An introduction to CZF

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Seminário de Lógica Matemática XXIX #10
Lisbon, 5th Jan 2018

Constructive set theory

Aim

- To provide a foundation for constructive mathematics, retaining set-theoretic language.

Origin

- J. Myhill, Constructive Set Theory, JSL 1975

Note. It is closely related to other approaches to constructive foundations:

- Martin-Löf type theory
- Predicative versions of elementary toposes

Outline of the talk

1. The axiom system & basic facts
2. Mathematics in CZF
3. Proof-theoretic aspects

Some references

General

1. P. Aczel, The type-theoretic interpretation of constructive set theory, 1977 & 1982 & 1986.
2. P. Aczel and M. Rathjen, Notes on Constructive Set Theory, 2010.

Mathematics in CZF

3. P. Aczel, Aspects of general topology in Constructive Set Theory, 2006.

Proof theory

4. M. Rathjen, The strength of some Martin-Löf type theories, 1994.
5. R. Lubarsky, Independence Results around Constructive ZF, 2005.
6. N. Gambino, Heyting-valued interpretations of Constructive Set Theory, 2006.
7. A. Swan, CZF does not have the existence property, 2014.

Part I:

Axiom system and basic facts

CZF (Aczel 1977)

Language: standard set-theoretic language.

As usual, we define the restricted quantifiers by:

$$\forall x \in a\, \phi(x) =_{\mathrm{def}} \forall x(x \in a \to \phi(x))$$
$$\exists x \in a\, \phi(x) =_{\mathrm{def}} \exists x(x \in a \wedge \phi(x))$$

$\rightsquigarrow$ Notion of a restricted formula (cf. Δ_0-formula).

Logic: standard first-order intuitionistic logic.

Axioms: a variant of Myhill's CST.

Axioms of CZF (I)

Axioms:

1. Extensionality
2. Set Induction
3. Pairing
4. Union
5. Infinity
6. Restricted Separation:

$$\forall a \, \exists b \, \forall x \left[x \in b \leftrightarrow x \in a \wedge \phi(x) \right]$$

where $\phi(x)$ is a restricted formula.

The axioms of CZF (II)

7. Strong Collection:

$$\forall x \in a\, \exists y\, \phi(x, y) \rightarrow$$
$$\exists b \Big[\forall x \in a\, \exists y \in b\, \phi(x, y) \ \wedge \ \forall y \in b\, \exists x \in a\, \phi(x, y) \Big]$$

Here, $\phi(x, y)$ is any formula.

8. Subset Collection:

$$\exists c\, \forall z\, \Big[\forall x \in a\, \exists y \in b\, \phi(x, y, z) \rightarrow \exists u \in c$$
$$\forall x \in a\, \exists y \in u\, \phi(x, y, z) \ \wedge \ \forall y \in u\, \exists x \in a\, \phi(x, y, z) \Big]$$

Here, $\phi(x, y, z)$ is any formula.

More on Subset Collection

Let A, B be sets.

Definition. A **multi-valued function** from A to B is a relation $r \subseteq A \times B$ that is a set and such that

$$\forall x \in A \, \exists y \in B \; (x, y) \in r$$

Let $\mathbf{mv}(A, B)$ be the class of multi-valued functions from A to B.

Fullness axiom:

$$\exists C \big[C \subseteq \mathrm{mv}(A, B) \wedge \forall r \in \mathrm{mv}(A, B) \; \exists s \in C \;\; s \subseteq r \big]$$

Proposition. Over the axioms (1)-(7), Subset Collection is equivalent to Fullness.

Basic facts

Proposition.

- Power Set $\Rightarrow$ Subset Collection $\Rightarrow$ Exponentiation
- Strong Collection $\Rightarrow$ Collection $\Rightarrow$ Replacement

$$\mathsf{CST} \ \subset \ \mathsf{CZF} \ \subset \ \mathsf{IZF} \ \subset \ \mathsf{ZF}$$

Theorem. $\mathsf{CZF} + \mathsf{EM} = \mathsf{ZF}$.

Proof. Using classical logic, we have that

- Collection implies Separation,
- Exponentiation implies Power Set.

Extensions of CZF

CZF is a very flexible setting for exploring intuitionistic principles.

We can consider extensions with

- choice principles:

  AC, CC, DC, RDC, ...

- intuitionistic principles:

  Fan Theorem, Continuity Principles, ...

- axioms for inductive definitions:

  Regular Extension Axiom, ...

- Large set axioms

Part II:

Mathematics in CZF

Basic notions

The axioms of CZF give us

- basic sets:

$$\emptyset\,, \quad 1\,, \quad \ldots\,, \quad \mathbb{N}\,, \quad \ldots$$

- binary operations:

$$A \times B\,, \quad A \cup B\,, \quad B^A\,, \quad ,\ldots$$

- set-indexed operations:

$$\bigcup_{i \in I} X_i\,, \quad \bigcap_{i \in I} X_i\,, \quad \ldots$$

- quotients of set-sized equivalence relations: X/R

Working in CZF

We compensate lack of Full Separation and Power Set via **classes**:

$$\{x \mid \phi(x)\}$$

But we need to distinguish carefully between sets and classes.

Examples.

- For a set A, we have the class

$$\mathcal{P}(A) =_{\mathrm{def}} \{S \mid S \subseteq A\}$$

  which is not a set in general.

- For a set A and a formula $\phi(x)$, we have the class

$$\{\, x \in A \mid \phi(x) \,\}$$

  which may not be a set if $\phi(x)$ is not restricted.

The real numbers in CZF

Many possible approaches: Cauchy reals, Dedekind reals, ...

Definition. A **left cut** is a subset $X \subseteq \mathbb{Q}$ such that:

1. $X = \{y \in \mathbb{Q} \mid \exists x \in X\,(y < x)\}$
2. X is inhabited
3. X is bounded above, i.e. $\mathbb{Q} \setminus X$ is inhabited
4. X is located above, i.e.

$$\forall x, y \in \mathbb{Q}\left[x < y \to (x \in X \vee y \notin X)\right].$$

Theorem (Aczel). The class $\mathbb{R}$ of left cuts is a set in CZF.

Topology

Problem

- Several standard results about topological spaces cannot be proved constructively, e.g. Heine-Borel theorem.

Solution

- Work with point-free spaces, i.e. the algebras of open sets.

Standard approach: notion of a frame/locale

$$(X, \leq, \bigvee, \wedge, 0, 1)$$

Studied extensively in topos theory (cf. Johnstone's Stone Spaces).

This development uses Power Set.

Pointfree topology in CZF

Definition. A **set-generated frame** consists of

$$(X,\ \leq,\ \bigvee,\ \wedge,\ 0,\ 1,\ G)$$

where

- $(X, \leq, \bigvee, \wedge, 0, 1)$ is a class-sized frame,
- G is a set such that for all $a \in X$

$$a = \bigvee \{x \in G \mid x \leq a\}$$

Set-generated frames allow us to develop:

- constructive pointfree topology (cf. formal topology)
- Heyting-valued models (cf. sheaves)

Part III:

Proof-theoretical aspects

Relation to Martin-Löf type theories

Theorem (Aczel). CZF admits a model in a Martin-Löf type theory with

- standard basic types
- Σ-types, Π-types
- a type universe, U
- a single W-type

$$V =_{\mathrm{def}} (Wx : \mathsf{U})x$$

Idea. Sets-as-trees, formulas-as-types.

Note.

- Various choice principles are validated.
- The interpretation can be extended.
- Valid formulas can be characterized (Rathjen & Tupailo)

Proof-theoretical strength

Theorem (Rathjen).

1. $\mathsf{CZF} \equiv \mathsf{KP} \equiv \mathsf{ID}_1$
2. $\mathsf{CZF} + \mathsf{REA} \equiv \mathsf{KPi} \equiv \Delta^1_2$-comprehension $+ \mathsf{BI}$

Theorem. CZF + Power Set > Z.

Theorem (Lubarsky). CZF + Full Separation $\equiv \mathsf{PA}^2$.

Some independence results

Theorem (Lubarsky). On the basis of CZF without Subset Collection, we have that

1. Subset Collection does not imply Power Set,
2. Exponentiation does not imply Subset Collection,
3. Exponentiation does not imply that $\mathbb{R}$ is a set.

Disjunction and existence properties

Definition. We say that T has

- the Disjunction Property if

$$T \vdash \phi \vee \psi \quad \Rightarrow \quad T \vdash \phi \text{ or } T \vdash \psi$$

- the Numerical Existence Property if

$$T \vdash \exists x \in \omega \, \phi(x) \quad \Rightarrow \quad \text{There is } n \text{ s.t. } T \vdash \phi(n)$$

- the Existence Property if

$$T \vdash \exists x \phi(x) \quad \Rightarrow \quad \text{There is } \theta \text{ s.t. } T \vdash \exists! x(\theta(x) \wedge \phi(x))$$

Disjunction and existence properties for CZF

Beeson showed that IZF has DP and NEP.

Theorem (Rathjen). CZF has DP and NEP.

Friedman and Scedrov showed that IZF does not have EP.

Their proof uses crucially Full Separation.

Theorem (Swan). CZF does not have EP.

Further aspects

1. Category-theoretic models of constructive set theories (Awodey, van den Berg, Moerdijk, Palmgren, Warren)
2. Constructive pointfree topology in CZF (Curi)
3. A setting for constructive reverse mathematics?