Polynomial functors and polynomial monads

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Example

A **natural numbers object** in a category $\mathcal{C}$ consists of

- $(\,\mathbb{N}\,,\ 1 + \mathbb{N} \to \mathbb{N}\,)$

such that for all

- $(\,X\,,\ 1 + X \to X\,)$

there exists a unique $\theta : \mathbb{N} \to X$ such that

$$\begin{CD} 1 + \mathbb{N} @>{1+\theta}>> 1 + X \\ @VVV @VVV \\ \mathbb{N} @>>{\theta}> X \end{CD}$$

commutes.

The theory of polynomial functors

- Similar analysis for a wide class of inductively-defined sets
- Applications to free constructions

Outline

1. Background

- Endofunctors and their algebras
- Locally cartesian closed categories

2. Polynomial functors in a single variable

3. Polynomial functors in many variables

4. Free monads

Endofunctors and their algebras

Let $P : \mathcal{C} \to \mathcal{C}$ be an endofunctor.

The category P-Alg is defined as follows.

- **Objects:** $(X\,,\; PX \to X)$
- **Maps:**

$$\begin{array}{ccc} PX & \xrightarrow{P\theta} & PX' \\ \downarrow & & \downarrow \\ X & \xrightarrow[\theta]{} & X' \end{array}$$

Forgetful functor $U : P\text{-Alg} \to \mathcal{C}$.

An **initial algebra** for P is an initial object in P-Alg. Explicitly:

- $(W\,,\ PW \to W)$

such that for all

- $(X\,,\ PX \to X)$

there exists a unique $\theta : W \to X$ such that

$$\begin{array}{ccc} PW & \xrightarrow{P\theta} & PX \\ \downarrow & & \downarrow \\ W & \xrightarrow[\theta]{} & X \end{array}$$

commutes.

Lambek's Lemma. $PW \xrightarrow{\cong} W\,.$

Example

The object of natural numbers is the initial algebra for

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{C} \\ X & \longmapsto & 1 + X \end{array}$$

Idea: $$1 + X \cong \underbrace{X^0}_{\text{0-ary operation}} + \underbrace{X^1}_{\text{1-ary operation}}$$

Locally cartesian closed categories

Let $\mathcal{E}$ be a category.

For $A \in \mathcal{E}$, the **slice category** $\mathcal{E}/A$ is defined as follows.

- **Objects:** $(X\,,\ X \to A)$
- **Maps:**

$$\begin{array}{ccc} X & \longrightarrow & X' \\ & \searrow \quad \swarrow & \\ & A & \end{array}$$

Definition. We say that $\mathcal{E}$ is locally cartesian closed if

- $\mathcal{E}$ has finite limits
- $\mathcal{E}/A$ is a cartesian closed category for all $A \in \mathcal{E}$.

We also assume that $\mathcal{E}$ has finite disjoint coproducts.

Examples:

- **Set**
- Variants of **Top**
- $\mathrm{Psh}(C)$
- $\mathrm{Sh}(C, J)$
- Every elementary topos

The **internal language** of $\mathcal{E}$ is an extensional dependent type theory with rules for the following forms of type:

$$0\,, \quad 1\,, \quad \mathrm{Id}_A(a,b)\,, \quad A \times B\,, \quad B^A\,, \quad A + B\,,$$

$$\sum_{a \in A} B_a\,, \quad \prod_{a \in A} B_a$$

Idea. Identify $(X\,,\ X \to A)$ with $(X_a \mid a \in A)$.

Given $f : B \to A$, we can define three functors.

- **Reindexing**:

$$(\, X_a \mid a \in A \,) \mapsto \left(\, X_{f(b)} \mid b \in B \,\right)$$

- **Sum**:

$$(X_b \mid b \in B) \mapsto \left(\sum_{b \in B_a} X_b \mid a \in A \right)$$

- **Product**:

$$(X_b \mid b \in B) \mapsto \left(\prod_{b \in B_a} X_b \mid a \in A \right).$$

Polynomial functors in a single variable

Given $f : B \to A$, we define the **polynomial functor**

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{P_f} & \mathcal{E} \\ X & \longmapsto & \sum_{a \in A} X^{B_a} \end{array}$$

Idea. $(B_a \mid a \in A)$ as a signature.

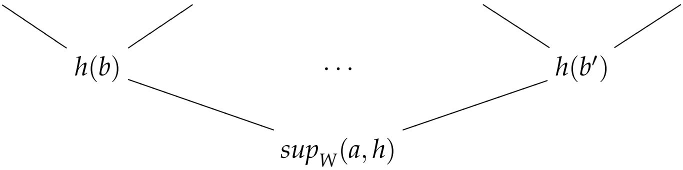
W-types

The initial algebra for $P_f : \mathcal{E} \to \mathcal{E}$

- $(W\,,\ sup_W : P_f(W) \to W\,)$

is called the **W-type** of $f : B \to A$.

For $a \in A$ and $h \in W^{B_a}$, we think of $sup_W(a, h) \in W$ as the tree



Examples of W-types

Binary trees

$$\begin{aligned} \mathcal{E} &\longrightarrow \mathcal{E} \\ X &\longmapsto 1 + X^2 \end{aligned}$$

Second number class

$$\begin{aligned} \mathcal{E} &\longrightarrow \mathcal{E} \\ X &\longmapsto 1 + X + X^{\mathbb{N}} \end{aligned}$$

List(A)

$$\begin{aligned} \mathcal{E} &\longrightarrow \mathcal{E} \\ X &\longmapsto 1 + A \times X \end{aligned}$$

Polynomial functors in many variables

Given

$$I \xleftarrow{\ \sigma\ } B \xrightarrow{\ f\ } A \xrightarrow{\ \tau\ } I$$

we define the **polynomial functor**

$$\mathcal{E}/I \xrightarrow{\quad P_f \quad} \mathcal{E}/I$$

$$\big(X_i \mid i \in I\big) \longmapsto \left(\textstyle\sum_{a \in A_i} \prod_{b \in B_a} X_{\sigma(b)} \mid i \in I \right)$$

Idea. $(B_a \mid a \in A)$ as an I-sorted signature

Examples

Polynomial functors in one variable

$$
\begin{array}{ccccc}
 & B & \xrightarrow{f} & A & \\
\swarrow & & & & \searrow \\
1 & & & & 1
\end{array}
$$

Linear functors

$$
\begin{array}{ccccc}
 & M & = & M & \\
{}^{\sigma}\swarrow & & & & \searrow^{\tau} \\
I & & & & I
\end{array}
$$

$$(X_i \mid i \in I) \longmapsto \Big(\sum_{m \in M_i} X_{\sigma(m)} \mid i \in I \Big)$$

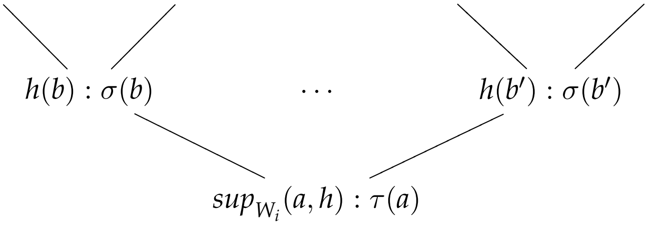
General tree types

The initial algebra for $P_f : \mathcal{E}/I \to \mathcal{E}/I$

- $$\left(\begin{array}{c} W \\ \downarrow \\ I \end{array} \quad , \quad \begin{array}{ccc} P_f(W) & \xrightarrow{sup_W} & W \\ & \searrow \quad \swarrow & \\ & I & \end{array} \right)$$

is called the **general tree type** associated to $f : B \to A$.

For $a \in A_i$ and $h \in \prod_{b \in B_a} W_{\sigma(b)}$, we think of $sup_{W_i}(a, h) \in W_{\tau(a)}$ as the tree



Note. $a \in A_i$ iff $\tau(a) = i$.

Examples of general trees

- $\big(\mathbb{N}_i \mid i \in 2 \big)$, where
$$\mathbb{N}_0 = \{n \in \mathbb{N} \mid n \text{ is even}\}, \quad \mathbb{N}_1 = \{n \in \mathbb{N} \mid n \text{ is odd}\}.$$
- $\big(\mathrm{List}_n(A) \mid n \in \mathbb{N}\big)$, where
$$\mathrm{List}_n(A) = \{t \in \mathrm{List}(A) \mid \mathrm{length}(t) = n\}.$$
- The free Grothendieck site generated by a coverage.

Theorem [G. & Hyland 2004]

Let $\mathcal{E}$ be a locally cartesian closed category with finite disjoint coproducts and W-types.

- Every polynomial functor $P : \mathcal{E}/I \to \mathcal{E}/I$ has an initial algebra.

Basic properties

1. Identity functors are polynomial
2. Composites of polynomial functors are polynomial
3. The functor

$$\begin{array}{ccc} \mathrm{Poly}(\mathcal{E}/I) & \longrightarrow & \mathcal{E}/I \\ P & \longmapsto & P(1) \end{array}$$

   is a Grothendieck fibration.

Free monads

Let $P : \mathcal{C} \to \mathcal{C}$ be an endofunctor.

We say that P admits a **free monad** if the forgetful functor

$$\begin{array}{c} P\text{-Alg} \\ \downarrow U \\ \mathcal{C} \end{array}$$

has a left adjoint $F : \mathcal{C} \to P\text{-Alg}$.

The monad (T, η, μ) resulting from $F \dashv U$ is called the free monad on P.

Theorem [G. and Kock 2009]

Let $\mathcal{E}$ be a locally cartesian closed category with finite disjoint coproducts and W-types.

1. Every polynomial functor $P : \mathcal{E}/I \to \mathcal{E}/I$ admits a free monad.
2. The free monad (T, η, μ) on a polynomial functor is a polynomial monad.

Proof of Part 1.

If $F : \mathcal{C} \to P$-Alg exists, it has to be

$$F(X) = \mu Y \,.\, X + PY \,.$$

But the endofunctor

$$\begin{array}{ccc} \mathcal{E}/I & \longrightarrow & \mathcal{E}/I \\ Y & \longmapsto & X + P(Y) \end{array}$$

is polynomial, since P is so.

Hence, it must have an initial algebra.

Sketch of the proof of Part 2.

We need to show that T is polynomial.

Let $P : \mathcal{E}/I \to \mathcal{E}/I$ be given by

$$\begin{array}{ccccc} & & B & \xrightarrow{f} & A & & \\ & \swarrow^{\sigma} & & & & \searrow^{\tau} & \\ I & & & & & & I \end{array}$$

Let us temporarily **assume** that $T : \mathcal{E}/I \to \mathcal{E}/I$ is given by

$$\begin{array}{ccccc} & & D & \xrightarrow{g} & C & & \\ & \swarrow^{\phi} & & & & \searrow^{\psi} & \\ I & & & & & & I \end{array}$$

We have

$$TX = \mu Y \,.\, X + P(Y)$$

Hence, by Lambek's Lemma, we must have

$$X + P(TX) \cong TX$$

Unfolding the definitions of P and T, we get equations. For example, we get

$$C_i \cong \{i\} + \sum_{a \in A_i} \prod_{b \in B_a} C_{\sigma(b)} \qquad (i \in I)$$

All of these equations can be solved via general tree types.

We also need to show that $\eta : \mathrm{Id} \Rightarrow T$ and $\mu : T^2 \Rightarrow T$ are cartesian. For this, use the following general fact.

Proposition. The following are equivalent.

1. $\phi : P_g \Rightarrow P_f$ cartesian natural transformation.

2. A diagram
$$\begin{array}{ccccccc} I & \longleftarrow & D & \xrightarrow{g} & C & \longrightarrow & I \\ \| & & \downarrow & \lrcorner & \downarrow & & \| \\ I & \longleftarrow & B & \xrightarrow[f]{} & A & \longrightarrow & I \end{array}$$

Further topics

- Polynomial functors $P : \mathcal{E}/I \to \mathcal{E}/J$
- The double category of polynomial functors
- Base change
- Relationship to operads and multicategories

Reference

- N. Gambino and J. Kock
  Polynomial functors and polynomial monads
  ArXiv, 2009