Surreal numbers, derivations and transseries

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Torino – 20 March 2015
Outline

1. Surreal numbers
2. Hardy fields and transseries
3. Surreal derivations
Conway’s games

A **Game** is a pair $L|R$ where $L, R$ are (w.f.) sets of Games and

1. $L$ are the legal moves for **Left** (called **left options**);
2. $R$ are the legal moves for **Right** (called **right options**).

Go, chess, checkers can be interpreted as Games.$^1$

Conway defined a partial order and a sum on Games.

He then noticed that some games behave as *numbers*.
I will omit the details of “numbers as Games” and directly jump to a more concrete description.

$^1$Ignoring draws, at least!
A surreal number $x \in \text{No}$ is a string of $+, -$ of ordinal length.

- $0 := ""$
- $1 := " + ", -1 := " - "$
- $2 := " ++ ", \frac{1}{2} := " + - ", \ldots$
- $\frac{3}{2} := " +++ ", \frac{3}{4} := " + -- ", \ldots$

Order on the surreal line:

Go on for all ordinals: $\omega := " +++ \ldots ", \frac{1}{\omega} := " + -- \ldots ", \ldots$

We get $\text{On} \subset \text{No}$, with $\alpha = \{ ++ + \ldots \}$

$\alpha$ times

Definition. $x$ is simpler than $y$, or $x <_s y$, if $x$ is a prefix of $y$. 
**Definition.** Given $L, R$ sets of numbers such that $L < R$, we say $x = L|R$ when $x$ is *simplest* such that $L < x < R$.

For instance: $1 = \{0\}|\{\}$, $2 = \{0, 1\}|\{\}$, $\frac{1}{2} = \{0\}|\{1\}$...

**Note.** For any $x \in \text{No}$, we can write $x = L|R$ where $L \cup R$ is the set of the numbers strictly simpler than $x$.

**Definition.** If $x = \{x'\}|\{x''\}$, $y = \{y'\}|\{y''\}$, then their sum is

$$x + y := \{x' + y, x + y'\}|\{x'' + y, x + y''\}.$$ 

(Idea: we want $(x' + y) < (x + y) < (x'' + y)$...)

**Fact.** $(\text{No}, +, <)$ is an ordered abelian group.
$(\text{On}, +)$ is a monoid and $+$ is the Hessenberg sum.
(The sum: \( x + y = \{x' + y, x + y'\} \{x'' + y, x + y''\} \).)

**Definition.** If \( x = \{x'\} \{x''\} \), \( y = \{y'\} \{y''\} \), their **product** is

\[
x \cdot y := \{x'y + xy' - x'y', x''y + xy'' - x''y''\} |
\]
\[
|\{x'y + xy'' - x'y'', x''y + xy' - x''y'\}\.
\]

(Idea: we want \((x - x')(y - y') > 0, (x - x'')(y - y'') > 0\)...)

**Fact.** \((\text{No}, +, \cdot, <)\) is a field containing \(\mathbb{R}\).
\((\text{On}^{>0}, \cdot)\) is a monoid and \(\cdot\) is the Hessenberg product.
No as field of Hahn series

Take some $R \supseteq \mathbb{R}$ and consider the Archimedean equivalence

$$x \asymp y \iff \frac{1}{n} |y| \leq |x| \leq n|y|$$

for some $n \in \mathbb{N}^>0$.

Let $\Gamma < (R^0, \cdot)$ be a group of representatives for $\asymp$.

Let $\mathbb{R}((\Gamma))$ be the field of Hahn series

$$r_0 \gamma_0 + r_1 \gamma_1 + \cdots + r_\omega \gamma_\omega + \cdots$$

where $r_\alpha \in \mathbb{R}$, $\gamma_\alpha \in \Gamma$, and $(\gamma_\alpha)_{\alpha < \gamma}$ decreasing.

**Theorem** (Hahn-Kaplansky). $R$ embeds into $\mathbb{R}((\Gamma))$.

The monomials $\mathcal{M}$ are the “simplest $\asymp$-representatives” in $\text{No}^0$.

**Theorem** (Conway). $(\mathcal{M}, \cdot) \cong (\text{No}, +)$ and $\text{No} \cong \mathbb{R}((\mathcal{M}))$.

**Corollary.** $\text{No}$ is a real closed field (in fact, $\text{Set}$-saturated).
Definition (Kruskal-Gonshor). Given \( x = \{x'\} \mid \{x''\} \), define

\[
\exp(x) := \{0, \exp(x') \cdot [x - x']_n, \exp(x'') [x - x'']_{2n+1}\} |
\]

\[
\{ \frac{\exp(x'')}{[x'' - x]_n}, \frac{\exp(x')}{[x' - x]_{2n+1}} \},
\]

where \( n \) ranges in \( \mathbb{N} \), \([y]_n := 1 + \frac{y}{1!} + \cdots + \frac{y^n}{n!}\), and \([y]_{2n+1}\) is to be considered only when \([y]_{2n+1} > 0\).

Theorem (Gonshor). \( \exp \) is a monotone isomorphism \( \exp: (\mathbb{N}_o, +) \overset{\sim}{\rightarrow} (\mathbb{N}_o^{>0}, \cdot) \) and \( \exp(x) \geq 1 + x \).
Monster model for $\mathbb{R}_{\text{an,exp}}$

Suppose $f$ analytic at $r \in \mathbb{R}$ with $f(r + x) = a_0 + a_1 x + a_2 x^2 + \ldots$. If $\varepsilon$ is infinitesimal, we define (after Alling)

$$f(r + \varepsilon) := a_0 + a_1 \varepsilon + a_2 \varepsilon^2 + \ldots$$

**Theorem** (van den Dries-Erlich).
$(\mathbb{N}o, +, \cdot, <, \{f\}_{f \text{ analytic}}, \exp)$ is an elem. extension of $\mathbb{R}_{\text{an,exp}}$.

By o-minimality and saturation, $\mathbb{N}o$ is a *monster model*. 
Take a family $\mathcal{F}$ of continuous functions $f : (u, \infty) \to \mathbb{R}$.
Take the ring $H(\mathcal{F})$ of germs at $\infty$: for each $f \in \mathcal{F}$,

$$[f] = \{ g \in \mathcal{F} \mid g(x) = f(x) \text{ for all } x \text{ sufficiently large} \}.$$

**Definition** (Bourbaki). $H(\mathcal{F})$ is a **Hardy field** if:

1. it is a field;
2. it is closed under differentiation.

**Fact.** A Hardy field $H(\mathcal{F})$ is always ordered (given $f \in \mathcal{F}$, either $f(x) > 0$, $f(x) < 0$ or $f(x) = 0$ for all $x$ sufficiently large).
Examples of Hardy fields

Some Hardy fields:

1. (germs of) rational functions $H(\mathbb{R}(x))$;
2. rational functions, exp and log $H(\mathbb{R}(x, \exp(x), \log(x)))$;
3. Hardy’s field $L$ of “logarithmico-exponential functions”.

Given an expansion $R$ of $\mathbb{R}$, we abbreviate with $H(R)$ the ring of germs at $\infty$ of unary definable functions $\mathbb{R} \to \mathbb{R}$.

**Fact.** $R$ is o-minimal if and only if $H(R)$ is a Hardy field.

$H(\mathbb{R}_{\text{an}, \exp})$ is a Hardy field which is also an elem. ext. of $\mathbb{R}_{\text{an}, \exp}$. 
**H-fields**

**H-fields** are an abstract version of Hardy fields. For simplicity, we work over \( \mathbb{R} \).

**Definition** (Aschenbrenner-van den Dries). An **H-field** is an ordered field with a derivation \( D \) such that:

1. if \( x > \mathbb{R} \), then \( D(x) > 0 \);
2. \( D(x) = 0 \) if and only if \( x \in \mathbb{R} \).

Hardy fields are obviously **H-fields**.

\( H(\mathbb{R}_{\text{an}, \exp}) \) is an elem. ext. of \( \mathbb{R}_{\text{an}, \exp} \) which is also an **H-field**. It satisfies \( D(\exp(f)) = \exp(f)D(f) \), \( D(\arctan(f)) = \frac{D(f)}{1+f^2} \), ...
Transseries

$H(\mathbb{R}_{\text{an,exp}})$ is an ordered field: it embeds into some $\mathbb{R}((\Gamma))$.

The field $\mathbb{R}((\Gamma))$ contains series such as

$$\log(\Gamma(t^{-1})) = \log(t) - \gamma t^{-1} + \sum_{n=2}^{\infty} q_n t^{-n}.$$  

This is a typical “transseries”.

There are many notions of “field of transseries”:

1. transseries by Dahn, Göring, Écalle;
2. “LE-series” by van den Dries, Macintyre and Marker;
3. “EL-(trans)series” by S. Kuhlmann, and Matusinski;
4. “grid-based transseries” by van der Hoeven;
5. “transseries” by M. Schmeling.
Several notions of transseries

The various fields are slightly different from one another. For instance, \( LE \) embeds into \( EL \), but \( EL \) contains also:

\[
\log(x) + \log(\log(x)) + \log(\log(\log(x))) \ldots
\]

All of them are naturally models of the theory of \( \mathbb{R}_{\text{an,exp}} \). They can be made into \( H \)-fields (with \( D(x) = 1 \) for (1)-(3)) such that \( D(\exp(t)) = \exp(t)D(t) \), \( D(\arctan(t)) = \frac{D(t)}{1+t^2} \), ...

**Conj./Theorem** (Aschenbrenner-van den Dries-van der Hoeven). \( LE \)-series are a model-companion of \( H \)-fields.
Theorem (Kuhlmann-Kuhlmann-Shelah). If $\Gamma$ is a set, $\mathbb{R}((\Gamma))$ cannot have a global exp “compatible with the series structure”. (We can close under either exp or “infinite sum”, but not both).

But $\mathbb{No}$ is a class, and $\mathbb{No} = \mathbb{R}((\mathcal{M}))$ has a global exp.

Questions (Aschenbrenner, van den Dries, van der Hoeven, S. Kuhlmann, Matusinski...).

1. Can we give $\mathbb{No}$ a natural structure of $H$-field and such that $D(\exp(x)) = \exp(x)D(x), D(\arctan(x)) = \frac{D(x)}{1+x^2}$, ...?

2. Can we give $\mathbb{No}$ a natural structure of transseries?

Van der Hoeven hinted at a candidate for (2). S. Kuhlmann and Matusinski made a conjecture for (1)-(2).
**Definition.** A **surreal derivation** is a $D : \mathbb{No} \to \mathbb{No}$ such that:

1. **Leibniz’ rule:** $D(xy) = xD(y) + yD(x)$;
2. **strong additivity:** $D\left(\sum_{i \in I} a_i\right) = \sum_{i \in I} D(a_i)$;
3. **compatibility with exp:** $D(\exp(x)) = \exp(x)D(x)$;
4. **constant field $\mathbb{R}$:** $\ker(D) = \mathbb{R}$;
5. **H-field:** if $x > \mathbb{R}$ then $D(x) > 0$.

Let us try to construct $D$ and see what happens...
Let $\mathbb{J}$ be the ring of purely infinite numbers $\mathbb{R}[[\mathcal{M}^>1]]$.

**Theorem** (Gonshor). $\exp(\mathbb{J}) = \mathcal{M}$.

Since $\mathbb{No} = \mathbb{R}((\mathcal{M}))$, for any $x \in \mathbb{No}$ we can write

$$x = r_0 e^{\gamma_0} + r_1 e^{\gamma_1} + \cdots + r_\omega e^{\gamma_\omega} + \ldots$$

where $r_\alpha \in \mathbb{R}$ and $\gamma_\alpha \in \mathbb{J}$, with $(\gamma_\alpha)_{\alpha<\gamma}$ decreasing.

We call this the **Ressayre representation** of $x$. 
First attempt using the Ressayre representation:

\[ D(x) = D(r_0 e^{\gamma_0} + r_1 e^{\gamma_1} + \ldots) = r_0 e^{\gamma_0} D(\gamma_0) + r_1 e^{\gamma_1} D(\gamma_1) + \ldots \]

However, this is not inductive, in a very strong sense.

An \( x \in \mathbb{N}o \) is \textbf{log-atomic} if \( m_0 := x \in M \) and 
\( m_{i+1} := \log(m_i) \in M \) for all \( i \in \mathbb{N} \).

Let \( \mathbb{L} \) be their class \((\omega \in \mathbb{L}, \varepsilon_0 \in \mathbb{L}, \kappa_{\mathbb{N}o} \subseteq \mathbb{L} \ldots)\).

The formula is not informative if \( x = m_0 \) is log-atomic:

\[ D(m_0) = m_0 \cdot D(m_1) = m_0 \cdot m_1 \cdot D(m_2) = \ldots =? \]

And log-atomic numbers are rather frequent:

\textbf{Proposition} (Berarducci-M.). \( \mathbb{L} \) is the “class of levels” of \( \mathbb{N}o \).
The simplest pre-derivation $\partial_{\mathbb{L}}$

Start with a $D_{\mathbb{L}} : \mathbb{L} \rightarrow \mathbb{N}^0$. Axioms (1)-(5) imply

$$\log(D_{\mathbb{L}}(\lambda)) - \log(D_{\mathbb{L}}(\mu)) < \frac{1}{k} \max\{\lambda, \mu\} \text{ for all } \lambda, \mu \in \mathbb{L},\ k \in \mathbb{N}^0.$$

Call pre-derivation a function satisfying the above inequality.

**Proposition** (Berarducci-M.). The “simplest” pre-derivation is

$$\partial_{\mathbb{L}}(\lambda) := \exp \left( - \sum_{\alpha \in \text{On}} \sum_{\exists n : \exp_n(\kappa_{-\alpha}) > \lambda}^{\infty} \log_i(\kappa_{-\alpha}) + \sum_{i=1}^{\infty} \log_i(\lambda) \right).$$

“Simplest” refers to the simplicity relation $\leq_s$.

$\kappa_{-\alpha}$ are the $\kappa$-numbers of S. Kuhlmann and Matusinski.
Let us make inductive the formula

\[ D(x) = D\left(r_0 e^{\gamma_0} + r_1 e^{\gamma_1} + \ldots\right) = r_0 e^{\gamma_0} D(\gamma_0) + r_1 e^{\gamma_1} D(\gamma_1) + \ldots \]

**Proposition** (Berarducci-M.). No \( R : \text{No} \rightarrow \text{On} \) satisfies

1. \( R(x) = 0 \) if \( x \in \mathbb{L} \cup \mathbb{R} \);
2. otherwise, \( R(x) = R\left(\sum_{\gamma} r_{\gamma} e^{\gamma}\right) > R(\gamma) \) for \( r_{\gamma} \neq 0 \).

**Theorem** (Berarducci-M.). There is \( R : \text{No} \rightarrow \text{On} \) such that

1. \( R(x) = 0 \) if \( x \in \mathbb{L} \cup \mathbb{R} \);
2. \( R(x) = R\left(\sum_{\gamma} r_{\gamma} e^{\gamma}\right) \geq R(\gamma) \) for \( r_{\gamma} \neq 0 \), and if the equality holds then \( \gamma \) is *minimal* such that \( r_{\gamma} \neq 0 \) (and \( r_{\gamma} = \pm 1 \)).
Extending $\partial_{\mathbb{L}}$ to $\partial : \text{No} \to \text{No}$

1. if $x \in \mathbb{L}, \partial(x) := \partial_{\mathbb{L}}(x)$; if $x \in \mathbb{R}$, $\partial(x) := 0$.

2. $\partial_0(x) := \sum_{R(\gamma) < R(x)} r_\gamma e^{\gamma} \partial(\gamma)$.

3. if there is a (unique!) $\gamma$ such that $r_\gamma \neq 0$ and $R(\gamma) = R(x)$,

   1. $\Delta_0(x) := r_\gamma e^{\gamma} \partial_0(\gamma)$,
   2. $\Delta_{n+1}(x) := r_\gamma e^{\gamma} \Delta_n(\gamma)$.

   otherwise $\Delta_n(x) := 0$.

4. $\partial(x) := \partial_0(x) + \sum_n \Delta_n(x)$.

Using the inequalities of $\partial_{\mathbb{L}}$ and the properties of $R$:

**Theorem** (Berarducci-M.). $\partial$ is a surreal derivation.

**Proposition.** $\partial$ sends infinitesimals to infinitesimals.

Using Rosenlicht “asymptotic integration” and Fodor’s lemma:

**Theorem.** $\partial$ is surjective (every number has an anti-derivative).
No as a field of transseries

In the PhD thesis of Schmeling:

**T4.** For all sequences \( m_i \in \mathcal{M} \), with \( i \in \mathbb{N} \), such that
\[
m_i = \exp(\gamma_{i+1} + r_{i+1}m_{i+1} + \delta_{i+1})
\]
we have eventually \( r_{i+1} = \pm 1 \) and \( \delta_{i+1} = 0 \).

**Theorem** (Berarducci-M.). No satisfies T4, and therefore it is a field of transseries as defined by Schmeling. This is roughly van der Hoeven’s conjecture.

**Theorem** (Fornasiero). Every model of the theory of \( \mathbb{R}_{\text{an},\exp} \) embeds “initially” in No (hence the image is truncation-closed). Therefore, the models have a structure of (Schmeling) transseries.
Open questions

1. Complete van der Hoeven’s picture.
2. Relationship with $LE$, $EL$, ...
3. Differential equations solved in $(\mathbb{No}, \partial)$?
4. Pfaffian functions?
5. Elementary extension of $LE$?
6. Transexponential functions?
7. ...

Thanks for your attention