Algebraic models of homotopy type theory

Nicola Gambino

School of Mathematics

University of Leeds

CT2016

Halifax, August 9th

Theme: property vs structure

Fundamental distinction:

- satisfaction of a property
- the existence of additional structure.

Examples:

- categories with finite products
- fibrations.

Sometimes ignoring this distinction is not harmful.

But sometimes things become more subtle:

- choices are unique up to higher and higher homotopies
- coherence issues
- constructivity issues.

Outline

Part I: Context and motivation

- Homotopy type theory
- The simplicial model

Part II: Uniform fibrations

- Algebraic weak factorization systems
- Uniform (trivial) fibrations

Part III: The Frobenius property

- Strong homotopy equivalences

Reference

N. Gambino and C. Sattler
Uniform fibrations and the Frobenius condition
arXiv:1510.00669, 2016.

Part I: Context and motivation

Homotopy type theory

Martin-Löf's type theory (**ML**) + Voevodsky's univalence axiom (**UA**)

Key aspects

- Isomorphic structures can be identified
- Concise definition of weak n-groupoid

Note

- Does not admit set-theoretic models
- Known models are homotopy-theoretic

Motivation from mathematical logic

Relative consistency problem

- Assume that **ML** is consistent. Is **ML** + **UA** consistent?

Line of attack

- Define a model of **ML** + **UA** working within **ML**

Issue

- Simplicial model of **ML** + **UA** is defined working in **ZFC**
- Can we redevelop it constructively?

Motivation from homotopy theory

Problem

- Can we develop homotopy theory without minimal fibrations?

Minimal fibrations are not very explicit, and often just tools.

Examples

(1) Existence of model structure for Kan complexes

(2) Right properness

Recent progress

(1) Proof by Christian Sattler

(2) Today

Categorical setting

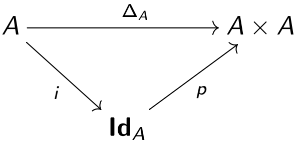
Definition. A *model* of homotopy type theory consists of

- a category $\mathcal{E}$ with a terminal object 1,
- a class **Fib** of maps, called *fibrations*, subject to axioms.

Some axioms

(1) Pullbacks of fibrations exist and are fibrations, i.e. for every map $f : B \to A$, we have $f^* : \mathbf{Fib}/A \to \mathbf{Fib}/B$

(2) For every fibrant object A, we have a factorization



where $p \in \mathbf{Fib}$ and $i \in {}^{\pitchfork}\mathbf{Fib}$.

(3) If $f : B \to A$ is a fibration, $f^* : \mathbf{Fib}/A \to \mathbf{Fib}/B$ has a right adjoint.

Examples from homotopy theory

Fix $\mathcal{E}$ with a model structure $(\mathbf{Weq}, \mathbf{Fib}, \mathbf{Cof})$.

Let $\mathbf{TrivFib} = \mathbf{Weq} \cap \mathbf{Fib}$, $\mathbf{TrivCof} = \mathbf{Weq} \cap \mathbf{Cof}$.

Is $(\mathcal{E}, \mathbf{Fib})$ a model of homotopy type theory?

Checking the axioms:

(1) ✓ (pullbacks exist and preserve fibrations)

(2) ✓ (given by factorization axiom)

(3) It would suffice to show that for a fibration $p : B \to A$, we have

$$\mathcal{E}/A \underset{p_*}{\overset{p^*}{\rightleftarrows}} \mathcal{E}/B \quad (p^* \dashv p_*)$$

where the right adjoint p_* (pushforward) preserves fibrations.

But we do not know this in general!

The Frobenius property

Remark. *Let $p : B \to A$ be a fibration. TFAE:*

(i) *pushforward $p_* : \mathcal{E}/B \to \mathcal{E}/A$ preserves fibrations*

(ii) *pullback $p^* : \mathcal{E}/A \to \mathcal{E}/B$ preserves trivial cofibrations.*

Note. The second statement refers only to $(\mathbf{TrivCof}, \mathbf{Fib})$.

Definition. We say that a wfs $(\mathbf{L}, \mathbf{R})$ has the *Frobenius property* if pullback along **R**-maps preserves **L**-maps.

Remark. *Assume that* $\mathbf{Cof} = \{$*monomorphisms*$\}$*. TFAE:*

(i) *The wfs* $(\mathbf{TrivCof}, \mathbf{Fib})$ *has the Frobenius property.*

(ii) *The model structure is right proper, i.e. pullback of weak equivalences along fibrations are weak equivalences.*

Example: simplicial sets

Let **SSet** be the category of simplicial sets.

We consider the model structure:

- **Weq** $=$ {weak homotopy equivalences}
- **Fib** $=$ {Kan fibrations}
- **Cof** $=$ {monomorphisms}

Proofs of right properness:

- via geometric realization (see Hovey, Hirschhorn)
- via minimal fibrations (Joyal and Tierney)

Constructivity issues

Theorem (Bezem, Coquand, Parmann)**.**

The right properness of **SSet** *cannot be proved constructively, i.e. without the law of excluded middle.*

How can we fix this?

Coquand's approach

- Switch from simplicial sets to cubical sets
- Work with uniform fibrations

Idea

- Work with uniform fibrations
- Keep simplicial sets

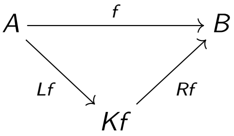
As a byproduct, we get a new proof of right properness of **SSet**.

Part II: Uniform fibrations

Algebraic weak factorization systems

Recall that in a weak factorization system $(\mathbf{L}, \mathbf{R})$, we often ask for

- *functorial factorizations*, i.e. functors such that



  gives the required factorization.

In an *algebraic weak factorization system*, we ask also that

- L has the structure of a comonad,
- R has the structure of a monad,
- a distributive law between L and R.

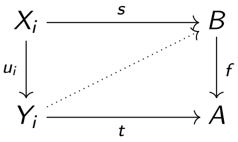
Grandis and Tholen (2006), Garner (2009).

The algebraic setting

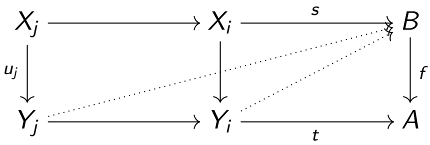
Fix a category $\mathcal{E}$. Let $u : \mathcal{I} \to \mathcal{E}^{\to}$ be a functor.

Definition. A *right $\mathcal{I}$-map* is a map $f : B \to A$ in $\mathcal{E}$ equipped with

- a function ϕ which assigns a diagonal filler



where $i \in \mathcal{I}$, subject to a uniformity condition:



$\mathcal{I}^{\pitchfork}$ = category of right $\mathcal{I}$-maps.

Goals

Working in **SSet**, we want:

1. To construct an algebraic weak factorization system $(\mathbf{TrivCof}, \mathbf{Fib})$
2. To show that $(\mathbf{TrivCof}, \mathbf{Fib})$ has the Frobenius property.

Preliminary step:

- To construct an algebraic weak factorization system $(\mathbf{Cof}, \mathbf{TrivFib})$

The approach is inspired by Cisinski's theory.

Uniform trivial fibrations

Let **Mono** be the sub-category of $\mathbf{SSet}^{\rightarrow}$ of

- monomorphisms
- pullback squares

Write $u : \mathbf{Mono} \rightarrow \mathbf{SSet}^{\rightarrow}$ for the inclusion.

Definition. A *uniform trivial fibration* is a right **Mono**-map, i.e. a map $f : B \rightarrow A$ in **SSet** together with a function which assigns fillers

$$\begin{array}{ccc} X & \xrightarrow{\;s\;} & B \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[\;t\;]{} & A \end{array}$$

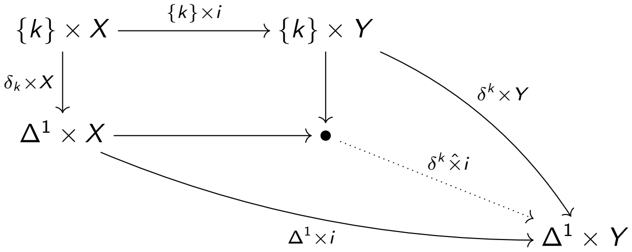
where $i : X \rightarrow Y$ is a monomorphism, subject to uniformity.

$$\mathbf{TrivFib} = \mathbf{Mono}^{\pitchfork} = \text{category of uniform trivial fibrations.}$$

Uniform fibrations (I)

Let $\delta^k : \{k\} \to \Delta^1$ be the endpoint inclusions, $k \in \{0, 1\}$.

For a monomorphism $i : X \to Y$, we have the **pushout product**



A subcategory **Cyl** $\subseteq$ **SSet**$^{\to}$ with objects the "cylinder inclusions"

$$\delta^k \hat{\times} i \; : \underbrace{(\Delta^1 \times X) \cup (\{k\} \times Y)}_{\bullet} \to \Delta^1 \times Y$$

Uniform fibrations (II)

Definition. A *uniform fibration* is a right **Cyl**-map, i.e. a map $p\colon B \to A$ in **SSet** together with a function which assigns fillers

$$\begin{array}{ccc}
(\Delta^1 \times X) \cup (\{k\} \times Y) & \longrightarrow & B \\
{\scriptstyle \delta^k \hat{\times} i}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\
\Delta^1 \times Y & \longrightarrow & A
\end{array}$$

where $k \in \{0,1\}$, $i\colon X \to Y$ is a monomorphism, subject to uniformity.

$$\mathbf{Fib} = \mathbf{Cyl}^{\pitchfork} = \text{category of uniform fibrations.}$$

Theorem (ZFC)**.** *A map is a (trivial) fibration in the usual sense if and only if it can be equipped with the structure of a uniform (trivial) fibration.*

The algebraic weak factorization systems

Theorem.

The category **SSet** *admits two algebraic weak factorization systems:*

1. $(\mathbf{Cof}, \mathbf{TrivFib})$
2. $(\mathbf{TrivCof}, \mathbf{Fib})$.

Proof.

- Use Garner's algebraic small object argument
- For this, isolate a small category $\mathcal{I}$ such that

$$\mathcal{I}^{\pitchfork} = \mathbf{Mono}^{\pitchfork} \quad (= \mathbf{TrivFib})$$

- E.g. $\mathcal{I} = \{\text{monomorphisms with representable codomain}\}$.

Note. Algebraic aspect is essential to work constructively.

Part III: The Frobenius property

Plan

For simplicity, we work in the non-algebraic setting.

We have:

- a weak factorization system $(\mathbf{TrivCof}, \mathbf{Fib})$
- a class of maps $\mathbf{Cyl}$ such that $\mathbf{Cyl}^{\pitchfork} = \mathbf{Fib}$

To show:

- for $p : B \to A$ in $\mathbf{Fib}$, pullback

  $$p^* : \mathbf{SSet}/A \to \mathbf{SSet}/B$$

  preserves trivial cofibrations, i.e. for all

  $$\begin{array}{ccc} Y & \longrightarrow & X \\ {\scriptstyle g}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle f} \\ B & \underset{p}{\longrightarrow} & A \end{array}$$

  we have $f \in \mathbf{TrivCof} \Rightarrow g \in \mathbf{TrivCof}$.

Outline of the proof

Define the class **SHeq** of strong homotopy equivalences

Step 1

- characterize strong homotopy equivalences as retracts

Step 2

- Show $\mathbf{SHeq} \cap \mathbf{Mono} \subseteq \overline{\mathbf{Cyl}}$
- Show $\mathbf{Cyl} \subseteq \mathbf{SHeq} \cap \mathbf{Mono}$

Step 3

- Prove the Frobenius property for $\mathbf{SHeq} \cap \mathbf{Mono}$

Strong homotopy equivalences

Definition. A map $f : X \to A$ is a *strong homotopy equivalence* if there exist

$$g : A \to X\,, \qquad \phi : 1_X \sim g \circ f \quad \psi : 1_A \sim f \circ g$$

such that

$$\begin{array}{ccc} \Delta^1 \times X & \xrightarrow{\Delta^1 \times f} & \Delta^1 \times A \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi} \\ X & \xrightarrow[f]{} & A \end{array}$$

Example. The endpoint inclusions $\delta^k : \{k\} \to \Delta^1$.

Step 1: a characterisation

Lemma. *A map $f : X \to A$ is a strong homotopy equivalence if and only if the canonical square*

$$
\begin{array}{ccccc}
X & \xrightarrow{\delta^0 \times X} & (\Delta^1 \times X) & \xrightarrow{\iota_1} & (\Delta^1 \times X) +_X A \\
{\scriptstyle f}\downarrow & & & & \downarrow{\scriptstyle \delta^1 \hat{\times} f} \\
A & & \xrightarrow{\delta^0 \times A} & & \Delta^1 \times A
\end{array}
$$

exhibits f as a retract of $\delta^1 \hat{\times} f$, i.e. we have

$$
\begin{array}{ccccc}
X & \longrightarrow & (\Delta^1 \times X) +_X A & \xrightarrow{s} & X \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \delta^1 \hat{\times} f} & & \downarrow{\scriptstyle f} \\
A & \xrightarrow{\delta^0 \times A} & \Delta^1 \times A & \xrightarrow{t} & A
\end{array}
$$

where the horizontal composites are identities.

Step 2

Lemma. *We have*

(i) $\mathbf{SHeq} \cap \mathbf{Mono} \subseteq \overline{\mathbf{Cyl}}$

(ii) $\mathbf{Cyl} \subseteq \mathbf{SHeq} \cap \mathbf{Mono}$

Proof.

(i) Let $f \in \mathbf{SHeq} \cap \mathbf{Mono}$.

Since $f \in \mathbf{SHeq}$, by Step 1, we have that f is a retract of $\delta^1 \hat{\times} f$.

Since $f \in \mathbf{Mono}$, we have $\delta^1 \hat{\times} f \in \mathbf{Cyl}$.

(ii) Each $\delta^1 \hat{\times} f \in \mathbf{Cyl}$ is both in $\mathbf{SHeq}$ and in $\mathbf{Mono}$.

Step 3

Theorem. *The weak factorization system* $(\mathbf{TrivCof}, \mathbf{Fib})$ *has the Frobenius property.*

Proof. We need to show that for every pullback

$$\begin{array}{ccc} Y & \longrightarrow & X \\ {\scriptstyle g}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle f} \\ B & \xrightarrow[p]{} & A \end{array}$$

where $p \in \mathbf{Fib}$, we have

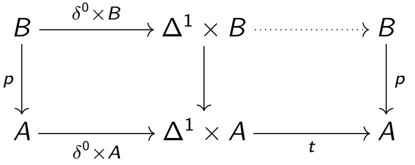
$$f \in \mathbf{TrivCof} \Rightarrow g \in \mathbf{TrivCof}$$

But by Step 2, it suffices to show

$$f \in \mathbf{SHeq} \cap \mathbf{Mono} \Rightarrow g \in \mathbf{SHeq} \cap \mathbf{Mono}$$

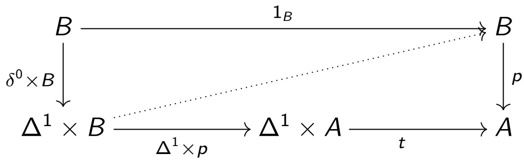
End of the proof

So let $f \in \mathbf{SHeq} \cap \mathbf{Mono}$. To show: $g \in \mathbf{SHeq} \cap \mathbf{Mono}$.
By Step 1 and some diagram-chasing, we need



Here t is part of the data making f into a retract of $\delta^1 \hat{\times} f$.

Such a map is given by a diagonal filler:



This exists since $\delta^0 \times B \in \mathbf{Cyl}$ and $p \in \mathbf{Fib}$. □

Remarks

Corollary. The model structure $(\mathbf{Weq}, \mathbf{Fib}, \mathbf{Cof})$ is right proper.

The argument in the algebraic setting is slighly more involved.

Everything generalizes from **SSet** to a presheaf category $\mathcal{E}$ with

- a functorial cylinder with contractions and connections
- a class of monomorphisms $\mathcal{M}$ satisfying some basic assumptions:
  1. Closed under pullback
  2. Closed under pushout product with endpoint inclusions
  3. All maps $0 \to X$ are in $\mathcal{M}$
  4. ...

Example

- uniform fibrations in cubical sets (Bezem, Coquand, Huber)

Conclusion

- Two algebraic weak factorization systems on **SSet**:
  1. (**Cof**, **TrivFib**)
  2. (**TrivCof**, **Fib**)

  such that (**TrivCof**, **Fib**) satisfies the Frobenius property.
- Hence, pushforward along fibrations preserves fibrations.
- We obtain new proof of right properness.

To get a model of the univalence axiom, one needs to construct a suitable universal fibration.

Again, there are constructivity issues in simplicial sets.

But, at least for cubical sets, it is possible to give an abstract account of the work of Coquand et al. (Sattler)