Lecture 3: Homotopical models of type theory

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What happened yesterday?

Type theory

- the type theory **T**
- Extensional vs intensional type theories

Homotopical algebra

- Weak factorisation systems and model structures
- Groupoids
- Simplicial sets

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Exercises

- $x, y : A, u : \mathrm{Id}_A(x, y) \vdash u^{-1} : \mathrm{Id}_A(y, x)$
- $x, y, z : A, u : \mathrm{Id}_A(x, y), v : \mathrm{Id}_A(y, z) \vdash v \circ u : \mathrm{Id}_A(x, z)$.

Problems with intensionality

The axioms for identity types do not seem to capture fully what we want.

Example

- we have

$$\mathsf{Id}_{A\times B}(\,c,d\,) \quad \longleftrightarrow \quad \mathsf{Id}_A\big(\,\pi_1(c),\pi_1(d)\,\big)\times \mathsf{Id}_B\big(\,\pi_2(c),\pi_2(d)\,\big)$$

- but only

$$\mathsf{Id}_{A\to B}(f,g) \quad \longrightarrow \quad (\Pi x:A)\,\mathsf{Id}_B(fx,gx)$$

Similar situation for Σ-types and Π-types.

What about the type universe?

Outline of Lecture 2

Part I: Models of type theory

Part II: Identity types

Part III: Π-types

Part IV: Universes

Part I: Models of type theory

Models of type theories

Question: What structure on a category **C** do we need to have a model of **T** ?

Idea: Look at the structure of the **syntactic category** of **T**.

- Objects: contexts

$$(x_1 : A_1)\,, \quad (x_1 : A_1, x_2 : A_2, \ldots)\,, \quad \ldots$$

- Maps: terms-in-context, e.g.

$$(a_1) : \Gamma \to (x_1 : A_1) \quad \text{if} \quad \Gamma \vdash a_1 : A_1$$

$$(a_1, a_2) : \Gamma \to (x_1 : A_1, x_2 : A_2) \quad \text{if} \quad \Gamma \vdash a_1 : A_1\,, \ \Gamma \vdash a_2 : A_2[a_1/x_1]$$

$$\ldots$$

Note.

$$\Gamma \vdash A : \text{type} \qquad \Longrightarrow \qquad \begin{array}{c} (\Gamma, x : A) \\ \downarrow{\scriptstyle p_A} \\ \Gamma \end{array}$$

Axiomatization

Fix

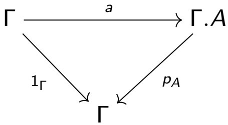
- category $\mathbf{C}$ with a terminal object 1
- a class $\mathbf{P} \subseteq \mathrm{Map}(\mathbf{C})$

Type Theory	**Syntactic category**	**Category Theory**
$A : \mathrm{type}$	$(x : A) \xrightarrow{p_A} (\,)$	$1.A \xrightarrow{p_A \in \mathbf{P}} 1$
$\Gamma \vdash A : \mathrm{type}$	$(\Gamma, x : A) \xrightarrow{p_A} \Gamma$	$\Gamma.A \xrightarrow{p_A \in \mathbf{P}} \Gamma$

Terms as sections

Type Theory | **Category Theory**

$\Gamma \vdash a : A$



$a : A$

$1 \xrightarrow{a} A$

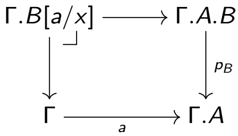
Substitution as pullback

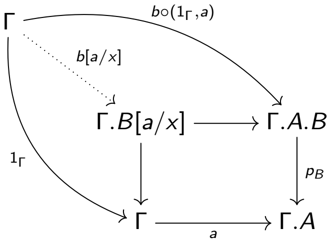
Type Theory

$$\frac{\Gamma \vdash a : A \qquad \Gamma, x : A \vdash B : \mathsf{type}}{\Gamma \vdash B[a/x] : \mathsf{type}}$$

$$\frac{\Gamma \vdash a : A \qquad \Gamma, x : A \vdash b : B}{\Gamma \vdash b[a/x] : B[a/x]}$$

Category Theory

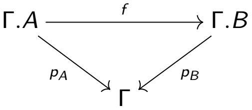




Weakening

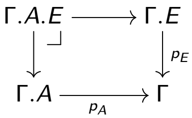
For $\Gamma \in \mathrm{Obj}(\mathbf{C})$, let $\mathbf{P}_{/\Gamma}$ be the category with

- objects: $\mathbf{P}$-maps $A : \Gamma.A \to \Gamma$
- maps: commutative triangles



$$\Gamma, x : A \vdash f(x) : B$$

Let $p_A : \Gamma.A \to \Gamma$ be in $\mathbf{P}$. Pullback



$$\frac{\Gamma \vdash E : \mathrm{type}}{\Gamma, x : A \vdash E : \mathrm{type}}$$

This is the ‘weakening functor’ $\Delta_A : \mathbf{P}_{/\Gamma} \to \mathbf{P}_{/\Gamma.A}$.

General setting

Let

- $\mathbf{C}$ be a category
- $\mathbf{P} \subseteq \mathrm{Map}(\mathbf{C})$

and assume

- $\mathbf{C}$ has a terminal object.
- The pullback of a $\mathbf{P}$-maps along any maps exists and is an $\mathbf{P}$-map.

Question

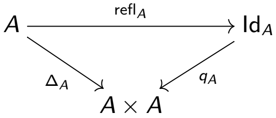
- What additional structure on $\mathbf{P}$ do we need to interpret type-formers?

Part II: Identity types

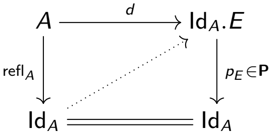
Identity types (I)

For simplicity, let us assume $\Gamma = 1$ and work with $p_A : 1.A \to 1$. We need

1. a **P**-map
$$q_A : \mathsf{Id}_A \to A \times A$$
2. a factorisation



3. a diagonal filler for every commutative diagram



Note. In fact, $\mathsf{refl}_A \pitchfork p$ for all $p \in \mathbf{P}$.

Identity types as path spaces

Idea

$$p : \mathrm{Id}_A(a, b) \quad \Longleftrightarrow \quad p \text{ is a path in } A \text{ from } a \text{ to } b$$

Note

- This explains several aspects of the behaviour of identity types

Theorem. The syntactic category of the type theory $\mathbf{T}$ admits a weak factorisation system $(\mathbf{L}, \mathbf{R})$, where

$$\mathbf{L} = \{i \mid (\forall p \in \mathbf{P})\ i \pitchfork p\} \qquad \mathbf{R} = \{p \mid (\forall i \in \mathbf{L})\ i \pitchfork p\}$$

where $\mathbf{P}$ is the set of projections $p_A : (\Gamma, x : A) \to \Gamma$.

Homotopical models of type theory

Idea

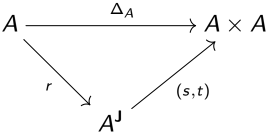
- Take $\mathbf{P} = \mathbf{Fib}$ in some model structure $(\mathbf{Weq}, \mathbf{Fib}, \mathbf{Cof})$.

We get a 'dictionary'

Type Theory	**Homotopical algebra**
$A : \text{type}$	fibrant object A
$x : A \vdash B(x) : \text{type}$	fibration $p : B \to A$
$x, y : A \vdash \mathsf{Id}_A(x, y)$	path space of A
$(\Pi x : A) B(x)$	space of sections of p
type universe U	a fibrant object U
$x \in \mathsf{U} \vdash \mathsf{El}(x) : \text{type}$	a generic fibration $\pi : \tilde{U} \to U$

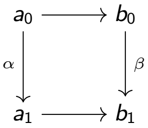
Example: Id-types in groupoids

Given by the (**Weq** $\cap$ **Cof**, **Fib**)-factorisation of diagonal



The groupoid $A^{\mathbf{J}}$ has

- objects: maps $\alpha : a_0 \to a_1$ in A
- maps: squares



Note Uniqueness of identity proofs fails in this model.

Warning. To define a model, one needs to take care of further aspects:

- mere existence vs structure
- coherence with respect to pullback (substitution)

Part III: Π-types

Π-types

Let $A : \Gamma.A \to \Gamma$ be in $\mathbf{P}$.

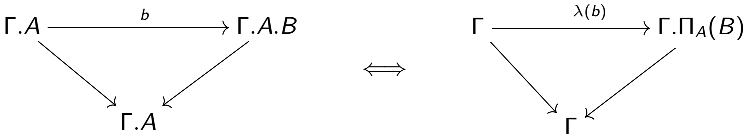
- Recall the 'weakening' functor

$$\Delta_A : \mathbf{P}_{/\Gamma} \to \mathbf{P}_{/\Gamma.A} \qquad \frac{\Gamma \vdash E : \text{type}}{\Gamma, x : A \vdash E : \text{type}}$$

- To interpret Π-types, we require a right adjoint

$$\Pi_A : \mathbf{P}_{/\Gamma.A} \to \mathbf{P}_{/\Gamma} \qquad \frac{\Gamma, x : A \vdash B : \text{type}}{\Gamma \vdash (\Pi x : A) B : \text{type}}$$

Idea



Π-types in groupoids

Theorem.

- For $p : \Gamma.A \to \Gamma$ an isofibration, the pullback functor

$$\Delta_p : \mathbf{Gpd}_{/\Gamma} \to \mathbf{Gpd}_{/\Gamma.A}$$

  has a right adjoint

$$\Pi_p : \mathbf{Gpd}_{/\Gamma.A} \to \mathbf{Gpd}_{/\Gamma}$$

- Furthermore, the right adjoint preserves isofibrations, and thus gives us

$$\begin{array}{cccc} \Pi_p : & \mathbf{Fib}_{/\Gamma.A} & \to & \mathbf{Fib}_{/\Gamma} \\ & \Gamma.A.B \xrightarrow{q} \Gamma.A & \mapsto & \Gamma.\Pi_A(B) \xrightarrow{\Pi_p(q)} \Gamma \end{array}$$

Example. When $\Gamma = 1$, the objects of the groupoid $\Pi_A(B)$ are

$$\begin{array}{ccc} A & \xrightarrow{\quad b \quad} & A.B \\ & {\scriptstyle 1_A} \searrow \quad \swarrow {\scriptstyle q} & \\ & A & \end{array}$$

i.e. sections of q.

Π-types in simplicial sets

Theorem.

- For any map $p: B \to A$, pullback along p has a right adjoint

$$\Pi_p : \mathbf{SSet}_{/B} \to \mathbf{SSet}_{/A}$$

- If p is a Kan fibration, then Π_p preserves Kan fibrations, and hence gives

$$\Pi_p : \mathbf{Fib}_{/B} \to \mathbf{Fib}_{/A}$$

Proof. By duality, it suffices to show that

$$\Delta_p : \mathbf{SSet}_{/A} \to \mathbf{SSet}_{/B}$$

preserves $(\mathbf{Weq} \cap \mathbf{Cof})$-maps. But

- **Cof**-maps are monomorphisms, so are always preserved.
- The preservation of weak equivalences by pullback along Kan fibrations is the so-called **right properness** of the model structure.

Note. Constructivity issues.

Part IV: Universes

Generic **P**-maps

We need a notion of 'smallness' for **P**-maps, e.g. fibers having cardinality $< \kappa$.

Then need a **P**-map

$$\pi : \tilde{U} \to U$$

that weakly classifies 'small' **P**-maps, i.e. for every such $p : B \to A$ there exists a pullback diagram

$$\begin{array}{ccc} B & \longrightarrow & \tilde{U} \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi} \\ A & \longrightarrow & U\,. \end{array}$$

Note. Given $a : 1 \to U$, we can form a pullback

$$\begin{array}{ccc} \mathrm{El}(a) & \longrightarrow & \tilde{U} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi} \\ 1 & \underset{a}{\longrightarrow} & U \end{array}$$

We think of a as the 'name' in U of the object $\mathrm{El}(a)$.

The type universe in groupoids and simplicial sets

Fix an inaccessible cardinal κ.

- In **Gpd**, it is not difficult to construct a universe. For example, one can consider the groupoid of all small (discrete) groupoids.

- In **SSet**, there exists a fibration

$$\pi : \tilde{U} \to U$$

that weakly classifies fibrations with fibers of cardinality $< \kappa$, i.e. for every such $p : B \to A$ there exists a pullback diagram

$$\begin{array}{ccc} B & \longrightarrow & \tilde{U} \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi} \\ A & \longrightarrow & U\,. \end{array}$$

Here,

$$U_n = \{\, p : B \to \Delta^n \mid p \text{ Kan fibration } \,\}$$

Problem. But U needs to be fibrant!

The type universe in simplicial sets

Theorem.

- The base U of the generic Kan fibration $\pi : \tilde{U} \to U$ is a Kan complex.

Proof. We need to show that U is a Kan complex. So show

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{\forall b} & U \\ {\scriptstyle h^n_k}\downarrow & \overset{\exists b'}{\nearrow} & \\ \Delta^n & & \end{array}$$

This reduces to the problem of extending fibrations along horn inclusions:

$$\begin{array}{ccc} B & \dashrightarrow & B' \\ {\scriptstyle p}\downarrow & & \vdots \\ \Lambda^n_k & \xrightarrow[h^n_k]{} & \Delta^n \end{array}$$

This can be done using the theory of **minimal fibrations** (AC).

Minimal fibrations extend along (**Weq** ∩ **Cof**)-maps

Lemma 1. Let

- $m : X \to A$ be a minimal fibration
- $i : A \to A'$ be a $(\mathbf{Weq} \cap \mathbf{Cof})$-map.

Then there exists $m' : M' \to A'$ a minimal fibration and

$$\begin{array}{ccc} X & \xrightarrow{j} & X' \\ {\scriptstyle m}\downarrow & \lrcorner & \downarrow{\scriptstyle m'} \\ A & \xrightarrow[i]{} & A' \end{array}$$

($\mathbf{Weq} \cap \mathbf{Fib}$)-maps can be extended along cofibrations

Lemma 2. Let

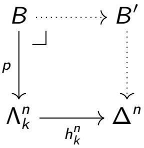
- $t : B \to X$ be a $(\mathbf{Weq} \cap \mathbf{Fib})$-map
- $j : X \to X'$ a cofibration

Then there exists $t' : E' \to X'$ a $(\mathbf{Weq} \cap \mathbf{Fib})$-map and

$$\begin{array}{ccc} B & \longrightarrow & B' \\ {\scriptstyle t}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle t'} \\ X & \underset{j}{\longrightarrow} & X' \end{array}$$

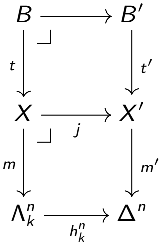
Proof of the theorem

Recall that we need to complete the diagram



It suffices to

- factor p as a (**Weq** $\cap$ **Fib**)-map t followed by a minimal fibration m
- apply Lemma 1 and Lemma 2 so as to get



Conclusions

The homotopical models of type theory suggest:

1. To use type theory as a language for speaking about spaces
2. To develop mathematics using this language; in particular, to define

   sets $=_{\mathrm{def}}$ **discrete spaces**

3. To add axioms to type theory motivated by homotopy theory