Lecture 2: Homotopical Algebra

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Homotopical algebra

Motivation

- Axiomatic development of homotopy theory
- Addressing size issues in localizations

Key notion

- Model category

Outline

Part I: Model categories

Part II: Groupoids

Part III: Simplicial sets

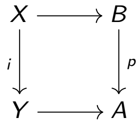
Part I: Model categories

Lifting problems

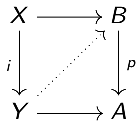
Fix a category **C**.

Definition. Let $p : B \to A$ and $i : X \to Y$.

- We say p has the **right lifting property** w.r.t. i if for every diagram



there exists a diagonal filler



Notation: $i \pitchfork p$

Examples

- In **Set**, if i injective and p surjective then $i \pitchfork p$

$$\begin{array}{ccc} X & \longrightarrow & B \\ {\scriptstyle i}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ Y & \longrightarrow & A \end{array}$$

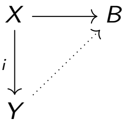
- In **Top**, a map $p\colon B \to A$ is a **Hurewicz fibration** if $i_X \pitchfork p$

$$\begin{array}{ccc} X \times \{0\} & \longrightarrow & B \\ {\scriptstyle i_X}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ X \times \mathbf{I} & \longrightarrow & A \end{array}$$

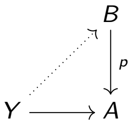
for all X. The case $X = \{*\}$ is a path-lifting property.

Lifting problems: special cases

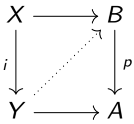
1. If $A = 1$, then we have an 'extension property' (cf. injective objects):



2. If $X = 0$, then we have a 'lifting property' (cf. projective objects):



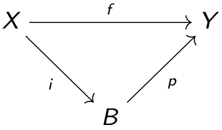
Note. General case is a combination of these:



Weak factorisation systems

Definition. A **weak factorisation system** on $\mathbf{C}$ is a pair $(\mathbf{L}, \mathbf{R})$ of classes of maps such that:

1. $\mathbf{L} = \{i \mid (\forall p \in \mathbf{R})\, i \pitchfork p\}$
2. $\mathbf{R} = \{p \mid (\forall i \in \mathbf{L})\, i \pitchfork p\}$
3. Every $f : X \to Y$ admits a factorisation



with $i \in \mathbf{L}$ and $p \in \mathbf{R}$.

Example

- $(\,\mathbf{Inj}\,,\mathbf{Surj}\,)$ is a weak factorisation system on **Set**.

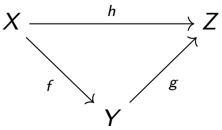
Model structures

Definition. A **Model structure** on **C** consists of three classes of maps,

$$(\mathbf{Weq}\,,\mathbf{Fib}\,,\mathbf{Cof})\,,$$

such that

1. **Weq** satisfies 3-for-2, i.e. for all



   if two out of f, g, h are in **Weq**, then so is the third.

2. $(\mathbf{Weq} \cap \mathbf{Cof}, \mathbf{Fib})$ is a weak factorisation system.

3. $(\mathbf{Cof}, \mathbf{Weq} \cap \mathbf{Fib})$ is a weak factorisation system

Model structures (II)

Examples

1. Any category **C** admits a model structure where

$$\mathbf{Weq} = \{ \text{ isomorphisms } \}\,, \quad \mathbf{Fib} = \{ \text{ all maps } \}\,, \quad \mathbf{Cof} = \{ \text{ all maps } \}$$

2. The category **Top** admits a model structure where

$$\mathbf{Weq} = \{ \text{ homotopy equivalences } \}\,, \quad \mathbf{Fib} = \{ \text{ Hurewicz fibrations } \}$$

3. The category **Top** admits a model structure where

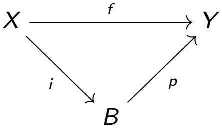
$$\mathbf{Weq} = \{ \text{ weak homotopy equivalences } \}\,, \quad \mathbf{Fib} = \{ \text{ Serre fibrations } \}$$

Terminology. An object $X \in \mathbf{C}$ is said to be

- **fibrant** if $X \to 1$ is in **Fib**
- **cofibrant** if $0 \to X$ is in **Cof**.

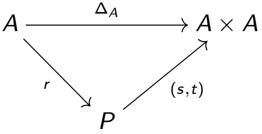
Model structures: factorisations

Remark 1. Every $f : X \to Y$ admits two factorisations



1. $i \in \mathbf{Weq} \cap \mathbf{Cof}$, $p \in \mathbf{Fib}$
2. $i \in \mathbf{Cof}$, $p \in \mathbf{Weq} \cap \mathbf{Fib}$

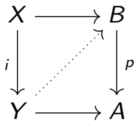
Example. The ‘path object’ factorisation



with $r \in \mathbf{Weq} \cap \mathbf{Cof}$ and $(s, t) \in \mathbf{Fib}$.

Model structures: lifting problems

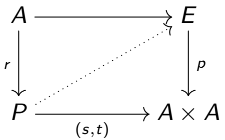
Remark 2. We have diagonal fillers



in two cases:

1. $i \in \mathbf{Weq} \cap \mathbf{Cof}$, $p \in \mathbf{Fib}$
2. $i \in \mathbf{Cof}$, $p \in \mathbf{Weq} \cap \mathbf{Fib}$

Example. We have diagonal fillers for



for all $p \in \mathbf{Fib}$.

Part II: Groupoids

Example: groupoids

The category **Gpd**

- objects: groupoids, i.e. categories in which every arrow is invertible
- maps: functors

Examples

1. Sets and bijections.
2. A group G is a groupoid with one object, $*$, and $\mathbf{Map}(*, *) = G$.
3. Every topological space has a **fundamental groupoid**, $\Pi_1(X)$ of points and homotopy classes of maps.

Isofibrations

Definition. A functor $p\colon B \to A$ between groupoids is a **isofibration** if it has the following **path lifting** property:

$$\begin{array}{ccc} B & \qquad & b_0 \xrightarrow{\ \exists\beta\ } \exists b_1 \\ \downarrow p & & \downarrow \qquad\qquad \vdots \\ A & & a_0 \xrightarrow[\ \forall\alpha\ ]{} a_1 \end{array}$$

Note. $p\colon B \to A$ is isofibration iff

$$\begin{array}{ccc} \{0\} & \xrightarrow{\ b\ } & B \\ {\scriptstyle i^0}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbf{J} & \xrightarrow[\ a\ ]{} & A \end{array}$$

has a diagonal filler, where

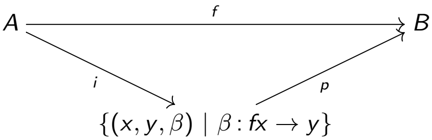
$$\mathbf{J} \quad = \quad 0 \rightleftarrows 1$$

The model structure on groupoids

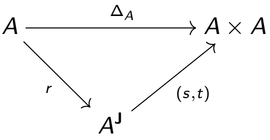
Theorem. The category **Gpd** admits a model structure

- **Weq** = equivalence of categories
- **Fib** = isofibrations
- **Cof** = functors injective on objects

Note. The $(\mathbf{Weq} \cap \mathbf{Cof}, \mathbf{Fib})$-factorisation of $f : A \to B$ is given by



In particular



Part III: Simplicial sets

Simplicial sets

The **simplicial category** Δ has

- objects: $[n] = \{0 < \ldots < n\}$, non-empty finite linear orders
- morphisms: order-preserving functions

Definition. A **simplicial set** is a functor

$$\begin{array}{rccc} A: & \Delta^{\mathrm{op}} & \to & \mathbf{Set} \\ & [n] & \mapsto & A_n \end{array}$$

The category **SSet**

- objects: simplicial sets
- maps: natural transformations

Simplicial sets as spaces

Idea. We think of a simplicial set as a set of instructions to construct a space:

- For $n \geq 0$, define the **topological standard n-simplex**

$$|\Delta^n| = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0\,, x_0 + \ldots + x_n = 1\}$$

- For $A \in \mathbf{SSet}$, define its **geometric realization**

$$R(A) = \Big(\bigsqcup_{[n] \in \Delta} A_n \times |\Delta^n| \Big) /_{\simeq}$$

This gives a functor $R \colon \mathbf{SSet} \to \mathbf{Top}$.

Note. For $[n] \in \Delta$, there is $\Delta^n \in \mathbf{SSet}$ such that

$$R(\Delta^n) \cong |\Delta^n|$$

This is called the **(simplicial) standard n-simplex**.

Examples: nerve of a groupoid

Given a groupoid G, its **nerve** is the simplicial set

$$NG : \Delta^{\text{op}} \to \mathbf{Set}$$

defined by

$$(NG)_n = \text{ set of strings of } n\text{-composable arrows in } G$$

$$= \{ \; x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_1} x_2 \longrightarrow \dots \xrightarrow{f_n} x_n \; \}$$

Note.

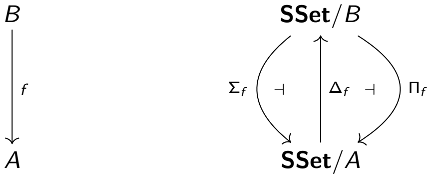
- NG captures objects and maps of G, not composition.
- This gives a functor $N : \mathbf{Gpd} \to \mathbf{SSet}$.

The category of simplicial sets

SSet is a presheaf category $\Rightarrow$

- it has all small limits and colimits
- it is locally cartesian closed: all slices are cartesian closed.

  Equivalently:



Kan fibrations

Definition. A map $p\colon B \to A$ is a **Kan fibration** if every diagram

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & B \\ {\scriptstyle h^n_k}\downarrow & & \downarrow{\scriptstyle p} \\ \Delta^n & \longrightarrow & A \end{array}$$

has a diagonal filler. Here, Λ^n_k is obtained by removing from Δ^n its interior and the interior of the face opposite the k-th vertex, and h^n_k the inclusion.

Examples.

$$\begin{array}{ccc} \Lambda^2_1 & \longrightarrow & B \\ {\scriptstyle h^1_0}\downarrow & & \downarrow{\scriptstyle p} \\ \Delta^1 & \longrightarrow & A \end{array} \qquad\qquad \begin{array}{ccc} \Lambda^1_0 & \longrightarrow & B \\ {\scriptstyle h^1_0}\downarrow & & \downarrow{\scriptstyle p} \\ \Delta^2 & \longrightarrow & A \end{array}$$

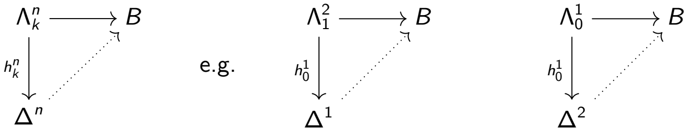
Note. $p\colon B \to A$ is an isofibration in **Gpd** $\Rightarrow$ $Np\colon NB \to NA$ Kan fibration.

The model structure on simplicial sets

Theorem. The category **SSet** admits a model structure where

- **Weq** = weak homotopy equivalences
- **Fib** = Kan fibrations
- **Cof** = monomorphisms

Note. The fibrant objects are the **Kan complexes**:



Note. G groupoid $\Rightarrow$ NG Kan complex (using the composition and inverses)

Kan complexes can be seen as weak ∞-groupoids.

Summary

Part I: Model structures

Part II: Groupoids

Part III: Simplicial sets

Tomorrow: the type theory **T** has models in groupoids and simplicial sets.