Barely locally presentable categories

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joint work with L. Positselski

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\[ \bigcup_{j \in J} K_j \to \bigcup_{i \in I} K_i, \] where \( J \) are subsets of \( I \) of cardinality less than \( \lambda \) will be called coproduct \( \lambda \)-directed colimits.

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This means that \( \mathcal{K}(A, -) \) sends coproduct \( \lambda \)-directed colimits to \( \lambda \)-directed colimits and not to coproduct \( \lambda \)-directed ones (because \( \mathcal{K}(A, -) \) does not preserve coproducts).
\( \lambda \)-directed colimits \( \bigsqcup_{j \in J} K_j \to \bigsqcup_{i \in I} K_i \), where \( J \) are subsets of \( I \) of cardinality less than \( \lambda \) will be called \textit{coproduct} \( \lambda \)-directed colimits.

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If coproduct injections are monomorphisms then \( A \) is barely \( \lambda \)-presentable if and only if for every morphism \( f : A \to \bigsqcup_{i \in I} K_i \) there is a subset \( J \) of \( I \) of cardinality less than \( \lambda \) such that \( f \) factorizes as \( A \to \bigsqcup_{j \in J} K_j \to \bigsqcup_{i \in I} K_i \) where the second morphism is the subcoproduct injection.
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Coproduct injections are very often monomorphisms, for instance in any pointed category. However, in the category of commutative rings, the coproduct is the tensor product and the coproduct injection \(\mathbb{Z} \to \mathbb{Z} \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2\) is not a monomorphism.
Any $\lambda$-presentable object is barely $\lambda$-presentable. We say that $A$ is \textit{barely presentable} if it is barely $\lambda$-presentable for some $\lambda$. 
Any $\lambda$-presentable object is barely $\lambda$-presentable. We say that $A$ is *barely presentable* if it is barely $\lambda$-presentable for some $\lambda$.

A cocomplete category $\mathcal{K}$ will be called *barely locally $\lambda$-presentable* if it is strongly co-wellpowered and has a strong generator consisting of barely $\lambda$-presentable objects.

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This concept was introduced by L. Positselski and J. Šťovíček for abelian categories. Since any abelian category with a generator is co-wellpowered, strong co-wellpoweredness does not need to be assumed there.
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Any locally $\lambda$-presentable category is barely locally $\lambda$-presentable.
A category $\mathcal{K}$ has $\lambda$-directed unions if for any $\lambda$-directed set of subobjects $(K_i)_{i \in I}$ of $K$ the induced morphism $\text{colim}_{i \in I} K_i \to K$ is a monomorphism. The following result was proved by Positselski and Šťovíček for abelian categories and for $\lambda = \aleph_0$.

Proposition 1. Any barely locally $\lambda$-presentable regular category has $\lambda$-directed unions.

We say that $\mathcal{K}$ has coproduct $\lambda$-directed unions if for every coproduct $\lambda$-directed colimit $\bigvee_{j \in J} K_j \to \bigvee_{i \in I} K_i$, every morphism $\bigvee_{i \in I} K_i \to K$ whose compositions with $\bigvee_{j \in J} K_j \to \bigvee_{i \in I} K_i$ are monomorphisms is a monomorphism.

Proposition 2. Any barely locally $\lambda$-presentable category has coproduct $\lambda$-directed unions.

Proposition 3. Let $\mathcal{K}$ be a locally presentable category such that $\mathcal{K}^{\text{op}}$ has coproduct $\lambda$-directed unions for some regular cardinal $\lambda$. Then $\mathcal{K}$ is equivalent to a complete lattice. Thus a non-trivial locally presentable category cannot have the barely locally presentable dual.
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**Proposition 3.** Let $\mathcal{K}$ be a locally presentable category such that $\mathcal{K}^{\text{op}}$ has coproduct $\lambda$-directed unions for some regular cardinal $\lambda$. Then $\mathcal{K}$ is equivalent to a complete lattice.

Thus a non-trivial locally presentable category cannot have the barely locally presentable dual.
Theorem 1. Any barely locally $\lambda$-presentable abelian category is locally presentable.
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Conversely, from the negation of Vopěnka’s principle, we construct artificial examples of regular barely locally presentable categories which are not locally presentable.
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**Problem.** Is there a barely locally presentable category which is not locally presentable in ZF?
Let $\textbf{Prox}$ be the category of proximity spaces and proximally continuous mappings and $\textbf{Unif}$ the category of uniformity spaces and uniformly continuous mappings.
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$\mathbf{Prox}$ is isomorphic to the full subcategory of $\mathbf{Unif}$ consisting of totally bounded uniformity spaces. A uniformity space is totally bounded if every uniform cover has a finite subcover.

Lemma 1. Every separated proximity space is barely presentable in $\mathbf{Prox}^\text{op}$. In fact, $A$ is barely $\lambda+$-presentable where $\lambda$ is its uniform character (Hušek 1973), i.e., the smallest cardinality of a base of uniform covers of $A$. 
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\textbf{Prox}^{\text{op}} \text{ is regular but not barely locally presentable because trivial proximity spaces (any two non-empty subsets are near) are not barely presentable in } \textbf{Prox}^{\text{op}} \text{ and}
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**Lemma 2.** Let \(\mathcal{K}\) be a barely locally presentable category with pullbacks such that coproduct injections are monomorphisms. Then any object of \(\mathcal{K}\) is barely presentable.
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\( \mathbb{R} \) is a cogenerator in \( \text{Prox}_0 \) because any separated proximity space is a subspace of powers of \( \mathbb{R} \). But it is not a strong cogenerator in \( \text{Prox}_0 \) because strong monomorphisms in \( \text{Prox}_0 \) are closed embeddings.
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Let \(\text{Prox}_\mathbb{R}\) be the full subcategory of \(\text{Prox}_0\) consisting of closed subspaces of powers of \(\mathbb{R}\).

**Proposition 4.** \(\text{Prox}_\mathbb{R}^{\text{op}}\) is barely locally \(\aleph_1\)-presentable.
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**Proposition 4.** \( \text{Prox}^{\text{op}}_\mathbb{R} \) is barely locally \( \aleph_1 \)-presentable.

\( \mathbb{R} \) is a strong cogenerator in \( \text{Prox}^{\text{op}}_\mathbb{R} \) and has uniform character \( \aleph_0 \).
Proposition 5. Assuming Vopěnka’s principle, $\text{Prox}^\text{op}_R$ is locally presentable.
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Any $K$ in $\text{Prox}_R$ induces a realcompact topological space. The category of proximity spaces is isomorphic to the category $\mathcal{K}$ whose objects are triples $(X, bX, f)$ where $f : X \to bX$ is an embedding of $X$ to its compactification, i.e., $f$ makes $X$ a dense subspace of a compact space $bX$. Consider the functor $G : \mathcal{K}^{\text{op}} \to \text{Ring}^{\to}$ sending $(X, bX, f)$ to the monomorphism $C(f) : C(bX) \to C(X)$ where $C(X)$ is the ring of continuous functions $X \to \mathbb{R}$. This makes $\mathcal{K}^{\text{op}}$ isomorphic to a full subcategory of the category $\text{Ring}^{\to}$ of morphisms of rings. Since the latter is locally presentable, Vopěnka’s principle implies that $\mathcal{K}^{\text{op}}$ is locally presentable.
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We do not know whether the local presentability of \( \text{Prox}_R^{\text{op}} \) depends on set theory.
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**Lemma 3.** Every separated uniform space is presentable in $\text{Unif}^{\text{op}}$. A uniform space is barely $\lambda^+$-presentable in $\text{Unif}^{\text{op}}$ where $\lambda$ is its uniform character. This follows from the fact that any uniformly continuous mappings from a subspace of a product depends on $\lambda$ many coordinates (Vidossich 1970). This is not true for proximity spaces (Hušek 1973).
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Let $\text{Unif}_1$ be the full subcategory of $\text{Unif}$ consisting of subspaces of powers of $\mathbb{R}$. Spaces from $\text{Unif}_1$ are rather special, any has the uniform character $< \aleph_1$. Let $\text{Unif}_\mathbb{R}$ be the full subcategory of $\text{Unif}_1$ consisting of closed subspaces of powers of $\mathbb{R}$. 
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**Proposition 6.** $\text{Unif}_{\mathbb{R}}^{\text{op}}$ is locally $\aleph_1$-presentable.
Proposition 7. Any barely locally presentable category with pullbacks is complete.
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**Proposition 8.** Let $\lambda_1 < \lambda_2$ be regular cardinals. Then any barely $\lambda_1$-presentable category is barely $\lambda_2$-presentable.
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Proposition 8. Let $\lambda_1 < \lambda_2$ be regular cardinals. Then any barely $\lambda_1$-presentable category is barely $\lambda_2$-presentable.

Proposition 9. Let $\mathcal{K}$ be a barely locally $\lambda$-presentable category and $\mathcal{C}$ be a small category. Then the functor category $\mathcal{K}^\mathcal{C}$ is barely locally $\lambda$-presentable.