On the bicategory of operads and analytic functors

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Reference

N. Gambino and A. Joyal
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Main result

Let $\mathcal{V}$ be a symmetric monoidal closed presentable category.

Theorem. The bicategory $\mathbf{Opd}_{\mathcal{V}}$ that has

- 0-cells = operads (= symmetric many-sorted $\mathcal{V}$-operads)
- 1-cells = operad bimodules
- 2-cells = operad bimodule maps

is cartesian closed.

Note. For operads A and B, we have

$$\mathbf{Alg}(A \sqcap B) = \mathbf{Alg}(A) \times \mathbf{Alg}(B)\,, \quad \mathbf{Alg}(B^A) = \mathbf{Opd}_{\mathcal{V}}[A, B]\,.$$

Plan of the talk

1. Symmetric sequences and operads

2. Bicategories of bimodules

3. A universal property of the bimodule construction

4. Proof of the main theorem

1. Symmetric sequences and operads

Single-sorted symmetric sequences

Let $\mathbf{S}$ be the category of finite cardinals and permutations.

Definition. A **single-sorted symmetric sequence** is a functor

$$\begin{aligned} F \colon \quad \mathbf{S} &\to \quad \mathcal{V} \\ n &\mapsto \quad F[n] \end{aligned}$$

For $F \colon \mathbf{S} \to \mathcal{V}$, we define the **single-sorted analytic functor**

$$F^\sharp \colon \mathcal{V} \to \mathcal{V}$$

by letting

$$F^\sharp(T) = \sum_{n \in \mathbb{N}} F[n] \otimes_{\Sigma_n} T^n \, .$$

Single-sorted operads

Recall that:

1. The functor category $[\mathbf{S}, \mathcal{V}]$ admits a monoidal structure such that

$$\begin{aligned} (G \circ F)^\sharp &\cong G^\sharp \circ F^\sharp\,, \\ I^\sharp &\cong \mathrm{Id}_{\mathcal{V}} \end{aligned}$$

2. Monoids in $[\mathbf{S}, \mathcal{V}]$ are exactly single-sorted operads.

See [Kelly 1972] and [Joyal 1984].

Symmetric sequences

For a set X, let $S(X)$ be the category with

- objects: $(x_1, \dots, x_n)$, where $n \in \mathbb{N}$, $x_i \in X$ for $1 \leq i \leq n$
- morphisms: $\sigma \colon (x_1, \dots, x_n) \to (x'_1, \dots, x'_n)$ is $\sigma \in \Sigma_n$ such that $x'_i = x_{\sigma(i)}$.

Definition. Let X and Y be sets. A **symmetric sequence** indexed by X and Y is a functor

$$\begin{array}{rccc} F\colon & S(X)^{\mathrm{op}} \times Y & \to & \mathcal{V} \\ & (x_1, \dots, x_n, y) & \mapsto & F[x_1, \dots, x_n; y] \end{array}$$

Note. For $X = Y = 1$ we get single-sorted symmetric sequences.

Analytic functors

Let $F\colon S(X)^{\mathrm{op}} \times Y \to \mathcal{V}$ be a symmetric sequence.

We define the **analytic functor**

$$F^\sharp\colon \mathcal{V}^X \to \mathcal{V}^Y$$

by letting

$$F^\sharp(T, y) = \int^{(x_1,\ldots,x_n)\in S(X)} F[x_1, \ldots, x_n; y] \otimes T(x_1) \otimes \ldots T(x_n)$$

for $T \in \mathcal{V}^X$, $y \in Y$.

Note. For $X = Y = 1$, we get single-sorted analytic functors.

The bicategory of symmetric sequences

The bicategory $\mathbf{Sym}_{\mathcal{V}}$ has

- 0-cells = sets
- 1-cells = symmetric sequences, i.e. $F \colon S(X)^{\mathrm{op}} \times Y \to \mathcal{V}$
- 2-cells = natural transformations.

Note. Composition and identities in $\mathbf{Sym}_{\mathcal{V}}$ are defined so that

$$(G \circ F)^\sharp \cong G^\sharp \circ F^\sharp$$

$$(\mathrm{Id}_X)^\sharp \cong \mathrm{Id}_{\mathcal{V}^X}$$

Monads in a bicategory

Let $\mathcal{E}$ be a bicategory.

Recall that a **monad** on $X \in \mathcal{E}$ consists of

- $A : X \to X$
- $\mu : A \circ A \Rightarrow A$
- $\eta : 1_X \Rightarrow A$

subject to associativity and unit axioms.

Examples.

- monads in $\mathbf{Ab}$ = monoids in $\mathbf{Ab}$ = commutative rings
- monads in $\mathbf{Mat}_{\mathcal{V}}$ = small $\mathcal{V}$-categories
- monads in $\mathbf{Sym}_{\mathcal{V}}$ = (symmetric, many-sorted) $\mathcal{V}$-operads

An analogy

$\mathbf{Mat}_{\mathcal{V}}$	$\mathbf{Sym}_{\mathcal{V}}$
Matrix $F\colon X \times Y \to \mathcal{V}$	Symmetric sequence $F\colon S(X)^{\mathrm{op}} \times Y \to \mathcal{V}$
Linear functor	Analytic functor
Category	Operad
Bimodule/profunctor/distributor	Operad bimodule

Categorical symmetric sequences

The bicategory $\mathbf{CatSym}_{\mathcal{V}}$ has

- 0-cells = small $\mathcal{V}$-categories
- 1-cells = $\mathcal{V}$-functors

  $$F \colon S(\mathbb{X})^{\mathrm{op}} \otimes \mathbb{Y} \to \mathcal{V}\,,$$

  where $S(\mathbb{X})$ = free symmetric monoidal $\mathcal{V}$-category on $\mathbb{X}$.
- 2-cells = $\mathcal{V}$-natural transformations

Note. We have $\mathbf{Sym}_{\mathcal{V}} \subseteq \mathbf{CatSym}_{\mathcal{V}}\,.$

Theorem 1. The bicategory $\mathbf{CatSym}_{\mathcal{V}}$ is cartesian closed.

Proof. Enriched version of main result in [FGHW 2008].

- Products:

$$\mathbb{X} \sqcap \mathbb{Y} =_{\text{def}} \mathbb{X} \sqcup \mathbb{Y} \,,$$

- Exponentials:

$$[\mathbb{X}, \mathbb{Y}] =_{\text{def}} S(\mathbb{X})^{\text{op}} \otimes \mathbb{Y} \,.$$

2. Bicategories of bimodules

Bimodules

Let $\mathcal{E}$ be a bicategory.

Let $A\colon X \to X$ and $B\colon Y \to Y$ be monads in $\mathcal{E}$.

Definition. A **(B, A)-bimodule** consists of

- $M : X \to Y$
- a left B-action $\lambda : B \circ M \Rightarrow M$
- a right A-action $\rho : M \circ A \Rightarrow M$.

subject to a commutation condition.

Examples.

- bimodules in $\mathbf{Ab}$ = ring bimodules
- bimodules in $\mathbf{Mat}_{\mathcal{V}}$ = bimodules/profunctors/distributors
- bimodules in $\mathbf{Sym}_{\mathcal{V}}$ = operad bimodules

Bicategories with local reflexive coequalizers

Definition.

We say that a bicategory $\mathcal{E}$ has **local reflexive coequalizers** if

(i) the hom-categories $\mathcal{E}[X, Y]$ have reflexive coequalizers,

(ii) the composition functors preserve reflexive coequalizers in each variable.

Examples.

- $(\mathbf{Ab}, \otimes, \mathbb{Z})$
- $\mathbf{Mat}_{\mathcal{V}}$
- $\mathbf{Sym}_{\mathcal{V}}$ and $\mathbf{CatSym}_{\mathcal{V}}$

The bicategory of bimodules

The bicategory $\mathrm{Bim}(\mathcal{E})$ has

- 0-cells = (X, A), where $X \in \mathcal{E}$ and $A\colon X \to X$ monad
- 1-cells = bimodules
- 2-cells = bimodule morphisms

Composition: for $M : (X, A) \to (Y, B)\,,\ N : (Y, B) \to (Z, C)$,

$$N \circ_B M : (X, A) \to (Z, C)$$

is given by

$$N \circ B \circ M \underset{\rho \circ M}{\overset{N \circ \lambda}{\rightrightarrows}} N \circ M \longrightarrow N \circ_B M\,.$$

Identities: $\mathrm{Id}_{(X,A)}\colon (X, A) \to (X, A)$ is $A\colon X \to X$.

Examples

1. The bicategory of ring bimodules

$$\mathrm{Bim}(\mathbf{Ab})$$

   - 0-cells = rings
   - 1-cells = ring bimodules
   - 2-cells = bimodule maps

2. The bicategory of bimodules/profunctors/distributors

$$\mathrm{Bim}(\mathbf{Mat}_{\mathcal{V}})$$

   - 0-cells = small $\mathcal{V}$-categories
   - 1-cells = $\mathcal{V}$-functors $\mathbb{X}^{\mathrm{op}} \otimes \mathbb{Y} \to \mathcal{V}$
   - 2-cells = $\mathcal{V}$-natural transformations.

3. The bicategory of operads

$$\mathbf{Opd}_{\mathcal{V}} =_{\text{def}} \text{Bim}(\mathbf{Sym}_{\mathcal{V}})$$

- 0-cells = $\mathcal{V}$-operads
- 1-cells = operad bimodules
- 2-cells = operad bimodule maps.

Note. The composition operation of $\mathbf{Opd}_{\mathcal{V}}$ obtained in this way generalizes Rezk's circle-over construction.

Remark. For an operad bimodule $F\colon (X, A) \to (Y, B)$, we define the analytic functor

$$\begin{array}{rccc} F^\sharp\colon & \mathbf{Alg}(A) & \to & \mathbf{Alg}(B) \\ & M & \mapsto & F \circ_A M \end{array}$$

These include restriction and extension functors.

Cartesian closed bicategories of bimodules

Theorem 2. Let $\mathcal{E}$ be a bicategory with local reflexive coequalizers. If $\mathcal{E}$ is cartesian closed, then so is $\mathrm{Bim}(\mathcal{E})$.

Idea.

- Products

$$(X, A) \times (Y, B) = (X \times Y, A \times B)$$

- Exponentials

$$\big[(X, A), (Y, B)\big] = \big([X, Y], [A, B]\big)$$

Note. The proof uses a homomorphism

$$\mathrm{Mnd}(\mathcal{E}) \to \mathrm{Bim}(\mathcal{E})\,,$$

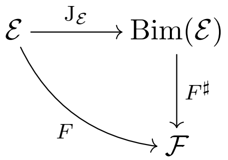
where $\mathrm{Mnd}(\mathcal{E})$ is Street's bicategory of monads.

3. A universal property of the bimodule construction

Eilenberg-Moore completions

Let $\mathcal{E}$ be a bicategory with local reflexive coequalizers.

The bicategory $\mathrm{Bim}(\mathcal{E})$ is the Eilenberg-Moore completion of $\mathcal{E}$ as a bicategory with local reflexive coequalizers:



Note.

- This was proved independently by Garner and Shulman, extending work of Carboni, Kasangian and Walters.
- Different universal property from the Eilenberg-Moore completion studied by Lack and Street.

Theorem 3. The inclusion

$$\mathrm{Bim}(\mathbf{Sym}_{\mathcal{V}}) \subseteq \mathrm{Bim}(\mathbf{CatSym}_{\mathcal{V}})$$

is an equivalence.

Idea. Every 0-cell of $\mathbf{CatSym}_{\mathcal{V}}$ is an Eilenberg-Moore object for a monad in $\mathbf{Sym}_{\mathcal{V}}$.

4. Proof of the main theorem

Theorem. The bicategory $\mathbf{Opd}_{\mathcal{V}}$ is cartesian closed.

Proof. Recall

$$\begin{array}{ccc} \mathbf{Sym}_{\mathcal{V}} & \rightarrowtail & \mathbf{CatSym}_{\mathcal{V}} \\ \downarrow & & \downarrow \\ \mathbf{Opd}_{\mathcal{V}} & \rightarrowtail & \mathrm{Bim}(\mathbf{CatSym}_{\mathcal{V}}) \end{array}$$

Theorem 1 says that $\mathbf{CatSym}_{\mathcal{V}}$ is cartesian closed.

So, by Theorem 2, $\mathrm{Bim}(\mathbf{CatSym}_{\mathcal{V}})$ is cartesian closed.

But, Theorem 3 says

$$\mathbf{Opd}_{\mathcal{V}} = \mathrm{Bim}(\mathbf{Sym}_{\mathcal{V}}) \simeq \mathrm{Bim}(\mathbf{CatSym}_{\mathcal{V}})\,.$$