Univalent Foundations of Mathematics and Homotopical Algebra

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The univalent foundations of mathematics programme

Origin:

- formulated around 2009 by Vladimir Voevodsky
- related to Homotopy Type Theory

Motivation:

- to facilitate computer-assisted verification of proofs

Main feature: it combines ideas from

- type theory
- homotopy theory

Overview of the talk

Part I: Simplicial sets

- simplicial sets
- univalent fibrations
- simplicial univalence

Part II: Univalent foundations

- type theory
- the univalence axiom
- univalent foundations

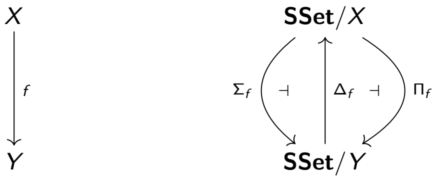
Part I: Simplicial sets

Simplicial sets as a category

SSet = category of simplicial sets.

SSet is a presheaf topos ⇒

- all small limits and colimits exist
- it is locally cartesian closed: all slices are cartesian closed. Equivalently:



Simplicial sets as a model category

SSet has a model structure $(\mathcal{W}, \mathcal{F}, \mathcal{C})$, where

- $\mathcal{W}$ = weak homotopy equivalences
- $\mathcal{F}$ = Kan fibrations
- $\mathcal{C}$ = monomorphisms

In particular, every diagram

$$\begin{array}{ccc} X & \longrightarrow & E \\ {\scriptstyle i}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ Y & \longrightarrow & B \end{array} \qquad \text{where} \begin{cases} i \in \mathcal{W} \cap \mathcal{C} \\ p \in \mathcal{F} \end{cases}$$

has a diagonal filler.

The fibrant objects are exactly the Kan complexes.

The model structure is cofibrantly generated and (left and right) proper:

- right proper = pullback along fibrations preserves weak equivalences
- left proper = pushout along cofibrations preserves weak equivalences

Homotopy theory in simplicial sets

The rich structure of **SSet** allows us to internalize a lot of constructions.

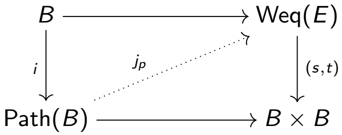
Example. Let $p\colon E \to B$ a fibration. There is a fibration

$$(s,t)\colon \mathrm{Weq}(E) \to B \times B$$

such that the fiber over (x,y) is

$$\mathrm{Weq}(E)_{x,y} = \{ w\colon E_x \to E_y \mid w \in \mathcal{W} \}$$

Note. Given $p\colon E \to B$, we have



Univalent fibrations

Definition. A fibration $p : E \to B$ is said to be **univalent** if

$$j_p : \mathrm{Path}(B) \to \mathrm{Weq}(E)$$

is a weak equivalence.

Idea. Weak equivalences between fibers are 'witnessed' by paths in the base.

Proposition (Kapulkin, Lumsdaine and Voevodsky)**.** A fibration $p : E \to B$ is univalent if and only if for every fibration $p' : E' \to B'$, the space of squares

$$\begin{array}{ccc} E' & \xrightarrow{u} & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow[v]{} & B \end{array}$$

such that

$$(u, p') : E' \to B' \times_B E$$

is a weak equivalence, is either empty or contractible.

Idea. Essential uniqueness of u, v (when they exist).

The generic Kan fibration

Fix an inaccessible cardinal κ. There exists a fibration

$$\pi : \tilde{U} \to U$$

that weakly classifies fibrations with fibers of cardinality $< \kappa$, i.e. for every such $p : E \to B$ there exists a pullback diagram

$$\begin{array}{ccc} E & \longrightarrow & \tilde{U} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ B & \longrightarrow & U\,. \end{array}$$

Note. Given $x : 1 \to U$, we can form a pullback

$$\begin{array}{ccc} \mathrm{El}(x) & \longrightarrow & \tilde{U} \\ \downarrow & & \downarrow \\ 1 & \underset{x}{\longrightarrow} & U \end{array}$$

We think of x as the 'name' of the Kan complex $\mathrm{El}(x)$.

Simplicial univalence

Theorem (Voevodsky). The generic fibration $\pi : \tilde{U} \to U$ is univalent.

Several proofs:

- Voevodsky
- Lumsdaine, Kapulkin, Voevodsky (using simplifications by Joyal)
- Moerdijk (using fiber bundles)
- Cisinski (general setting)

Note. The fibration $\pi : \tilde{U} \to U$ is therefore

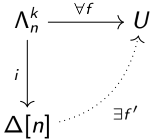
versal & univalent

for the class of κ-small fibrations.

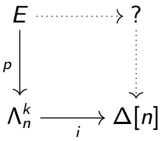
The universe is Kan

Theorem. The codomain of $\pi : \tilde{U} \to U$ is a Kan complex.

Proof. Show



This reduces to the problem of extending fibrations along horn inclusions:



which can be done using the theory of minimal fibrations.

Note. A similar unfolding is possible for the univalence of $\pi : \tilde{U} \to U$.

Part II: Univalent Foundations

General idea

Type theories are formal systems.

They have axioms for manipulating types and their elements:

$$A \text{ type}\,, \quad a \in A\,.$$

Many type theories in the literature.

Key fact. The axioms of Martin-Löf type theories correspond quite closely to a fragment of the structure of **SSet**.

Original inspiration for Martin-Löf type theories comes from proof theory and theoretical computer science (cf. implementation in Coq, Agda).

We have axioms for

$$0\,, \quad 1\,, \quad \ldots\,, \quad \mathbb{N}$$

$$X \times Y\,, \quad X + Y\,, \quad Y^X\,, \quad \ldots$$

$$\mathsf{Id}_X(x, y)\,, \quad \sum_{b \in B} E(b)\,, \quad \prod_{b \in B} E(b)\,, \quad \mathsf{U}$$

A dictionary

Simplicial sets	Type Theory
Kan complexes	Types
$x : 1 \to X$	$x \in X$
p is a path from x to y in X	$p \in \mathrm{Id}_X(x, y)$
fibration $p : E \to B$	$b \in B \vdash E(b)$ type
total space of $p : E \to B$	$\sum_{b \in B} E(b)$
space of sections of $p : E \to B$	$\prod_{b \in B} E(b)$
base of the generic fibration $\pi : \tilde{U} \to U$	type universe U
the generic fibration $\pi : \tilde{U} \to U$	$x \in \mathrm{U} \vdash \mathrm{El}(x)$ type

Note. We do not have counterparts for all the structure of simplicial sets. This is a limitation, but ensures good proof-theoretical and computational properties.

Type theory and homotopical algebra (I)

Theorem (Awodey and Warren)**.** The axioms for identity types can be stated equivalently as follows:

1. For every type X and $x, y \in X$, we have a type

$$\mathrm{Id}_X(x, y)\,.$$

2. For every X, the diagonal map $\Delta_X : X \to X \times X$ has a factorisation

$$X \xrightarrow{\;i\;} \textstyle\sum_{x,y \in X} \mathrm{Id}_X(x, y) \xrightarrow{\;p\;} X \times X\,.$$

3. Every commutative diagram of the form

$$\begin{array}{ccc} X & \longrightarrow & \sum_{b \in B} E(b) \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle \pi_1} \\ \sum_{x,y \in X} \mathrm{Id}_X(x, y) & \longrightarrow & B\,. \end{array}$$

has a diagonal filler.

Type theory and homotopical algebra (II)

Theorem (Gambino and Garner)**.** There is a weak factorization system $(\mathcal{L}, \mathcal{R})$, where

$$\mathcal{L} = \text{ functions with LLP w.r.t. every } \pi_1 : \sum_{b \in B} E(b) \to B\,.$$

Theorem (Voevodsky)**.** The category **SSet** is a model of Martin-Löf type theory and of the Univalence Axiom.

Theorem (Garner and van den Berg; Lumsdaine)**.** Every type X determines a weak ∞-groupoid $\Pi_\infty(X)$ (in the sense of Batanin-Leinster) in which

- objects are elements of X,
- 1-cells $p : x \to y$ are $p \in \mathsf{Id}_X(x, y)$,
- 2-cells $\alpha : p \to q$ are $\alpha \in \mathsf{Id}_{\mathsf{Id}_X(x,y)}(p, q)$,
- ...

Univalent foundations of mathematics

These connections suggest to:

1. use type theory as a language for speaking about homotopy types,
2. develop mathematics using this language; in particular

$$\text{sets} =_{\mathrm{def}} \text{discrete homotopy types},$$

3. add axioms to type theory motivated by homotopy theory

Homotopy-theoretic notions in type theory

- A type X is said to be **contractible** if the type

$$\mathsf{iscontr}(X) =_{\mathrm{def}} \sum_{x \in X} \prod_{y \in X} \mathsf{Id}_X(x, y)$$

is inhabited.

- The **homotopy fiber** of $f : X \to Y$ at $y \in B$ is the type

$$\mathsf{hfiber}(f, y) =_{\mathrm{def}} \sum_{x \in X} \mathsf{Id}_Y(fx, y)\,.$$

- We say that a function $f : X \to Y$ is a **weak equivalence** if for every $y \in Y$, the homotopy fiber of f at y is contractible.

- For types X and Y, there is a type

$$\mathsf{Weq}(X, Y)$$

of weak equivalences from X to Y.

The hierarchy of homotopy levels in type theory

Definition. We say that a type X has

- **homotopy level 0** if it is contractible,
- **homotopy level $\mathbf{n+1}$** if $\mathsf{Id}_X(x, y)$ is an n-type for all $x, y \in X$.

Example. Let X be a type.

$$X \text{ has level } 1 \quad \Leftrightarrow \quad \text{for all } x, y \in X,\ \mathsf{Id}_X(x, y) \text{ is contractible}$$

$$\Leftrightarrow \quad \text{if } X \text{ is inhabited, then it is contractible.}$$

Idea:

Level	0	1	2	3
Types	$*$	$\varnothing, \{*\}$	sets	groupoids
Mathematics		"logic"	"algebra"	"category theory"

The univalence axiom

The type universe U is such that if $x \in U$ then $\mathsf{El}(x)$ is a type.

Idea.

$$\begin{array}{ccc} \mathsf{El}(x) & \longrightarrow & \sum_{x \in U} \mathsf{El}(x) \\ \big\downarrow & & \big\downarrow {\scriptstyle \pi_1} \\ 1 & \underset{x}{\longrightarrow} & U \end{array}$$

Note. For $x, y \in \mathrm{U}$, we have

- the type of paths from x to y, $\mathsf{Id}_{\mathrm{U}}(x, y)$
- the type of weak equivalences from $\mathsf{El}(x)$ to $\mathsf{El}(y)$, $\mathsf{Weq}(\mathsf{El}(x), \mathsf{El}(y))$

and there is a canonical map

$$j_{x,y} : \mathsf{Id}_{\mathrm{U}}(x, y) \to \mathsf{Weq}(\mathsf{El}(x), \mathsf{El}(y))\,.$$

Univalence Axiom. For all $x, y \in \mathrm{U}$, the map $j_{x,y}$ is a weak equivalence.

Remarks on the univalence axiom

The univalence axiom is valid in **SSet** by the univalence of $\pi : \tilde{U} \to U$.

This axiom has several interesting aspects from a logical point of view:

- it forces the type universe U not to be of level 2 (the level of "sets")
- it is not valid in the set-theoretical model
- it allows us to treat isomorphic structures as if they were equal
- it has useful consequences, such as function extensionality (Voevodsky)
- it gives the 'Rezk completion' of a category (Ahrens, Kapulkin, Shulman)

Further aspects

- Synthetic homotopy theory (Shulman, Lumsdaine, Licata, Brunerie, ...)
- Higher inductive types (Shulman, Lumsdaine)
- Homotopy-initial algebras in type theory (Awodey, Gambino and Sojakova)
- Other models of univalent foundations (Coquand et al., Shulman, Cisinski)
- Univalent maps in $(\infty, 1)$-categories (Gepner and Kock)