Then by the universal property of tensor algebras, $\mathrm{Mod}_S(T_SV,M) \cong \mathrm{Hom}_S(V\otimes_S M,M)$.

(2) Suppose that A is a path algebra KQ of a quiver Q with vertex set I. Let $S = K\times\dots\times K$ spanned by the trivial paths e_i. If $\alpha \in \mathbb{N}^I$ define

$$K^{\alpha} = \oplus_{i\in I} K^{\alpha_i}.$$

This is an S-module, with multiplication by e_i acting as projection onto the i-th summand. Then we can identify

$$\mathrm{Mod}_S(KQ,K^{\alpha}) = \mathrm{Rep}(Q,\alpha), \quad G(K^{\alpha}) = G(\alpha)$$

(3) If A is a quotient of a path algebra, let S be the subalgebra generated by the trivial paths as before. Then $\mathrm{Mod}_S(A,K^{\alpha})$ is a closed subvariety of $\mathrm{Rep}(Q,\alpha)$. In particular,

$$\mathrm{Mod}_S(\Pi^{\lambda}(Q),K^{\alpha}) = \mu_{\alpha}^{-1}(\lambda).$$

3.2 TRACE FUNCTIONS. If $a\in A$ and X is a f.d. A-module then define

$$\mathrm{tr}(a,X) = \text{trace of the map } X \longrightarrow X,\ x \longmapsto ax.$$

This defines a function $\mathrm{tr}(a,-) : \mathrm{Mod}_S(A,M)\longrightarrow K$. It is a G(M)-invariant, so $\mathrm{tr}(a,-) \in K[\mathrm{Mod}_S(A,M)]^{G(M)}$.

Given a module X, we write gr X for the semisimple module which is the direct sum of the composition factors of X (with the same multiplicities).

(1) $\mathrm{tr}(a,X) = \mathrm{tr}(a,\mathrm{gr}\,X)$. The trace of a matrix with block form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ is the sum of the traces of the diagonal blocks, so if $0\longrightarrow X_1\longrightarrow X\longrightarrow X_2\longrightarrow 0$ is an exact sequence of A-modules, then $\mathrm{tr}(a,X) = \mathrm{tr}(a,X_1)+\mathrm{tr}(a,X_2) = \mathrm{tr}(a,X_1\oplus X_2)$. The assertion follows by induction.

(2) If $\mathrm{tr}(a,X) = \mathrm{tr}(a,Y)$ for all a, then $\mathrm{gr}\,X \cong \mathrm{gr}\,Y$. This is character theory! To prove it we may assume that X,Y are semisimple. Replacing A by $A/\mathrm{ann}_A(X\oplus Y)$ we may assume that A is semisimple. Now if

$$A \cong \mathrm{Mat}(n_1,K) \times \dots \times \mathrm{Mat}(n_r,K),$$

a_i is the i-th identity element, and S_j is the j-th simple module, then $\mathrm{tr}(a_i,S_j) = \delta_{ij}n_j$. The claim follows.

3.3. PROPOSITION. The closure of any orbit $\mathcal{O}(X)$ contains a unique closed orbit, $\mathcal{O}(\mathrm{gr}\ X)$. Thus the closed orbits in $\mathrm{Mod}_S(A,M)$ are exactly those of semisimple A-modules.

PROOF. We show first that if $Y \subseteq X$ then $\mathcal{O}(Y \oplus X/Y) \subseteq \overline{\mathcal{O}(X)}$. By definition $X = M$ with an A-module structure. Let C be an S-module complement to Y in M. The action of any $a \in A$ on X is an element of $\mathrm{End}_K(Y \oplus C)$, so can be written as a 2×2 matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$$

with $a_{11} \in \mathrm{End}_K(Y)$, $a_{12} \in \mathrm{Hom}_K(C,Y)$, $a_{22} \in \mathrm{End}_K(C)$. For $t \in K^*$, let $g_t \in G(M)$ correspond to the automorphism of M which is multipliction by t on Y and the identity on C. The action of $a \in A$ on $g_t X$ is given by the matrix

$$\begin{pmatrix} a_{11} & ta_{12} \\ 0 & a_{22} \end{pmatrix}$$

Thus the closure of the orbit of X must contain the element given by matrices

$$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}$$

That is, $Y \oplus X/Y$.

Now by induction $\mathcal{O}(\mathrm{gr}\ X) \subseteq \overline{\mathcal{O}(X)}$. In particular, if $\mathcal{O}(X)$ is closed then X must be semisimple. On the other hand, the closure of any orbit is a union of orbits, so always contains a closed orbit, eg one of minimal dimension. Finally note that if $\mathcal{O}(Y)$ is a closed orbit in $\overline{\mathcal{O}(X)}$ then by continuity X,Y have the same trace functions, so $\mathrm{gr}\ Y \cong \mathrm{gr}\ X$.

3.4. QUOTIENTS. If a reductive group G acts on an affine variety V then the quotient V / G (the set of orbits) is not usually well-behaved topologically. However the set V // G of closed orbits is. It is natually an affine variety with coordinate ring $K[V]^G$.

This applies to G(M) acting on $Mod_S(A,M)$. Since M is semisimple, $M \cong \oplus_i S_i^{\alpha_i}$ with the S_i non-isomorphic simples, and $Aut_S(M) \cong \prod_i GL_{\alpha_i}(K)$. Thus G(M) is reductive. Now by Proposition 3.3,

$$Mod_S(A,M) \,//\, G(M) = \left\{ \begin{array}{l} \text{Isomorphism classes of semisimple} \\ \text{A-modules X with } {}_SX \cong M \end{array} \right\}$$

and there is a natural map

$$Mod_S(A,M) \longrightarrow Mod_S(A,M) \,//\, G(M), \quad X \longmapsto gr\, X.$$

More general quotients can be constructed as part of "Geometric Invariant Theory". Let θ be an additive function {semisimple S-modules} $\longrightarrow \mathbb{Q}$. One says that X is

θ-<u>semistable</u> if $\theta(X) = 0$ and $\theta(Y) \geq 0$ for all A-submodules $Y \leq X$.

θ-<u>stable</u> if $\theta(X) = 0$ and $\theta(Y) > 0$ for all A-submodules $0 \neq Y \subset X$.

A θ-semistable module X naturally has associated to it a module $gr_\theta X$ which is a direct sum of θ-stables, and one says X,X′ are S-<u>equivalent</u> if $gr_\theta X \cong gr_\theta X'$.

There is a GIT quotient $Mod_S(A,M) \,//\, (G(M),\theta)$ whose points correspond to {θ-semistables X with ${}_SX \cong M$} / S-equivalence, and there is a proper map $Mod_S(A,M) \,//\, (G(M),\theta) \longrightarrow Mod_S(A,M) \,//\, G(M)$.

3.5. MORITA EQUIVALENCE. If $e \in A$ is an idempotent then eAe is an algebra with identity e, and there is a functor

$$\text{A-modules} \longrightarrow \text{eAe-modules}, \quad X \longmapsto eX.$$

To apply this to module varieties we assume that $e \in S$. Then the functor induces a morphism

$$\Phi : Mod_S(A,M) \,//\, G(M) \longrightarrow Mod_{eSe}(eAe,eM) \,//\, G(eM).$$

In case AeA = A the the functor is an equivalence, so that A and eAe are Morita equivalent. It follows that the morphism is a bijection. In fact we have:

THEOREM. If AeA = A then Φ is an isomorphism of varieties.

We prove this in the next subsection. Here we verify it the special case when $S = \mathrm{Mat}(n_1,K) \times \dots \times \mathrm{Mat}(n_r,K)$, and $e = \sum_i e_i$, where e_i is the elementary matrix

$$\begin{pmatrix} 1 & 0 & & 0 \\ 0 & 0 & & 0 \\ & & \ddots & \\ 0 & 0 & & 0 \end{pmatrix}$$

in the i-th factor. Then SeS = S and $eSe \cong K\times\dots\times K$. It is easy to see that A is isomorphic to the algebra of $r\times r$ matrices (C_{ij}) where each C_{ij} is an $n_i\times n_j$ matrix of elements of e_iAe_j, and that eAe is isomorphic to the algebra of $r\times r$ matrices (D_{ij}) with each $D_{ij} \in e_iAe_j$. Thus Φ has an inverse coming from the map which sends an eAe-module structure on $eM \cong \oplus_{i=1}^r e_iM$ to the A-module structure on $\oplus_{i=1}^r (e_iM)^{n_i}$ given by the action of the block matrices.

3.6. THEOREM. The ring of invariants $K[\mathrm{Mod}_S(A,M)]^{G(M)}$ is generated by the trace functions tr(a,-).

PROOF. Presumably there is a direct proof of this. Definitely it needs characteristic zero in this generality. It is proved by Le Bruyn and Procesi for path algebras. If G acts on X and $Y \subseteq X$ is G-stable closed subset, then the restriction map $K[X]^G \longrightarrow K[Y]^G$ is surjective by the Reynolds operator. Thus it follows for quotients of path algebras, or equivalently for algebras A in which the semisimple subalgebra S is isomorphic to $K\times\dots\times K$. Now the general case can be reduced to this by the special case of Morita equivalence proved above.

We now prove that Φ is an isomorphism in the general case when AeA=A. Since Φ is a bijection, it is sufficient to prove that it is a closed embedding, or equivalently that the natural map

$$K[\mathrm{Mod}_{eSe}(eAe,eM)]^{G(eM)} \longrightarrow K[\mathrm{Mod}_S(A,M)]^{G(M)}$$

is surjective. Now the space on the right hand size is generated by trace functions tr(a,-) with $a\in A$. Since AeA = A we can write $a = \sum a_k e a_k'$. Then

$$tr(a,M) = \sum tr(a_k e a_k',M) = \sum tr(e a_k' a_k e,M) = tr(b,eM),$$

where $b = \sum e a_k' a_k e \in eAe$, so tr(a,-) is in the image of the map.

3.7. SKEW GROUP ALGEBRAS. If A is an algebra and Γ is a finite group acting as automorphisms of A, then the skew group algebra $A\#\Gamma$ consists of the formal sums $\sum_{g\in\Gamma} a_g\, g$ with the multiplication satisfying

$$(a\, g)(a'\, g') = a({}^g a')\, gg'.$$

Let $K\Gamma$ be the group algebra of Γ. Observe that an $A\#\Gamma$-module consists of an A-module X which is also a $K\Gamma$-module, and such that $g(ax) = ({}^g a)(gx)$.

If X is an A-module then $(A\#\Gamma)\otimes_A X$ is isomorphic as an A-module to $\oplus_{g\in\Gamma}\, {}^g X$, where ${}^g X$ denotes the module X with the action of A twisted by g, and the action of Γ permutes the factors.

Henceforth we write e for the idempotent

$$e = \frac{1}{|\Gamma|} \sum_{g\in\Gamma} g \in K\Gamma.$$

It has the property that $eK\Gamma = K\Gamma e = Ke$ and $e(A\#\Gamma)e = A^\Gamma e \cong A^\Gamma$.

3.8. LEMMA. gl.dim $A\#\Gamma$ = gl.dim A.

PROOF. Using a projective resolution of X it is easy to see that

$$\mathrm{Ext}^i_{A\#\Gamma}(A\#\Gamma \otimes_A X,Y) \cong \mathrm{Ext}^i_A(X,{}_A Y)$$

when X is an A-module and Y is an $A\#\Gamma$-module.

Now if Z is an A-module, then it is a direct summand of ${}_AY$ where $Y = (A\#\Gamma)\otimes_A Z$. It follows that gl.dim A ≤ gl.dim A#Γ.

On the other hand, if X is an A#Γ-module then it is isomorphic to a summand of $A\#\Gamma \otimes_A X$, since the multiplication map $A\#\Gamma \otimes_A X \longrightarrow X$ has a section

$$x \longmapsto \frac{1}{|\Gamma|} \sum_{g\in\Gamma} g\otimes g^{-1}x$$

This implies that gl.dim A#Γ ≤ gl.dim A.

3.9. SIMPLE MODULES. Let Γ be a finite group acting on an affine variety V, so also on its coordinate ring K[V] via

$$(gf)(v) = f(g^{-1}v)$$

for g∈Γ, v∈V and f∈K[V]. The quotient V/Γ is an affine variety with coordinate ring $K[V]^\Gamma$ (since all orbits are closed).

We are interested in simple modules for K[V]#Γ. Now $K[V]^\Gamma$ is contained in the centre of K[V]#Γ, with equality if the action of Γ on V is faithful. Thus any simple module for K[V]#Γ is annihilated by a unique maximal ideal $\mathfrak{m}$ for $K[V]^\Gamma$, so it is a module for $(K[V]/K[V]\mathfrak{m})\#\Gamma$. Thus it is a module for

$$\left(K[V] \,/\, \sqrt{K[V]\mathfrak{m}} \right)\#\Gamma \;\cong\; (K^{\mathcal{O}})\#\Gamma$$

where $\mathcal{O}$ is the orbit in V corresponding to $\mathfrak{m}$, $K^{\mathcal{O}}$ is the space of functions from $\mathcal{O}$ to K, and Γ acts by permuting the factors. This is a semisimple algebra. Special cases are:

(1) Γ acts freely on $\mathcal{O}$. That is $|\mathcal{O}| = |\Gamma| = N$, say. In this case $(K^{\mathcal{O}})\#\Gamma \cong \mathrm{Mat}(N,K)$. Thus there is a unique simple module S for K[V]#Γ which is annihilated by $\mathfrak{m}$. Note that ${}_{K\Gamma}S \cong K\Gamma$ and that $eS \neq 0$.

(2) $\mathcal{O}$ consists of one fixed point. Then $(K^{\mathcal{O}})\#\Gamma \cong K\Gamma$, so each simple KΓ-module induces a simple module for K[V]#Γ annihilated by $\mathfrak{m}$.

3.10. MORITA EQUIVALENCE. There is a functor

$$K[V]\#\Gamma\text{-modules} \longrightarrow K[V]^{\Gamma}\text{-modules}, \quad X \longmapsto eX$$

and hence (since $eK\Gamma \cong K$), a morphism

$$\Phi : \mathrm{Mod}_{K\Gamma}(K[V]\#\Gamma,K\Gamma) \,/\!/\, G(K\Gamma) \longrightarrow \mathrm{Mod}_{K}(K[V]^{\Gamma},K) \,/\!/\, G(K) \cong V/\Gamma.$$

LEMMA. If Γ acts freely on V then $(K[V]\#\Gamma)e(K[V]\#\Gamma) = K[V]\#\Gamma$. Thus the functor is a Morita equivalence, and Φ is an isomorphism.

PROOF. If $K[V]\#\Gamma \,/\, (K[V]\#\Gamma)e(K[V]\#\Gamma) \neq 0$ then there is a non-zero $K[V]\#\Gamma$-module M with $eM=0$. It follows that there is a simple module with this property. But this is not the case.

3.11. ISOLATED SINGULARITIES. Let Γ act on a smooth irreducible variety V. Let $0 \in V$ be a fixed point, and assume that Γ acts freely on $V\setminus\{0\}$. Then V/Γ is an isolated singularity.

THEOREM. In this case Φ is also an isomorphism.

PROOF. For each orbit of Γ on $V\setminus\{0\}$ there is a simple $K[V]\#\Gamma$-module structure on $K\Gamma$. In addition, each simple $K\Gamma$-module gives a simple $K[V]\#\Gamma$-module corresponding to the point $0\in V$. It follows that for each point of V/Γ there is a unique semisimple $K[V]\#\Gamma$-module structure on $K\Gamma$. Thus Φ is a bijection.

For each $v\in V$ we consider $K\Gamma$ as a $K[V]\#\Gamma$-module, with $fg = f(gv)g$ for $f\in K[V]$, $g\in\Gamma$. This induces a map

$$V \longrightarrow \mathrm{Mod}_{K\Gamma}(K[V]\#\Gamma,K\Gamma) \,/\!/\, G(K\Gamma)$$

which is constant on Γ-orbits so factors through V/Γ. The resulting map

$$V/\Gamma \longrightarrow \mathrm{Mod}_{K\Gamma}(K[V]\#\Gamma,K\Gamma) \,/\!/\, G(K\Gamma)$$

is an inverse for Φ.

3.12. REMARK. Provided Γ acts faithfully on V, so that the general orbit of Γ on V is free, The same argument shows that even if V/Γ is not an isolated singularity, the quotient $Mod_{K\Gamma}(K[V]\#\Gamma,K\Gamma) // G(K\Gamma)$ has an irreducible component isomorphic to V/Γ.

Now using other Geometric Invariant Theory quotients

$$Mod_{K\Gamma}(K[V]\#\Gamma,K\Gamma) // (G(K\Gamma),\theta)$$

one can hope to obtain a desingularization of V/Γ. This was used by Cassens and Slodowy to construct the minimal desingularization for Kleinian singularities, and it is essentially the "Γ-Hilbert scheme" $Hilb^{\Gamma}(V)$ considered by Ito and Nakamura, Nakajima, and others.

3.13. LEMMA. In the isolated singularities case, $K[V]\#\Gamma/(K[V]\#\Gamma)e(K[V]\#\Gamma)$ is finite-dimensional.

PROOF. It is a f.g. $K[V]^{\Gamma}$-module, whose only composition factor is $K[V]^{\Gamma}/\mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal corresponding to the singular point.

3.14. FURTHER READING. Module varieties have been extensively studied for finite dimensional algebras. See for example P.Gabriel, Finite representation type is open, in: Representations of algebras, Proc Ottawa 1974, eds V. Dlab and P. Gabriel, SLN 488; C.Geiß, Geometric methods in representation theory of finite dimensional algebras, in: Canadian Math. Soc. Conf. Proc., 19, 1996; K.Bongartz, Some geometric aspects of representation theory, in: Canadian Math. Soc. Conf. Proc., 23, 1998.

Geometric Invariant Theory quotients are discussed in A.D.King, Moduli of representations of finite dimensional algebras, Quart. J. Math. Oxford 45 (1994), 515-530. The special case $\theta=0$ also covers the affine quotients. Note that King assumes that A is finite-dimensional in his Section 4, but this is only necessary for Proposition 4.3.

The fact that the invariants for representations of quivers are generated by traces of oriented cycles is in L.Le Bruyn, and C.Procesi, Semisimple representations of quivers, Trans. Amer. Math. Soc. 317 (1990), 585-598.

Skew group algebras are a classical topic. See for example J.C.McConnell and J.C.Robson, Noncommutative noetherian rings.

For the construction of desingularizations of Kleinian singularities in this way see H.Cassens and P.Slodowy, On Kleinian singularities and quivers, Singularities (Oberwolfach, 1996), Birkhauser, Basel, 1998, 263-288.

The higher dimensional McKay correspondence comes from M.Reid, McKay correspondence, math.AG/9702016. The Γ-Hilbert scheme is in Y.Ito and I.Nakamura, Hilbert schemes and simple singularities, in: Algebraic Geometry (Proc. Warwick, 1996), eds K. Hulek et al. (Cambridge Univ. Press 1999), 151-233. Also significant is Y.Ito and H.Nakajima, McKay correspondence and Hilbert schemes in dimension three, math.AG/9803120.

Lecture 4. Deforming skew group algebras

4.1. SYMPLECTIC FORMS. Recall that a symplectic form on a vector space V is a bilinear form $\omega:V\times V\longrightarrow K$ which is skew symmetric and non-degenerate in the sense that $\omega(u,v)=0$ for all v implies u=0.

One can think of ω as a skew symmetric element of $V^*\otimes V^*$.

One can choose symplectic coordinates $p_i,q_i:V\longrightarrow K$ such that $\omega = \sum p_i\otimes q_i-q_i\otimes p_i$. In particular dim V must be even.

Observe that ω induces an isomorphism $V\longrightarrow V^*$, $v\longmapsto\omega(v,-)$ so V^* also gets a symplectic form ω^*.

4.2. GROUPS PRESERVING A SYMPLECTIC FORM. We give some examples of group actions which preserve a symplectic form.

(1) (Kleinian case) Let $V = K^2$ with $\omega(x,y) = x_1y_2 - x_2y_1$. Then a subgroup $\Gamma \le GL_2(K)$ preserves ω if and only if $\Gamma \le SL_2(K)$.

(2) If Γ acts on U, then it also acts on the cotangent bundle $T^*U = U^*\oplus U$ preserving the symplectic form ω defined by $\omega(f\oplus u,f'\oplus u') = f'(u)-f(u')$.

(3) An irreducible representation V of a finite group Γ with Frobenius-Schur indicator -1 preserves a skew symmetric bilinear form on V. It must be a symplectic form since V is irreducible.

4.3. DEFORMING SKEW GROUP ALGEBRAS. Suppose that a finite group Γ acts linearly on a vector space V preserving a symplectic form ω. Observe that

$$K[V]\#\Gamma = (T(V^*) / (\theta\phi-\phi\theta:\theta,\phi\in V^*))\#\Gamma \cong (T(V^*)\#\Gamma)/(\theta\phi-\phi\theta:\theta,\phi\in V^*),$$
$$K[V]^\Gamma = e\, K[V]\#\Gamma\, e.$$

Here $T(V^*)$ is the tensor algebra of V^* (over K) and $e = \frac{1}{|\Gamma|} \sum_{g\in\Gamma} g$.
For $\lambda \in Z(K\Gamma)$, define

$$S^\lambda = T(V^*)\#\Gamma / (\theta\phi-\phi\theta-\lambda\omega^*(\theta,\phi) : \theta,\phi\in V^*),$$
$$\mathcal{O}^\lambda = eS^\lambda e.$$

Observe that if $c\in K^*$ then $S^\lambda \cong S^{c\lambda}$ and $\mathcal{O}^\lambda \cong \mathcal{O}^{c\lambda}$, using the automorphism of $T(V^*)\#\Gamma$ which multiplies each element of V^* by $\sqrt{c}$.

4.4. ASSOCIATED GRADED ALGEBRAS. Suppose that an algebra A is generated over a subalgebra A_0 by finitely many elements x_i. There is a "standard" filtration

$$0 = A_{-1} \subseteq A_0 \subseteq A_1 \subseteq \ldots$$

where A_n = span of elements $a_0 x_{i_1} a_1 \ldots x_{i_k} a_k$ with $a_i\in A_0$ and $k\le n$.

Whenever A is a filtered ring there is an associated graded algebra

$$\mathrm{gr}\, A = \oplus_{i=0}^{\infty} A_i/A_{i-1}.$$

For the standard filtration, gr A is generated over A_0 by elements $\bar{x}_i$.

It is well known that if gr A has one of the following properties, then so does A:

- domain
- prime
- noetherian
- finite global dimension

4.5. LEMMA. S^λ is filtered, with associated graded ring $K[V]\#\Gamma$. Thus it is prime, noetherian of finite global dimension. $\mathcal{O}^\lambda$ is filtered, with associated graded ring $K[V]^\Gamma$. Thus it is a noetherian domain.

PROOF. S^λ is generated over $K\Gamma$ by a basis θ_i of V^*. This gives the filtration. Now gr S^λ is generated over $K\Gamma$ by $\bar\theta_i$. These elements satisfy $\bar\theta_i\bar\theta_j = \bar\theta_j\bar\theta_i$. Thus there is a surjection $K[V]\#\Gamma \twoheadrightarrow \mathrm{gr}\, S^\lambda$.

Is it an isomorphism? You can use the relation in S^λ to reorder monomials, modulo lower degree. Thus S^λ has basis the elements $\theta_1^{n_1}\dots\theta_m^{n_m}g$. The rest follows.

(When is $\mathcal{O}^\lambda$ commutative, and how does its global dimension depend on λ? I can only answer these questions in the Kleinian case, when these properties are related to preprojective algebras.)

4.6. FINITE GENERATION. The following lemma shows that eS^λ is a f.g. $\mathcal{O}^\lambda$-module and $\mathcal{O}^\lambda$ is a f.g. K-algebra.

LEMMA (Montgomery and Small). If A is an algebra, $e\in A$ is idempotent and AeA is a f.g. left ideal in A (eg A is noetherian), then eA is a f.g. eAe-module. If in addition A is a f.g. K-algebra then eAe is a f.g. K-algebra.

PROOF. Let $AeA = \sum Ax_i$ and $x_i = \sum v_{ij}ew_{ij}$ with $v_{ij},w_{ij}\in A$. Then $\sum_{ij} Aew_{ij} = AeA$, so $\sum_{ij} eAew_{ij} = eAeA = eA$, so the elements ew_{ij} generate eA as an eAe-module.

Now suppose that $t_1,\dots,t_n$ generate A and let $eA = \sum eAex_i$.
Write $et_j = \sum_i ey_{ij}ex_i$ and $ex_kt_j = \sum_i ez_{ijk}ex_i$ with $y_{ij},z_{ijk}\in A$.
We claim that eAe is generated by the elements ex_ie, $ey_{ij}e$ and $ez_{ijk}e$.
For, every element of eAe is a linear combination of terms $et_{j_1}t_{j_2}\dots t_{j_\ell}e$, and, for example, $et_1t_2e = \sum_i ey_{i1}ex_it_2e = \sum_{i\ell} ey_{i1}e\, ez_{\ell 2i}ex_\ell e$.

4.7. LEMMA. There is a surjective homomorphism

$$T(V^*)\#\Gamma\ /\ (\omega-\lambda) \longrightarrow S^\lambda.$$

PROOF. Recall that $S^\lambda = T(V^*)\#\Gamma\ /\ (\theta\phi-\phi\theta-\lambda\omega^*(\theta,\phi))$.

If V has symplectic coordinates p_i, q_i, then $\omega = \sum p_i \otimes q_i - q_i \otimes p_i$. Now $\omega^*(p_i,q_i) = 1$, so

$$\omega - \lambda = \sum (p_i \otimes q_i - q_i \otimes p_i - \lambda' \ \omega(p_i,q_i)).$$

where $\lambda' = \frac{2}{\dim V} \lambda$. Thus there is a natural map $T(V^*)\#\Gamma/(\omega-\lambda) \longrightarrow S^{\lambda'}$. But $S^{\lambda'} \cong S^{\lambda}$.

4.8. McKAY QUIVERS. Let Γ be a finite group acting linearly on a vector space V. Let N_i $(i \in I)$ be the simple $K\Gamma$-modules, with N_0 the trivial module. The McKay quiver Δ for Γ and V has vertex set I, and by definition the number of arrows $i \longrightarrow j$ is the multiplicity of N_i in $V \otimes N_j$.

HENCEFORTH SUPPOSE that Γ preserves a symplectic form ω. Identify $\lambda \in Z(K\Gamma)$ with an element of K^I via λ_i = trace of λ on N_i. The rest of this lecture is devoted to proving that $T(V^*)\#\Gamma \ / \ (\omega-\lambda)$ is Morita equivalent to $\Pi^{\lambda}(Q)$ for some quiver Q with vertex set I and $\overline{Q} = \Delta$.

Some more notation. Let $\dim N_i = \delta_i$, so that $K\Gamma \cong \prod_i \mathrm{Mat}(\delta_i,K)$. Let the elements $E^i_{pq} \in K\Gamma$ $(i \in I,\ 1 \le p,q \le \delta_i)$ correspond to the elementary matrices.

Let $f_i = E^i_{11}$ and let $f = \sum f_i$. Clearly $f_0 = e = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} g$. Also

$$f \ K\Gamma \ f = \oplus \ K \ f_i \cong K \times \ldots \times K.$$

Observe that $N_i \cong K\Gamma f_i$, which has basis E^i_{p1}. Dually, $f_i K\Gamma$ has basis E^i_{1q}.

$$1 = \sum_{i,p} E^i_{p1} \ E^i_{1p} = \sum_{i,p} E^i_{p1} \ f \ E^i_{1p} \in K\Gamma \ f \ K\Gamma.$$

Thus $fK\Gamma f$ is Morita equivalent to $K\Gamma$.

4.9. LEMMA. If B is a $K\Gamma$-$K\Gamma$-bimodule, then there is an isomorphism $f \ T_{K\Gamma}(B) \ f \cong T_{fK\Gamma f}(fBf)$ which is the identity on $fK\Gamma f$, and sends an element $b \otimes c \in fB \otimes_{K\Gamma} Bf$ to the element $\sum_{ip} bE^i_{p1} \otimes E^i_{1p} c \in fBf \otimes fBf$.

PROOF. Follows from the Morita equivalence.

4.10. LEMMA. If Γ acts linearly on a vector space W preserving a symplectic form τ, then the restriction σ of τ to V^Γ is a symplectic form σ, and if $\tau^* = \sum w_k \otimes v_k \in W \otimes W$ then $\sigma^* = \sum ew_k \otimes ev_k \in W^\Gamma \otimes W^\Gamma$.

PROOF. σ is non-degenerate since if $0 \neq w \in W^\Gamma$ then there is $v \in W$ with $\tau(w,v) \neq 0$, but then $\tau(w,ev) = \tau(w,v) \neq 0$ and $ev \in W^\Gamma$.

Say $\theta, \theta' \in (W^\Gamma)^*$, so $\theta = \sigma(x,-)$, $\theta' = \sigma(x',-)$ for some $x, x' \in W^\Gamma$. Then

$$\begin{aligned}
\sigma^*(\theta,\theta') &= \sigma(x,x') \text{ by definition of } \sigma^* \\
&= \tau(x,x') \\
&= \tau^*(\tau(x,-),\tau(x',-)) \text{ by definition of } \tau^* \\
&= \textstyle\sum_k \tau(x,w_k)\, \tau(x',v_k) \text{ since } \tau^* = \sum w_k \otimes v_k \\
&= \textstyle\sum_k \tau(x,ew_k)\tau(x',ev_k) \text{ since } x,x' \in W^\Gamma \\
&= \textstyle\sum_k \sigma(x,ew_k)\sigma(x',ev_k) \\
&= \textstyle\sum_k \theta(ew_k)\, \theta'(ev_k).
\end{aligned}$$

4.11. PROPOSITION. Let $B = V^* \otimes K\Gamma$ considered as a $K\Gamma$-$K\Gamma$-bimodule via $g(v \otimes g')g'' = gv \otimes gg'g''$. Then fBf has a symplectic form σ with

$$\sigma^* = \sum_{ijpqk} E^i_{1p}(\theta_k \otimes E^j_{q1}) \otimes E^j_{1q}(\phi_k \otimes E^i_{p1}) \in fBf \otimes fBf$$

where $\omega = \sum \theta_k \otimes \phi_k \in V^* \otimes V^*$. Moreover σ respects the decomposition $fBf = \oplus_{ij} f_i B f_j$ in the sense that $\sigma(b,b')=0$ if $b \in f_i B f_j$ and $b' \in f_{i'} B f_{j'}$ with $i \neq j'$ or $j \neq i'$.

PROOF. Let $W = fK\Gamma \otimes V^* \otimes K\Gamma f$, as a $K\Gamma$-module via $g(a \otimes \theta \otimes b) = ag^{-1} \otimes g\theta \otimes gb$.
Consider the map $m: W \longrightarrow fBf$, $a \otimes v \otimes b \longmapsto a(v \otimes b)$.
Observe that $m(e(a \otimes \theta \otimes b)) = m(a \otimes \theta \otimes b)$.
By dimensions it follows that m induces a vector space isomorphism $(fK\Gamma \otimes V^* \otimes K\Gamma f)^\Gamma \longrightarrow fBf$.

Now $fK\Gamma \otimes V^* \otimes K\Gamma f$ has a symplectic form τ given by

$$\tau(a \otimes \theta \otimes b, a' \otimes \theta' \otimes b') = \alpha\beta\ \omega^*(\theta,\theta')$$

for $a \in f_i K\Gamma$, $b \in K\Gamma f_j$, $a' \in f_{i'} K\Gamma$, $b' \in K\Gamma f_{j'}$, where $\alpha, \beta \in K$ are given by $ab' = \alpha f_i$

and $a'b = \beta f_{i'}$. The action of Γ preserves τ. Thus τ gives a symplectic form on fixed points, so on fBf. Call it σ. Clearly this respects the decomposition.

Now τ^* as an element of the tensor square of $fK\Gamma\otimes V\otimes K\Gamma f$ is

$$\tau^* = \textstyle\sum_{ijpqk} \ (E^i_{1p}\otimes\theta_k\otimes E^j_{q1}) \otimes (E^j_{1q}\otimes\phi_k\otimes E^i_{p1})$$

Thus, using that $m(e(a\otimes\theta\otimes b)) = a(\theta\otimes b)$, we get

$$\sigma^* = \textstyle\sum_{ijpqk} \ E^i_{1p}(\theta_k\otimes E^j_{q1}) \otimes E^j_{1q}(\phi_k\otimes E^i_{p1})$$

4.12. THEOREM. $f\ T(V)\#\Gamma \ / \ (\omega-\lambda)\ f \cong T_{fK\Gamma f}(fBf) \ / \ (\sigma^* - \lambda_i f_i)$

PROOF. First observe that $T(V^*)\#\Gamma \cong T_{K\Gamma}B$
- As vector spaces the LHS is $K\Gamma \oplus V^*\otimes K\Gamma \oplus V^*\otimes V^*\otimes K\Gamma \oplus \ldots$
- The RHS is $K\Gamma \oplus V^*\otimes K\Gamma \oplus (V^*\otimes K\Gamma)\otimes_{K\Gamma}(V^*\otimes K\Gamma) \oplus \ldots$

Thus $T(V)\#\Gamma \ / \ (\omega-\lambda) \cong T_{K\Gamma}(B) \ / \ (\zeta-\lambda)$ where, if $\omega = \sum \theta_k\otimes\phi_k \in V^*\otimes V^*$, then $\zeta = \sum (\theta_k\otimes 1)\otimes(\phi_k\otimes 1) \in B\otimes_{K\Gamma}B$.

Now $f\ T(V)\#\Gamma \ / \ (\omega-\lambda)\ f \cong f\ T_{K\Gamma}(B)\ f \ / \ I$ where I is the ideal

$$f\ T_{K\Gamma}(B)\ (\zeta-\lambda)\ T_{K\Gamma}(B)\ f = f\ T_{K\Gamma}(B)\ K\Gamma fK\Gamma\ (\zeta-\lambda)\ K\Gamma fK\Gamma\ T_{K\Gamma}(B)\ f$$

This is generated as an ideal in $f\ T_{K\Gamma}(B)\ f$ by $fK\Gamma\ (\zeta-\lambda)\ K\Gamma f$.

Now if $g\in\Gamma$ then

$$g\zeta = \textstyle\sum (g\theta_k\otimes g)\otimes(\phi_k\otimes 1) = \sum (g\theta_k\otimes 1)\otimes(g\phi_k\otimes g) = \sum (g\theta_k\otimes 1)\otimes(g\phi_k\otimes 1)g = \zeta g$$

since $\omega = \sum g\theta_k\otimes g\phi_k$ because ω is Γ-invariant. Thus

$$E^i_{1p}\ (\zeta-\lambda)\ E^j_{q1} = (\zeta-\lambda)\ E^i_{1p}E^j_{q1} = \begin{cases} (\zeta-\lambda)f_i & (i=j \text{ and } p=q) \\ 0 & (\text{else}) \end{cases}$$

Now $\lambda f_i = \lambda_i/\delta_i\ f_i$ since λ acts on N_i as multiplication by λ_i/δ_i.

Now we consider the isomorphism $f\ T_{K\Gamma}(B)\ f \longrightarrow T_{fK\Gamma f}(fBf)$.

We have $\zeta f_i = 1/\delta_i \sum_p E^i_{1p}\ \zeta\ E^i_{p1}$ since one can carry the E^i_{1p} across.
The isomorphism thus sends ζf_i to

$$1/\delta_i \sum_{jpqk} E^i_{1p}\ (\theta_k \otimes E^j_{q1}) \otimes E^j_{1q} (\phi_k \otimes 1)\ E^i_{p1} = 1/\delta_i\ \sigma^* f_i.$$

Thus the relation $(\zeta-\lambda)f_i$ is sent to $1/\delta_i(\sigma^* f_i - \lambda_i f_i)$.

4.13. COROLLARY. The McKay quiver is the double of a quiver Q, and there is an isomorphism $f\ T(V)\#\Gamma\ /\ (\omega-\lambda)\ f \cong \Pi^\lambda(Q)$ sending f_i to e_i. Thus $T(V^*)\#\Gamma\ /\ (\omega-\lambda)$ is Morita equivalent to $\Pi^\lambda(Q)$.

PROOF. fBf has a symplectic form σ which respects the decomposition $fBf = \oplus_{ij}\ f_i B f_j$.

It follows that you can choose a basis $\{a,a^*\}$ with each a belonging to some $f_i B f_j$ and a^* to $f_j B f_i$ in such a way that $\sigma^* = \sum_a a \otimes a^* - a^* \otimes a$.

Let Q be the quiver with arrows the a. Since

$$\dim f_i B f_j = \dim \mathrm{Hom}_{K\Gamma}(K\Gamma f_i, V^* \otimes K\Gamma f_j) = \dim \mathrm{Hom}_{K\Gamma}(V \otimes N_i, N_j),$$

it follows that the double of Q is the McKay quiver.

4.14. NOTES. For a discussion about associated graded rings see J.C.McConnell and J.C.Robson, Noncommutative noetherian rings, §1.6. Theorem 4.12 and Corollary 4.13 are new.

Lecture 5. The Kleinian case

5.1. SETUP. Let Γ be a finite subgroup of $SL_2(K)$ acting on $V = K^2$. It preserves the symplectic form $\omega(x,y) = x_1y_2 - x_2y_1$. It acts freely on $V\setminus\{0\}$.

An element $\lambda \in Z(K\Gamma)$ gives rise to rings S^λ and $\mathcal{O}^\lambda$.

The Gelfand-Kirillov dimension of a f.g. K-algebra is denoted GK A. More generally GK dimension is defined for f.g. A-modules. The GK dimension of gr A is equal to the GK dimension of A. Moreover, for commutative rings, the GK dimension is equal to the usual Krull dimension. It follows easily that S^λ and $\mathcal{O}^\lambda$ have GK dimension 2.

The McKay quiver Δ is equal to $\overline{Q}$ for some Q. Moreover $f\ S^\lambda\ f \cong \Pi^\lambda(Q)$ (so that S^λ and $\Pi^\lambda(Q)$ are Morita equivalent) since when considering the generators

$$\theta\phi - \phi\theta - \lambda\omega^*(\theta,\phi)$$

of the ideal defining S^λ, because V is 2-dimensional there is only one element here, and it is essentially $\omega - \lambda$.

Thus also $\mathcal{O}^\lambda = eS^\lambda e \cong e_0\Pi^\lambda(Q)e_0$.

5.2. LEMMA (McKay!). Q is an extended Dynkin quiver. The element $\delta \in \mathbb{N}^I$ defined by $\delta_i = \dim N_i$ is the radical generator for Q.

PROOF. Since dim V = 2, it follows that $(\delta,\varepsilon_i) = 0$ for all i. Now $\delta_0 = 1$ since N_0 is the trivial module.

5.3. LEMMA. $S^\lambda/S^\lambda eS^\lambda$ is Morita equivalent to $\Pi^{\lambda^\circ}(Q^\circ)$ where Q° is the Dynkin part of Q and λ° is the restriction of λ. These algebras are f.d., and are zero if and only if $\lambda\cdot\alpha \neq 0$ for all Dynkin roots α (ie roots with $\alpha_0=0$).

PROOF. $S^\lambda/S^\lambda eS^\lambda$ is Morita equivalent to $\Pi^\lambda(Q)/\Pi^\lambda(Q)e_0\Pi^\lambda(Q) \cong \Pi^{\lambda^\circ}(Q^\circ)$.

Now $S^{\lambda}/S^{\lambda}eS^{\lambda}$ is finite dimensional, for it is filtered, and the associated graded algebra is a quotient of S^0/S^0eS^0, which is f.d. by Lemma 3.13.

Finally a f.d. algebra is zero if and only if it has no f.d. modules, so §2.6 applies.

5.4. ORDERS AND REFLEXIVE MODULES. A prime Goldie ring A has a simple artinian quotient ring Q, and then A is an order in Q, meaning that every $q \in Q$ can be written as as^{-1} and as $t^{-1}b$ with $a,b,s,t \in A$, s,t units in Q.

An order is said to be maximal if $A \subseteq B$ and $xBy \subseteq A$ for some units $x,y \in Q$, imply that A=B. A commutative integral domain is a maximal order if and only if it is integrally closed.

If A is an order in Q, simple artinian, and $e \in A$ is a non-zero idempotent, then you can identify $\mathrm{End}_{eAe}(eA) = \{q \in Q : eAq \subseteq eA\}$, for

$$\mathrm{End}_{eAe}(eA) \xrightarrow{1} \mathrm{End}_{eAe}(eQ) \overset{2}{\supseteq} \mathrm{End}_{eQe}(eQ) \xleftarrow{3} Q$$

1 is 1-1. It comes from the fact that $eA \otimes_A Q \cong eQ$.

2 is equality since by general theory eAe is an order in eQe.

3 the homothety is an isomorphism since Q is simple artinian.

If in addition A is a maximal order then $A \cong \mathrm{End}_{eAe}(eA)$:

We have $A \subseteq \{q \in Q : eAq \subseteq eA\} \subseteq Q$.
Now QeQ = Q, so AeA contains a unit s of Q. Then $eAq \subseteq eA \Rightarrow sq \subseteq A$.
Thus maximality implies $A = \{q \in Q : eAq \subseteq eA\}$

Also eA is a reflexive eAe-module. (Recall that M is a reflexive R-module if $M \xrightarrow{\cong} \mathrm{Hom}_R(\mathrm{Hom}_R(M,R),R)$. Namely, there are isomorphisms

$$Ae \longrightarrow \mathrm{Hom}_{eAe}(eA,eA)e \cong \mathrm{Hom}_{eAe}(eA,eAe)$$
$$eA \longrightarrow e\,\mathrm{Hom}_{eAe}(Ae,Ae) \cong \mathrm{Hom}_{eAe}(Ae,eAe).$$

If A is a maximal order then so is $A\#\Gamma$. Thus $S^0 \cong K[V]\#\Gamma$ is a maximal order. Also $\mathcal{O}^0 \cong K[V]^{\Gamma}$ is integrally closed, so it too is a maximal order.

Finally we need to use a theorem of Van den Bergh & Van Oystaeyen, that the property of being a maximal order in a simple artinian ring passes from gr A to A. Thus S^λ and $\mathcal{O}^\lambda$ are maximal orders.

Thus $\mathrm{End}_{\mathcal{O}^\lambda}(eS^\lambda) \cong S^\lambda$ and eS^λ is a f.g. reflexive $\mathcal{O}^\lambda$-module.

5.5. HOMOLOGICAL PROPERTIES. If A is an algebra which is noetherian and has finite injective dimension, then the <u>grade</u> of a non-zero f.g. module M is defined by

$$j({}_A M) = \inf \{i : \mathrm{Ext}^i_A(M,A)\neq 0\}.$$

One says that A is <u>Auslander-Gorenstein</u> if it is noetherian, finite injective dimension, and for every f.g. A-module M and every submodule $N_A \leq \mathrm{Ext}^i_A(M,A)$ one has $j(N_A) \geq i$.

One says that A is <u>Cohen-Macaulay</u> if j(M) + GK M = GK A for all nonzero f.g. A-modules M.

Björk shows that the Auslander-Gorenstein and Cohen-Macaulay properties pass from gr A to A.

Now Kleinian singularities are Auslander-Gorenstein and Cohen-Macaulay.
Thus $\mathcal{O}^\lambda$ is Auslander-Gorenstein and Cohen-Macaulay.
Of course GK $\mathcal{O}^\lambda$ = GK S^λ = 2.

5.6. LEMMA. If A is a f.g. noetherian Auslander-Gorenstein and Cohen-Macaulay algebra of finite global dimension (eg $\mathcal{O}^\lambda$), then

$$\mathrm{gl.dim}\ A = \max \{ GK(A) - GK(M) : M \text{ a f.g. nonzero module} \}$$

PROOF. Since A is noetherian, gl.dim. A is its injective dimension as a right (or left) module. Say μ.

This is the maximal value of j(M) for a non-zero f.g. module M.

- If $j(M)=j$ then $Ext^j(M,A)\neq 0$ so inj.dim $R \geq j$, so $\mu \geq j$.

- There is some right module N with $Ext^\mu(N,A)\neq 0$.
  We may assume that N is f.g.
  Now if $M = Ext^\mu(N,A)$ then by the Auslander condition $j(M) \geq \mu$.

Thus μ = maximal value of GK(A) - GK(M) where M is f.g. nonzero module.

5.7. THEOREM.

$$\text{gl.dim } \mathcal{O}^\lambda = \begin{cases} 1 & (\text{if } \lambda\cdot\alpha \neq 0 \text{ for all roots } \alpha) \\ 2 & (\text{else}) \\ \infty & (\text{if } \lambda\cdot\alpha = 0 \text{ for some Dynkin root } \alpha) \end{cases}$$

and S^λ is Morita equivalent to $\mathcal{O}^\lambda \Leftrightarrow$ gl.dim $\mathcal{O}^\lambda < \infty$.

PROOF. If $\lambda\cdot\alpha \neq 0$ for all Dynkin roots α then $\mathcal{O}^\lambda$ is Morita equivalent to S^λ by Lemma 5.3.

If $\mathcal{O}^\lambda$ is Morita equivalent to S^λ then gl.dim $\mathcal{O}^\lambda$ = gl.dim $S^\lambda < \infty$.

Conversely, if gl.dim $\mathcal{O}^\lambda < \infty$, then it clearly can't be 0, so it is 1 or 2 according to whether or not there are any f.d. modules. In any case, since it is ≤ 2, any f.g. reflexive module is projective. Thus eS^λ is a projective left $\mathcal{O}^\lambda$-module. Since it has $\mathcal{O}^\lambda$ as a summand, it is a progenerator. Thus $S^\lambda e S^\lambda = S^\lambda$. This implies that $\lambda\cdot\alpha\neq 0$ for all Dynkin roots α by Lemma 5.3. It also implies that $\mathcal{O}^\lambda$ and S^λ are Morita equivalent. Thus $\mathcal{O}^\lambda$ has a f.d. module if and only if S^λ has a f.d. module, and this is if and only if $\lambda\cdot\alpha = 0$ for some root α.

5.8. LEMMA. If Q is extended Dynkin and $\lambda\cdot\delta = 0$ then $\mu_\delta^{-1}(\lambda)//G(\delta)$ has dimension 2.

PROOF. Let $\mathfrak{h} = \{\lambda\in K^I : \lambda\cdot\delta = 0\}$. Since μ_δ is flat, so is the pullback $\mu_\delta^{-1}(\mathfrak{h}) \longrightarrow \mathfrak{h}$. Thus by the Reynolds operator, the map

$$f : \mu_\delta^{-1}(\mathfrak{h})//G(\delta) \longrightarrow \mathfrak{h}$$

is also flat. (This is a lift through the Weyl group of the semi-universal deformation of the Kleinian singularity).

Also f is surjective, and the fibre over $\lambda \in \mathfrak{h}$ is $\mu_\delta^{-1}(\lambda)//G(\delta)$, which is irreducible. Since f is flat, has irreducible fibres, and $\mathfrak{h}$ is irreducible, it follows that $\mu_\delta^{-1}(\mathfrak{h})//G(\delta)$ is irreducible. Flatness now implies that all fibres of f have the same dimension. But

$$\mu_\delta^{-1}(0)//G(\delta) \cong \mathrm{Mod}_S(\Pi^\lambda(Q),K^\delta)//G(K^\delta) \cong \mathrm{Mod}_{K\Gamma}(K[V]\#\Gamma,K\Gamma)//G(K\Gamma) \cong V/\Gamma,$$

has dimension 2.

5.9. THEOREM. If $\lambda\cdot\delta = 0$ then $\mathcal{O}^\lambda \cong K[\mu_\delta^{-1}(\lambda)//G(\delta)]$, so $\mathcal{O}^\lambda$ is commutative.

PROOF. Let $S = K^I$ and identify

$$K[\mu_\delta^{-1}(\lambda)//G(\delta)] \cong K[\mathrm{Mod}_S(\Pi^\lambda(Q),K^\delta)]^{G(K^\delta)}$$

The map $a \longmapsto \mathrm{tr}(a,-)$ defines a map

$$K\overline{Q} \longrightarrow \Pi^\lambda(Q) \longrightarrow K[\mathrm{Mod}_S(\Pi^\lambda(Q),K^\delta)]^{G(K^\delta)}.$$

Recall that the invariants are generated by the trace functions.
For generators one can take oriented cycles in $\overline{Q}$.

- The invariant given by any cycle that doesn’t pass through 0 factors

$$\mathrm{Mod}_S(\Pi^\lambda(Q),K^\delta) \xrightarrow{a} \mathrm{Mod}_S(\Pi^{\lambda^\circ}(Q^\circ),K^{\delta^\circ}) \xrightarrow{b} K$$

where Q° is the Dynkin quiver obtained by deleting the vertex 0, and λ° and δ° are the restrictions of λ and δ. But this Dynkin deformed preprojective algebra is finite-dimensional, so

$$\mathrm{Mod}_S(\Pi^{\lambda^\circ}(Q^\circ),K^{\delta^\circ})//G(K^\delta)$$

is finite. Thus any polynomial invariant is constant on the connected components of $\mathrm{Mod}_S(\Pi^{\lambda^\circ}(Q^\circ),K^{\delta^\circ})$. Now a must map into one connected component, so the invariant is constant.

- The invariant given by any cycle that passes through 0 is in the image of the map

$$\phi : e_0\Pi^\lambda(Q)e_0 \longrightarrow K[\mathrm{Mod}_S(\Pi^\lambda(Q),K^\delta)]^{G(K^\delta)}$$

Now ϕ is an algebra homomorphism since trace is multiplicative for 1×1 matrices. It follows that ϕ is surjective. Now the left hand side is $\mathcal{O}^\lambda$, a domain of GK dimension 2. The right hand side is of dimension 2. Thus ϕ must be an isomorphism.

5.10. PROPOSITION. If $\lambda\cdot\delta \neq 0$ then $\Pi^\lambda(Q)$ has only finitely many f.d. simple modules. The same holds for $\mathcal{O}^\lambda$, so it is noncommutative.

PROOF. If there is a simple module of dimension α then by a sequence of reflection functors one can pass from (λ,α) to (λ',α') with α' a coordinate vector or in the fundamental region.

Since δ is invariant under reflections, $\lambda'\cdot\delta = \lambda\cdot\delta \neq 0$, so the latter possibility is ruled out.

For the former, note that there is a unique simple module. Thus we just need there to be only finitely many roots α with $\lambda\cdot\alpha=0$. Now if α is any root then so is $\alpha+n\delta$ for all n since $q(\alpha) = q(\alpha+n\delta)$. It follows that any root is of the form $\beta+n\delta$ where β is a Dynkin root. Now there are only finitely many possible β, and for any β, at most one of the roots $\beta+n\delta$ has $\lambda\cdot(\beta+n\delta)=0$.

Now any f.d. $\mathcal{O}^\lambda$-module M is isomorphic to eL for some f.d. $\Pi^\lambda(Q)$-module L (for example take $L = \Pi^\lambda(Q)e_0 \otimes_{\mathcal{O}^\lambda} M$), and if M is simple one can take L simple.

5.11. PROPOSITION. The following are equivalent.
(1) every non-Dynkin root α has $\lambda\cdot\alpha \neq 0$.

(2) There is non nonzero f.d. $\mathcal{O}^\lambda$-module.
(3) $\mathcal{O}^\lambda$ is simple.

PROOF. (1)⇔(2) As above, every f.d. $\mathcal{O}^\lambda$-module M is isomorphic to e_0L for some f.d. $\Pi^\lambda(Q)$-module L. Now the dimensions of $\Pi^\lambda(Q)$-modules are sums of roots α with $\lambda\cdot\alpha = 0$.

(2)⇔(3) If I is a nonzero ideal in $\mathcal{O}^\lambda$ then $\mathcal{O}^\lambda/I$ has GK dimension ≤ 1, so it is a PI ring by (Stafford, Small and Warfield, Math Proc Cam Phil Soc 97(1985),407-414. A f.g K-algebra with GK(R)=1 is PI, N(R) is nilpotent, and R/N(R) is module-finite over noetherian centre). Thus it has f.d. modules.

5.12. SOME BIJECTIONS. M. P. Holland pointed out to me some work of G.Wilson. Consider the set

{Right ideals of the first Weyl algebra} / isomorphism

By work of Cannings and Holland the elements of this set are in 1-1 correspondence with the a certain set

{Primary decomposable subspaces of $\mathbb{C}[x]$} / a certain equivalence

Wilson observed that this set in naturally identified with a certain "adelic Grassmannian",

$$\mathrm{Gr}^{\mathrm{ad}}.$$

By work of Segal and Wilson its points are in 1-1 correspondence with rational solutions of the KP heierarchy

$$\tfrac{3}{4}u_{yy} = (u_t - \tfrac{1}{4}(u_{xxx}+6uu_x))_x \ .$$

Wilson also proved that the points of $\mathrm{Gr}^{\mathrm{ad}}$ are in 1-1 correspondence with

$$\bigcup_{n\geq 0} C_n$$

where C_n is a certain completed phase space for the rational Calogero-Moser system of n particles moving on the complex line with the Hamiltonian

$$\frac{1}{2} \sum_{i=1}^{n} p_i^2 - \sum_{i<j} 1/(x_i-x_j)^2 \ .$$

Now by inspection C_n is identified with the space

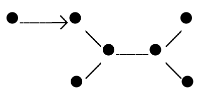
$$\mu_\alpha^{-1}(\lambda) \ // \ G(\alpha)$$

for the quiver with two vertices and two arrows

$$Q = \overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \circlearrowleft$$

and $\alpha = (1,n)$, $\lambda = (-n,1)$.

In my work on simple modules for deformed preprojective algebras I had dealt with generalizations of this quiver, in which one starts with an extended Dynkin quiver, and adds a new vertex connected by an edge to an extending vertex. For example



M. P. Holland and I conjecture that for generic λ there is a bijection between isomorphism classes of stably free right ideals in $\mathcal{O}^\lambda$ and elements of the sets $\mu_\alpha^{-1}(\lambda') \ // \ G(\alpha)$ where $\alpha=(1,n\delta)$, $\lambda'=(-n\lambda\cdot\delta,\lambda)$. Variations are also possible, in which one varies the extending vertex or allows other λ.

5.13. FURTHER READING. Most of the arguments come from W.Crawley-Boevey and M.P.Holland, Noncommutative deformations of Kleinian singularities, Duke Math. J. 92 (1998), 605-635.

For Gelfand-Kirillov dimension see J.C.McConnell and J.C.Robson, Noncommutative noetherian rings.

The right ideals in the first Weyl algebra are classified in R.C.Cannings and M.P.Holland, Right ideals of rings of differential operators, J. Algebra 167 (1994), 116-141.

The connection with Calogero-Moser phase spaces is G.Wilson, Collisions of Calogero-Moser particles and an adelic Grassmannian, Invent. Math. 133 (1998), 1-41.