ELEMENTARY CONSTRUCTIVE OPERATIONAL SET THEORY

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Dedicated to Prof. Wolfram Pohlers

Abstract. We introduce an operational set theory in the style of [5] and [17]. The theory we develop here is a theory of *constructive* sets and operations. One motivation behind constructive operational set theory is to merge a constructive notion of set ([1], [2]) with some aspects which are typical of explicit mathematics [14]. In particular, one has non-extensional operations (or rules) alongside extensional constructive sets. Operations are in general partial and a limited form of self–application is permitted. The system we introduce here is a fully explicit, finitely axiomatised system of *constructive* sets and operations, which is shown to be as strong as **HA**.

1. Introduction

This article is a follow-up of [9], where a constructive set theory with operations was introduced. Constructive operational set theory (**COST**) is a constructive theory of sets and operations which has similarities with Feferman's (classical) Operational Set Theory ([17], [18], [21], [22], [23]) and Beeson's Intuitionistic set theory with rules [5]. In this article a fully explicit fragment, called **EST**, of **COST** is singled out. This system is finitely axiomatized and is shown to be proof–theoretically as strong as Peano Arithmetic (section 5).

One motivation behind constructive operational set theory is to merge a constructive notion of set ([26], [1], [2]) with some aspects which are typical of explicit mathematics [14]. In particular, one has non-extensional operations (or rules) alongside extensional constructive sets. Operations are in general partial and a limited form of self–application is permitted.

The informal concept of rule plays a prominent role in constructive mathematics. Both Feferman and Beeson have repeatedly called attention to the distinction between rules and set–theoretic functions (see e.g. [15], [3]). There are several examples of intuitive rules which can not be represented by the set–theoretic concept of function. For example the operation of pair, which given two sets a and b enables us to form a new set, the set–theoretic pair of a and b. In operational set theory we have primitive operations corresponding to some set–theoretic rules, among which that of pair. In a sense, rules can be regarded as generalized algorithms or abstract rules. Without entering a detailed conceptual analysis of the notion of rule, we simply adopt the view that rules are represented by sets, and that it makes sense to *apply* a set c '*qua rule*' to another set b as input; and this possibly provides a result,

The research is part of a project supported by PRIN 2006 (Dimostrazioni, Operazioni, Insiemi). The second author gratefully acknowledges a grant by the John Templeton Foundation. She would also like to thank the School of Mathematics, University of Leeds, for the hospitality.

whenever the algorithm encoded by c produces a computation converging to b. The application structure is specified by a ternary application relation, which satisfies very general closure conditions, in that it embodies at least pure combinatory logic with a number of primitive set–theoretic rules. As Beeson has emphasised e.g. in [5] this approach has the advantage of allowing for a natural computation system based on set theory. The idea is that while functions as graphs are hardly of any use in programming, a notion of operation can be utilised to obtain a polymorphic computation system based on set theory. Such a computation system is the main motivation for the theory of sets and rules, called **IZFR**, introduced in [5]. This is an operational version of intuitionistic Zermelo–Fraenkel set theory, **IZF** (see [3]); in particular, like that theory it is fully impredicative.

Quite different is Feferman's motivation in developing *operational set theory*. Feferman observes that analogues of 'small large cardinal notions' (those consistent with $V = L$) have emerged in different contexts, like admissible set theory, admissible recursion theory, explicit mathematics, recursive ordinal notations and constructive set and type theory. His aim in defining operational set theory is to develop a common language in which such notions can be expressed and can be interpreted both in their original classical form and in their analogue form in each of these special constructive and semi-constructive cases. Feferman's system **OST** is inherently classical, due to the presence of a choice operator (see section 3.4).

We see the present paper, though founded on [9], as a preliminary and rather experimental attempt in studying *constructive* operational set theory. It is hoped that the results here presented will contribute to both Feferman and Beeson's aims. We stress, however, our more parsimonious approach to the foundations for (constructive) mathematics: constructive operational set theory is based on intuitionistic logic and also complies with a notion of generalised predicativity.

The system **COST** of [9] had urelements at the base of the set–theoretic universe, representing the elements of an applicative structure with natural numbers. The main idea was to carefully endow the whole universe of sets with a natural extension of the base application relation. In [9] **COST** was shown to be of the same strength as **CZF** ([1], [2], [12]). Furthermore, a subtheory was singled out and shown to be of the same proof–theoretic strength as **PA**. The theory **COST** and its subsystems were introduced so to resemble as much as possible the constructive set theory **CZF** (and subsystems). In particular, **COST** had schemata of strong and subset collection, thus retaining all the mathematical expressivity of **CZF**. However, the presence of implicit principles of collection was not entirely satisfactory if one wished to have an explicit theory of sets and operations. In addition, as already noted in the introduction to [9], an inspection of the proofs in that paper (especially sections 3 - 5) shows that many of them can already be conducted in an explicit fragment of **COST**. For this reason we here single out such a fragment, **EST**, and show that it has the same strength as **PA**. Note further that in this article we work with pure sets, i.e. we do not introduce urelements.[1] One could also say that with **COST** and its subsystems we aimed at expressive theories, though of limited proof–theoretic

[1] Urelements had a twofold motivation in [9]. On the one side, in the authors' opinion, including urelements at the ground of the set–theoretic universe appears as a constructively justified option. On the other side, urelements played a useful technical role, as they allowed for a separation between the principles of induction on the natural numbers and on sets. As a result we could define theories which had full induction on sets but bounded induction on the natural numbers. These theories had a considerable expressive power and a very limited proof–theoretic strength.

strength. With **EST** we single out a more elegant, finitely axiomatized theory, though at the price of a more limited expressivity. We wish to note, however, that Friedman's system **B** ([19]) can be interpreted in the theory **EST** plus bounded (or limited) Dependent Choice (**LDC**) (section 4.3), so that we are persuaded we have a theory which is foundationally meaningful.

One contribution of the present paper is the use of the technique of partial cut elimination and asymmetric interpretation ([6]) to determine the strength of **EST**. We are not aware of other attempts to introduce this technique to systems of constructive set theory (see [22] for an application of this technique in the context of a proof–theoretic analysis of strong systems of classical operational set theory).

As to the contents of this paper, section 2 describes language and axioms of the theory **EST**. Section 3 collects elementary facts linking the set–theoretic and the applicative structures. In particular, we show that extensionality and totality of operations can not be assumed in general in the present context. In addition, we study the relations between the notions of set–theoretic function and operation and also assess the status of some choice principles on the basis of **EST**.

Section 4 is dedicated to clarifying the relation between **EST** and Beeson's **IZFR**, Feferman's **OST** and Friedman's **B**, respectively.

Finally, section 5 shows that **EST** has the same proof–theoretic strength as **PA**. The lower bound is easily achieved. The upper bound is addressed by a series of steps. First an auxiliary constructive set theory, **ECST***, is introduced. This is reduced to a classical axiomatic theory of abstract self–referential truth, $\mathbf{T_c}$, which is conservative over PA. The interpretation is obtained by an appropriate modification of [9]'s realisability interpretation. The reduction of **EST** to **ECST*** is obtained by first introducing a Gentzen–style formulation of **EST** (in fact of a strengthening of it). A partial cut elimination theorem holds for such a system. Finally, we define an asymmetric interpretation of the operational set theory in **ECST***, which allows us to obtain the desired upper bound.

2. The theory EST

2.1. **Language and conventions.** The language of **EST** is the following applicative extension, $\mathcal{L}^O$, of the usual first order language of Zermelo–Fraenkel set theory, $\mathcal{L}$.

The language includes the predicate symbols $\in$ and $=$. The logical symbols are all the intuitionistic operators: $\bot$, $\wedge$, $\vee$, $\rightarrow$, $\exists$, $\forall$. We have in addition:

- the combinators K and S;
- a ternary predicate symbol, *App*, for application; $App(x, y, z)$ is read as x applied to y yields z;
- el for the ground operation representing membership;
- pair, un, im, sep, for set operations;
- $\emptyset$, ω, set constants;
- IT for ω–iterator.[2]

For convenience we also use the bounded quantifiers $\exists x \in y$ and $\forall x \in y$, as abbreviations for $\exists x\,(x \in y \wedge \ldots)$ and $\forall x\,(x \in y \rightarrow \ldots)$.

However, in this paper we look for a more fundamental and simpler theory, and thus focus on a pure subsystem of **COST** with no set–induction.

[2]The idea of postulating an iteration principle as primitive is already present in Weyl's *Das Kontinuum* (chapter 1, section 7).

As customary, we define $\varphi \leftrightarrow \psi$ by $(\varphi \to \psi) \wedge (\psi \to \varphi)$ and $\neg\varphi$ by $\varphi \to \bot$. We also write $a \subseteq b$ for $\forall z\,(z \in a \to z \in b)$.

Terms and formulas. Terms and formulas are inductively defined as usual.

To increase perspicuity, we consider a definitional extension of $\mathcal{L}^O$ with application terms, defined inductively as follows.

(i) Each variable and constant is an application term.
(ii) If t, s are application terms then ts is an application term.

Application terms will be used in conjunction with the following abbreviations.

(i) $t \simeq x$ for $t = x$ when t is a variable or constant.
(ii) $ts \simeq x$ for $\exists y\, \exists z\, (t \simeq y \;\wedge\; s \simeq z \;\wedge\; App(y, z, x))$.
(iii) $t\downarrow$ for $\exists x\, (t \simeq x)$.
(iv) $t \simeq s$ for $\forall x\, (t \simeq x \leftrightarrow s \simeq x)$.
(v) $\varphi(t, \dots)$ for $\exists x\, (t \simeq x \;\wedge\; \varphi(x, \dots))$.
(vi) $t_1 t_2 \dots t_n$ for $(\dots(t_1 t_2) \dots) t_n$.

To ease readability we sometimes use the notation $t(x, y)$ for txy.

In the language $\mathcal{L}^O$, the notion of *bounded* formula needs to be appropriately modified.

Definition 2.1 (Bounded formulas)**.** A formula of $\mathcal{L}^O$ is *bounded*, or Δ_0**,** if and only if all quantifiers occurring in it, if any, are bounded *and in addition it does not contain application App.*

Classes are introduced as usual in set theory, as abbreviations for abstracts $\{x : \varphi(x)\}$ for any formula φ of the language $\mathcal{L}^O$. In particular, we let $\mathbf{V} := \{x : x\downarrow\}$. For A and B sets or classes, we write $f : A \to B$ for $\forall x \in A\, (fx \in B)$ and $f : \mathbf{V} \to B$ for $\forall x\, (fx \in B)$. By $f : A^2 \to B$ and $f : \mathbf{V}^2 \to B$ we indicate $\forall x \in A\, \forall y \in A\, (fxy \in B)$ and $\forall x\, \forall y\, (fxy \in B)$, respectively. This can be clearly extended to arbitrary exponents $n > 2$. Finally, for set a, $f : a \to \mathbf{V}$ means that f is everywhere defined on a.

Truth values. We may represent false and truth by the empty set and the singleton empty set, respectively; that is we let $\bot := \emptyset$ and $\top := \{\emptyset\}$.

Let Ω be the class $\mathcal{P}\top$, the powerset of $\top$. Then $x \in \Omega$ is an abbreviation for $\bot \subseteq x \subseteq \top$. The class Ω intuitively represents the class of truth values (or of propositions). Note that in the presence of exponentiation if Ω is taken to be a set then full powerset follows (see Aczel [1], Proposition 2.3).

Relations and set–theoretic functions. The notions of relation between two sets, of domain and range of a relation can be defined in the obvious way in **EST**. In the following we write $Dom(R)$ and $Ran(R)$ to denote the domain and the range of a relation, respectively. In remark 3.9 we shall see that in **EST** there is an operator **opair** internally representing the ordered pair of two sets. In addition, also the range and the domain of a relation correspond to internal operations, respectively.

We also have a standard notion of *set–theoretic function* which we can express by a formula, $Fun(F)$, stating that F is a set encoding a total binary relation which satisfies the obvious uniqueness condition. We shall use upper case letters $F, G, \dots$ for set–theoretic functions and lower case letters $f, g, \dots$ for operations (that is if they formally occur as operators in application terms or as first coordinates in *App*–contexts). Given a set–theoretic function F, we write $\langle x, y\rangle \in F$ or also $F(x) = y$ for **opair** $xy \in F$. We shall investigate the relation between the notions of operation and set–theoretic function in section 3.3.

Finally, in defining the axiom of infinity we shall make use of the following successor operation.

Definition 2.2. Let $\mathsf{Suc} := \lambda x.\mathsf{un}\, x(\mathsf{pair}\, xx)$.

2.2. **Axioms of EST.**

Definition 2.3. EST is the $\mathcal{L}^O$ theory whose principles are all the axioms and rules of first order intuitionistic logic with equality, plus the following principles.

Extensionality

- $\forall x\,(x \in a \leftrightarrow x \in b) \to a = b$

General applicative axioms

- $App(x, y, z) \wedge App(x, y, w) \to z = w$
- $\mathsf{K}xy = x \wedge \mathsf{S}xy\downarrow \wedge \mathsf{S}xyz \simeq xz(yz)$

Membership operation

- $\mathsf{el} : \mathbf{V}^2 \to \Omega$ and $\mathsf{el}\, xy \simeq \top \leftrightarrow x \in y$

Set constructors

- $\forall x\,(x \notin \emptyset)$
- $\mathsf{pair}\, xy\downarrow \wedge \forall z\,(z \in \mathsf{pair}\, xy \leftrightarrow z = x \vee z = y)$
- $\mathsf{un}\, a\downarrow \wedge \forall z\,(z \in \mathsf{un}\, a \leftrightarrow \exists y \in a(z \in y))$
- $(f : a \to \Omega) \to \mathsf{sep}\, fa\downarrow \wedge \forall x\,(x \in \mathsf{sep}\, fa \leftrightarrow x \in a \wedge fx \simeq \top)$
- $(f : a \to V) \to \mathsf{im}\, fa\downarrow \wedge \forall x\,(x \in \mathsf{im}\, fa \leftrightarrow \exists y \in a(x \simeq fy))$

Strong infinity

- $(\omega 1)$ $\quad \emptyset \in \omega \wedge \forall y \in \omega\,(\mathsf{Suc}\, y \in \omega)$
- $(\omega 2)$ $\quad \forall x\,(\emptyset \in x \wedge \forall y(y \in x \to \mathsf{Suc}\, y \in x) \to \omega \subseteq x)$

ω–Iteration

- $$\begin{aligned} &\forall F[[Fun(F) \wedge Dom(F) = a \wedge Ran(F) \subseteq a] \\ &\quad \to \forall x \in a\, \exists z\,[IT(F, a, x) \simeq z \wedge Fun(z) \wedge Dom(z) = \omega \\ &\wedge Ran(z) \subseteq a \wedge z(\emptyset) = x \wedge \forall n \in \omega(z(\mathsf{Suc}\, n) = F(z(n)))]]. \end{aligned}$$

Remark 2.4. The principles ruling sep and im embody the explicit character of the separation and replacement schemata in the present operational context: sep provides – uniformly in any given $f : a \to \Omega$ – the set of all elements satisfying the "propositional function" defined by f; on the other hand, im yields – uniformly in any given operation f defined on a set a – the image of a under f.

Definition 2.5 (The theory **ESTE**)**.** Let **ESTE** be obtained from **EST** by removing ω–iteration and by adding a new constant exp to the language together with the following explicit version of Myhill's exponentiation axiom [26]:

$$\mathsf{exp}\, ab\downarrow \wedge \forall x(x \in \mathsf{exp}\, ab \leftrightarrow (Fun(x) \wedge Dom(x) = a \wedge Ran(x) \subseteq b)).$$

3. Elementary properties of **EST**

In this section we present some properties of **EST**. In particular, we aim at clarifying the status of extensionality and intensionality in **EST**. We also look at some aspects of the relationship between functions as operations and as graphs and the status of some choice principles. Finally, we show that the theory **ESTE** proves

ω–iteration. Part of this section draws on [9], however adapting the arguments to the present context. For the reader's convenience we shall recall some of the arguments of [9]. First of all, as a consequence of the axioms for combinators, the universe of sets is closed under abstraction and recursion for operations (see e.g. [30]).

Lemma 3.1. *(i) For each term t, there exists a term $\lambda x.t$ with free variables those of t other than x and such that*

$$\lambda x.t \downarrow \wedge (\lambda x.t)y \simeq t[x := y].$$

(ii) (Second recursion theorem) There exists a term **rec** *with*

$$\mathbf{rec} f \downarrow \wedge (\mathbf{rec} f = e \rightarrow ex \simeq fex).$$

We now show that the logical operations generating bounded formulas are mirrored by internal operations.

Lemma 3.2. *There are application terms* eq, and, all, exists, imp, or *such that*

(i) $\mathsf{eq} : \mathbf{V}^2 \rightarrow \Omega$ *and* $\mathsf{eq}\, xy \simeq \top \leftrightarrow x = y$;

(ii) $x \in \Omega \wedge y \in \Omega \rightarrow \mathsf{and}\, xy \in \Omega \wedge (\mathsf{and}\, xy \simeq \top \leftrightarrow (x \simeq \top \wedge y \simeq \top))$;

(iii) $(f : a \rightarrow \Omega) \rightarrow \mathsf{all}\, fa \in \Omega \wedge (\mathsf{all}\, fa \simeq \top \leftrightarrow \forall x \in a\, (fx \simeq \top))$;

(iv) $(f : a \rightarrow \Omega) \rightarrow \mathsf{exists}\, fa \in \Omega \wedge (\mathsf{exists}\, fa \simeq \top \leftrightarrow \exists x \in a\, (fx \simeq \top))$;

(v) $x \in \Omega \wedge (x = \top \rightarrow y \in \Omega) \rightarrow \mathsf{imp}\, xy \in \Omega \wedge (\mathsf{imp}\, xy \simeq \top \leftrightarrow (x \simeq \top \rightarrow y \simeq \top))$;

(vi) $x \in \Omega \wedge y \in \Omega \rightarrow \mathsf{or}\, xy \in \Omega \wedge (\mathsf{or}\, xy \simeq \top \leftrightarrow (x \simeq \top \vee y \simeq \top))$.

Proof. See Lemma 3.2 of [9]. □

Proposition 3.3. *(i) For each Δ_0 formula φ with free variables contained in $\{x_1, \dots, x_k\}$, there is an application term f_φ such that $f_\varphi \downarrow$, $f_\varphi : \mathbf{V}^k \rightarrow \Omega$ and*

$$f_\varphi\, x_1 \dots x_k \simeq \top \leftrightarrow \varphi(x_1, \dots, x_k).$$

(ii) To each Δ_0 formula $\varphi(x, y_1 \dots y_k)$, we can associate an application term c_φ such that

$$c_\varphi a y_1 \dots y_k \downarrow \wedge \forall u (u \in c_\varphi a y_1 \dots y_k \leftrightarrow u \in a \wedge \varphi(u, y_1, \dots, y_k)). \tag{1}$$

Proof. (i) A simple induction applies, making use of Lemma 3.2. (ii) follows from (i) and explicit separation. □

Remark 3.4.

(i) the schema (1) is naturally called *uniform bounded separation schema* (i.e. restricted to Δ_0-formulas, which *do not contain App*);

(ii) *uniform bounded separation* with application terms: we are allowed to use application terms as genuine terms insofar as they are defined. In the special case of separation, if t, s are application terms such that $t \downarrow$, $s \downarrow$ and $s : t \rightarrow \Omega$, then there exists an application term $r := \mathsf{sep}\, st$ such that

$$\forall u (u \in r \leftrightarrow u \in t \wedge su \simeq \top).$$

Instead of r, we write $\{u \in t : su \simeq \top\}$. Similarly, if φ is Δ_0 with free variables x, y, and t, s are application terms such that $t \downarrow$, $s \downarrow$, then there exists an application term $r_\varphi := c_\varphi ts$ such that

$$\forall u (u \in r_\varphi \leftrightarrow u \in t \wedge \varphi(u, s)).$$

Instead of r_φ, we again stick to the more familiar and perspicuous notation

$$\{u \in t : \varphi(u, s)\}.$$

The main tool in proving the results in the next subsection is the following Lemma. This is a consequence of proposition 3.3, and states that we can express an operator representing definition by cases on the universe for bounded predicates.

Lemma 3.5. *Let $\varphi(x, y)$ be Δ_0 (with the free variables shown). Then there exists an operation D_φ such that $D_\varphi uvab \downarrow$ and*

$$\varphi(u, v) \quad \to \quad D_\varphi uvab = a \tag{2}$$

$$\neg\varphi(u, v) \quad \to \quad D_\varphi uvab = b. \tag{3}$$

Proof. By uniform bounded separation (see proposition 3.3) and uniform union, there exists an operation D_φ such that

$$D_\varphi = \lambda u \lambda v \lambda a \lambda b.\{x \in a : \varphi(u, v)\} \cup \{x \in b : \neg\varphi(u, v)\}.$$

By λ-abstraction, $D_\varphi uvab \downarrow$. By extensionality, D_φ satisfies (2) – (3). □

Note that in the particular case in which a is $\top$ and b is $\bot$, even if $\varphi(u, v)$ is undecidable, then $D_\varphi uv\top\bot$ equals the proposition (the truth value) associated to $\varphi(u, v)$, i.e an element of Ω.

Indeed, as a special case we have the following.

Corollary 3.6. *There exists an operation EQ such that $\mathsf{EQ}uv \downarrow$ and*

$$u = v \quad \to \quad EQ(u, v) = \top$$

$$\neg u = v \quad \to \quad EQ(u, v) = \bot.$$

We stress again that $=$ is not decidable in general.

In the following we shall make use of the usual notation $\bigcup$ for the uniform operation of union, un, and write $\cup$ for the obvious definition of a uniform version of binary union.

3.1. **Non–extensionality and partiality of operations.** As observed in [9], the combination of operations and sets needs to be accomplished with care. The following argument shows that totality and extensionality can not be assumed in general.We also show that separation can not be extended to formulas with bounded quantifiers and *App*.

We say that two operations f and g are *extensional* if they satisfy the following:

$$\forall x\,(fx \simeq gx) \to f = g. \tag{4}$$

Proposition 3.7. EST *refutes extensionality for operations and totality of application:*

- $\neg[\forall x\,(fx \simeq gx) \to f = g]$;
- $\neg\forall x\,\forall y\,\exists z\,App(x, y, z)$.

Proof. The argument is standard. First of all, recall a (folklore) preliminary fact about *partial combinatory algebras* (pcas for short). By a pca we understand a non-empty set endowed with a partial binary function (i.e. application) and two special elements K and S satisfying the standard axioms for combinators (see definition 2.3). A pca is *extensional* if it satisfies extensionality for operations (4). Extensional pcas

satisfy the fixed point property for total operations: if g is a total operation, then for some e, $ge = e$ (for the proof see [9] Lemma 3.11).

Now, assume extensionality, define $\varphi(u, v) \equiv (u = v)$ and let $NOTu = D_\varphi u\top\bot\top$. Then

$$\begin{aligned} u = \top &\quad\rightarrow\quad NOTu = \bot \\ \neg u = \top &\quad\rightarrow\quad NOTu = \top. \end{aligned}$$

Note that NOT is total; hence by the previous remark, there exists a fixed point e such that $NOTe = e$ and

$$e = \top \quad\rightarrow\quad e = \bot \tag{5}$$

$$\neg e = \top \quad\rightarrow\quad e = \top. \tag{6}$$

The first implication implies $\neg e = \top$: if we assume $e = \top$, then by (5) $e = \bot$, which yields $\top = \bot$, i.e. $\emptyset \in \emptyset$, absurd. Hence by (6) we conclude $e = \top$: contradiction! On the other hand, if totality of application is assumed, the fixed point theorem of full lambda calculus holds and we can derive the inconsistency as well. □

Proposition 3.8. **EST** *with uniform separation for bounded conditions containing App*[3] *is inconsistent.*

Proof. By uniform separation including *App*-conditions, there would exist a total operation g such that

$$gfz = \{x \in \top : fz \simeq x\}.$$

By lemma 3.1 (second recursion theorem), there exists some e such that $gez \simeq ez$. Since g is total, e is total; hence $ee\downarrow$ and satisfies $ee = \{x \in \top : ee = x\}$. Were $x \in ee$, then $x = \emptyset \wedge x = ee$. Then $ee = \emptyset$ and hence $x \in \emptyset$: contradiction! □

3.2. $E_{\mathcal{P}}$**–recursion.** In [9] we noted that we can recast a form of set computability in a weak system of operational set theory. Already Beeson observed the link between his intuitionistic set theory with rules and a variant of set recursion (Beeson [5], see also [27]). In [28] Rathjen introduced a form of extended set recursion (inspired by [25]) named $E_{\mathcal{P}}$–computability. According to this form of set recursion, exponentiation is taken as one of the basic operations which are used to define set computability. Therefore, for a and b sets, the set ${}^a b$ of all set–theoretic functions from a to b, is computable. This notion of set recursion is used by Rathjen to develop an interpretation for **CZF** in itself which is a self validating semantics for that system of constructive set theory. This interpretation is called the formulas–as–classes interpretation. We showed in [9] that we can naturally capture $E_{\mathcal{P}}$–computability in a subsystem of **COST**. In particular, in operational set theory application is primitive and we can thus avoid the detour of [28] through coding and an inductive definition. In Proposition 4.3 of [9] we showed that the clauses defining $E_{\mathcal{P}}$-computability in Definition 4.1 of [28] can be carried out in a subsystem of **COST**. Here we note that the proof of the proposition can be carried out in the theory **ESTE**.[4]

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[3]This means the schema (1), where *App* is allowed to occur in the bounded formula φ; see also remark 3.4.

[4]Note, however, that due to the lack of set–induction, we can not prove in the present context Theorem 4.4 of [9] which showed that Rathjen's construction can be recast in **COST**. Note also

For the reader's convenience we now briefly recall the content of Lemma 4.1 and that part of Proposition 4.3 of [9] which is needed in the following.

Remark 3.9 (**EST**). (i) There are operations **int**, **prod**, **dom**, **ran**, **opair**, $\mathbf{proj_i}$ ($i = 0, 1$), representing: binary intersection, cartesian product, domain and range of a set–theoretic function, ordered pair and projections, respectively. (See Lemma 4.1 of [9]).

(ii) There is a term $\overline{\mathbf{fa}}$ such that for any set–theoretic function F and for any $x \in Dom(F)$, $\overline{\mathbf{fa}}\,Fx \simeq F(x)$. In fact, we can take $\overline{\mathbf{fa}}$ to be : $\lambda F.\lambda x. \bigcup\{y \in Ran(F) : \langle x, y\rangle \in F\}$ (by uniform pair, union, separation). In addition, there is an operation $\overline{\mathbf{ab}}$ such that, for each f which is defined (or total) on a, $\overline{\mathbf{ab}}\,fa \simeq H$, with H a set–theoretic function with domain a and such that $\forall x \in a\,(H(x) \simeq fx)$. In fact, if $f : a \to \mathbf{V}$, then by im we can find b such that $\forall x \in a\,\exists y \in b\,(y \simeq fx)$. By (i) we have an operator **prod** which gives the cartesian product of a and b. Thus we can form $\{\langle x, y\rangle \in \mathbf{prod}\,ab : \mathsf{eq}\,(fx)y \simeq \top\}$ (see Remark 3.4) and obtain the desired operation. Note that both (i) and (ii) hold in **EST**.

3.3. **Operations and functions.** In operational set theory we have set–theoretic functions and operations. We now wish to address the question of the relationship between them. Note that differences occur both with [9], where we had full replacement at our disposal, and with [18], where use is made of the choice operator.

According to Remark 3.9 (ii), in the theory **EST** to each set–theoretic function F there corresponds an operation which coincides with F on the common domain. In addition, for every operation total on a set a there is a set–theoretic function representing it.

We can consistently (see section 5.3) achieve a sort of "harmony" between functions and operations by assuming Beeson's axiom **FO** (see [5]). **FO** asserts that every set–theoretic function *is* an operation, more precisely[5]:

$$(\mathbf{FO}) \quad \forall f\,(Fun(f) \to \forall x \forall y\,(\langle x, y\rangle \in f \leftrightarrow fx \simeq y)).$$

From Remark 3.9 (ii), when working in the theory **ESTE**, the set $\mathsf{exp}\,ab$ contains a representative of each total operation $f : a \to b$. If we add **FO** to **ESTE** then every element of the set $\mathsf{exp}\,ab$ *is* an operation from a to b, that is

$$f \in \mathsf{exp}\,ab \to \forall x \in a \forall y \in b\,(\langle x, y\rangle \in f \leftrightarrow fx \simeq y).$$

One might now wonder if it is consistent to assume the existence of a set of *all operations* from a to b:

$$\mathsf{op}ab := \{\, f : \forall x \in a\,\exists y \in b\,(fx \simeq y)\}.$$

Pierluigi Minari has observed that if $\mathsf{op}ab$ is defined (and hence is a set), then one can reproduce the fixed point argument of Proposition 3.8.

Lemma 3.10. $\mathbf{EST} + \forall a \forall b\,\exists c(\mathsf{op}\,ab = c)$ *is inconsistent.*

The interaction between operations and functions is well exemplified in the section 3.5 on ω–Iteration in the theory **ESTE**.

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that the proof of the existence of dependent products in Proposition 4.3 of [9] needs exponentiation, and thus in the present context requires the theory **ESTE**.

[5]Unfortunately, in [9], section 5, the axiom **FO** appears to be stated incorrectly. However, the correct principle is used in the interpretation in Theorem 6.4.

3.4. **Choice principles.** The full axiom of choice is validated in constructive type theory, where the Curry–Howard correspondence holds. However, the axiom of choice is not constructively acceptable in the context of set theory with extensionality and (bounded) separation, since it implies the (bounded) law of excluded middle by a well known argument (see [13] and [20]).

It is thus natural to ask what is the status of choice principles for operations.

In addition, as Feferman's theory **OST** is formulated with a choice operator ([17]), it is also worth exploring what is the status of such an operator on the basis of **EST**.

First of all we consider two forms of choice *for operations*. Let **OAC** be the following principle:

$$\forall x \in a\, \exists y\, \varphi(x, y) \to \exists f\, \forall x \in a\, \varphi(x, fx). \tag{7}$$

Let **GAC** be its generalized class form:

$$\forall x\, (\varphi(x) \to \exists y\, \psi(x, y)) \to \exists f\, \forall x\, (\varphi(x) \to \psi(x, fx)). \tag{8}$$

Finally, let **GAC**! be **GAC** with the uniqueness restriction on the quantifier $\exists y$ in the antecedent of (8).

Lemma 3.11. *(i)* $\mathbf{EST} + \mathbf{OAC}$ *proves* $\varphi \vee \neg\varphi$ *for arbitrary bounded formulas.*
(ii) Moreover, $\mathbf{EST} + \mathbf{GAC}$ *and* $\mathbf{EST} + \mathbf{GAC}\,!$ *are inconsistent.*

Proof. (*i*) The standard argument, as presented for example by Goodman and Myhill [20], can be applied here, too. (*ii*) See Beeson [7, p. 228] or [9] Lemma 5.4. □

Let's consider Feferman's choice operator. Uniform choice is one of the principles of **OST** and is defined as follows (for a new constant **C**):

$$(\mathbf{C}) \qquad \exists x\, (fx \simeq \top) \to (\mathbf{C}f\downarrow \wedge f(\mathbf{C}f) \simeq \top).$$

In [21], Theorem 6, Jäger shows that the theory $\mathbf{KP}_\omega$ + (AC) is a subsystem of **OST** (where $\mathbf{KP}_\omega$ is Kripke–Platek set theory with infinity axiom). An essential part of the proof consists in showing that **OST** proves bounded collection and that it proves the axiom of choice. The axiom of Choice is here taken in the form

$$(AC) \qquad \forall x \in a \exists y (y \in x) \to \exists F (Fun(F) \wedge Dom(F) = a \wedge \forall x \in a\, (F(x) \in x)).$$

It is not difficult to see that Jäger's proof that bounded Collection and (AC) hold in **OST** carries through to **EST** plus (**C**).

Thus to conclude: **EST** *plus* (**C**) *proves bounded Collection and* (AC). *Due to the latter fact, this theory is constructively unacceptable.*

3.5. **The ω–iteration theorem in ESTE.** We now show that in the theory **ESTE** we can prove the existence of an operation of ω-iteration.

First of all, note that strong infinity allows us to derive bounded induction on the natural numbers. In the following we also write 0 for $\emptyset$.

$$(\Delta_0 - IND_\omega) \qquad \varphi(0) \wedge \forall x \in \omega\, (\varphi(x) \to \varphi(\mathsf{Suc}\, x)) \to \forall x \in \omega\, (\varphi(x)),$$

where $\varphi(x)$ is Δ_0.

Lemma 3.12. *The principle* $(\Delta_0 - IND_\omega)$ *holds in* **EST**.

Proof. This is proved by a simple application of proposition 3.3 and strong infinity. □

In the reminder of this section let F be a set–theoretic function with domain a and range $\subseteq a$, and $x \in a$. Let $Iter(H, F, a, x)$ be the *bounded* formula expressing the fact that $Fun(H)$, $Dom(H) = \omega$, $Ran(H) \subseteq a$ and H is defined by iterating F along ω with initial value x, i.e.

$$H(0) = x \wedge \forall n \in \omega(H(\mathsf{Suc}\, n) = F(H(n))).$$

By $(\Delta_0 - IND_\omega)$ we easily verify the following.

Lemma 3.13. EST *without ω-iteration proves:*

$$Iter(H, F, a, x) \wedge Iter(G, F, a, x) \to H = G.$$

Thus the IT-operator chooses the unique such H uniformly in the data F, a, x.

Let's now consider the following bounded formula: $Iter^*(H, F, \mathsf{Suc}\, m, a, x)$ expressing the fact that $Fun(H)$ and $Dom(H) = \mathsf{Suc}\, m$ and $Ran(H) \subseteq a$ and H is defined by iterating F along $\mathsf{Suc}\, m$ with initial value x.

By bounded induction on the natural numbers we also have the following analogue of lemma 3.13.

Lemma 3.14. EST *without ω-iteration proves:*

$$Iter^*(H, F, \mathsf{Suc}\, m, a, x) \wedge Iter^*(G, F, \mathsf{Suc}\, j, a, x) \to (\forall n \in m \cap j)(H(n) = G(n)).$$

In addition, the following holds by uniform exponentiation exp and $(\Delta_0 - IND_\omega)$.

Lemma 3.15. ESTE *proves:*

$$\forall m \in \omega(\exp(\mathsf{Suc}\, m)a) \downarrow \textit{(and hence is a set).}$$

We write $a^{\mathsf{Suc}\, m}$ for $\exp(\mathsf{Suc}\, m)a$.

Theorem 3.16. ESTE *proves ω-iteration.*

Proof. We first prove the following:

$$(9) \qquad \forall m \in \omega\, \exists G \in a^{\mathsf{Suc}\, m}\, Iter^*(G, F, \mathsf{Suc}\, m, a, x).$$

Observe that we can apply $(\Delta_0 - IND_\omega)$ to verify (9) (here it is essential to have a set bound for G). The case $m = 0$ is obvious; at the successor step $m = \mathsf{Suc}\, j$, we simply expand any function G' such that $Iter^*(F, G', \mathsf{Suc}\, j, a, x)$ (which exists by IH) with the pair $\langle \mathsf{Suc}\, j, F(G'(j)) \rangle$. The resulting set G satisfies $Iter^*(G, F, (\mathsf{Suc}\, m), a, x)$. For every $m \in \omega$ let

$$J(F, a, x, m) = \{G \in a^{\mathsf{Suc}\, m} : Iter^*(F, G, \mathsf{Suc}\, m, a, x)\}$$

is a set. By uniform bounded separation (see proposition 3.3), $J(F, a, x, m)$ can be regarded as an application term, as well as

$$H(F, a, x) = \bigcup \bigcup \{J(F, a, x, m) : m \in \omega\},$$

which is a set by explicit union, explicit replacement (im) and strong infinity. Now, $H(F, a, x)$ is a set (uniformly in F, a, x) and in fact a function with domain ω and range a, defined by iterating F along ω with initial value $x \in a$ (apply the uniqueness lemma above and (9)). Hence we can choose $IT = \lambda F \lambda a \lambda x. H(F, a, x)$. $\square$

4. Relations with other theories

As already mentioned, the theory **EST** may be regarded as the pure and explicit fragment of **COST** ([9]). In particular, there are no urelements, no $\in$–induction and no implicit principles, i.e. Strong Collection and Subset Collection.[6]

We now wish to explore the relations between **EST** and the operational theories **IZFR** of [5] and **OST** of [17]. We also clarify the relation of **EST** with Friedman's system **B** ([19]).

4.1. **Relation with Beeson's IZFR.** The theory **IZFR** is formulated on the basis of Beeson's logic of partial terms, LPT (see [4], [3]). We here consider a variant of **IZFR** with the application predicate *App* in place of LPT.

The theory has natural numbers as urelements, and is thus formulated in an extension of $\mathcal{L}^O$ with two predicates, S and N, for being a set and a natural number, respectively. In addition, there are constants 0, $\mathsf{Suc}_{\mathcal{N}}$, $\mathbf{d}$ for the natural number zero, successor and case distinction on the natural numbers, respectively. Finally, there are a new constant $\mathbf{P}$ for powerset and one c_φ for each primitive formula φ. A formula is *primitive* if it does not contain *App* or any constant c_ψ.

The theory **IZFR** is based on intuitionistic logic with equality and includes the following principles.

(1) **Applicative axioms and extensionality** as in **EST**.

(2) **Basic set–theoretic axioms:** empty set, pair, union, image, all essentially as in **EST**. Note that in the presence of urelements the axiom of pair, for example, is written as follows:

$$S(\mathsf{pair}\, yz) \wedge \forall x\, (x \in \mathsf{pair}\, yz \leftrightarrow x = y \vee x = z).$$

In addition:

$\in$-induction axiom schema:

$$(\in -IND) \qquad \forall x\, (\forall y \in x\, \varphi(y) \to \varphi(x)) \to \forall x\, \varphi(x).$$

The axiom of infinity, asserting the existence of a set of natural numbers as urelements.

(3) **Ontological axiom and Natural numbers:** The following axiom:

$$z \in x \to S(x).$$

In addition, principles expressing the desired properties of successor on the natural numbers and distinction by numerical cases and the schema of full induction on the natural numbers.

(4) **Separation:**

$$(SEP) \qquad S(c_\varphi(a, y_1, \ldots, y_n)) \wedge \forall x(x \in c_\varphi(a, y_1, \ldots, y_n) \leftrightarrow x \in a \wedge \varphi(x, y_1, \ldots, y_n)),$$

where φ is primitive.

(5) **Powerset**:

$$(POW) \qquad S(\mathbf{P}a) \wedge \forall x\, (x \in \mathbf{P}a \leftrightarrow S(x) \wedge \forall z \in x(z \in a)).$$

[6] As to the term 'implicit', we mean that strong collection and subset collection have no associated operation witnessing the sets asserted to exist, uniformly depending on the given data. For instance, if $\forall x \in a \exists y \varphi(x, y)$, by collection there exists some b, such that $\forall x \in a \exists y \in b\, \varphi(x, y, c)$; this schema is called implicit, since no operation coll_φ is assumed to exist, such that $\mathsf{coll}_\varphi(a, c) \downarrow$ and it yields d such that $\forall x \in a \exists y \in d\, \varphi(x, y, c)$.

It is well–known that intuitionistic set theory with natural numbers as urelements can be interpreted in the corresponding "pure" (i.e. set only) theory. See e.g. Beeson [3], p. 166 (exercises 7 and 8). As a consequence, we can prove the following proposition.

Proposition 4.1. **IZFR** *is interpretable in* **EST**+ *(SEP) + (POW) + (∈–Ind).*

Remark 4.2. The referee has asked about the converse direction of proposition 4.1. As far as we can see, there is no direct interpretation of the theory **EST**+ (SEP) + (POW) + (∈–Ind) in **IZFR** because of the membership operation el and its corresponding axiom.

4.2. **Relation with OST.** Let **OST** be the theory defined in [17], see also [21]. Briefly, **OST** may be formulated in an extension of $\mathcal{L}^O$ with constants $\top$, $\bot$, **non**, **dis**, **all** and **C**.[7] The theory **OST** is based on *classical logic* and includes the following principles.

1. **Applicative axioms and extensionality** as in **EST**.
2. **Basic set–theoretic axioms**: empty set, pair, union, infinity, ∈–induction (all formulated as in Zermelo–Fraenkel set theory).
3. **Logical operations axioms.** Let $\mathbb{B} := \{\top, \bot\}$ (which is a set by pair).
   (i) $\top \neq \bot$
   (ii) $(\mathsf{el} : \mathbf{V}^2 \to \mathbb{B}) \wedge \forall x \forall y\, (\mathsf{el}\, xy \simeq \top \leftrightarrow x \in y)$
   (iii) $(\mathbf{non} : \mathbb{B} \to \mathbb{B}) \wedge \forall x \in \mathbb{B}\, (\mathbf{non}(x) \simeq \top \leftrightarrow x \simeq \bot)$
   (iv) $(\mathbf{dis} : \mathbb{B}^2 \to \mathbb{B}) \wedge \forall x, y \in \mathbb{B}\, (\mathbf{dis} xy \simeq \top \leftrightarrow (x \simeq \top \vee y \simeq \top))$
   (v) $(f : a \to \mathbb{B}) \to (\mathbf{all} fa \in \mathbb{B} \wedge (\mathbf{all} fa \simeq \top \leftrightarrow \forall x \in a\, (fx \simeq \top)))$.
4. **Operational set–theoretic axioms**: uniform bounded separation and image (as in **EST**, with $\mathbb{B}$ replacing Ω) and the uniform choice principle (**C**) as defined in section 3.4.

Note first of all that ∈–induction implies full induction on the natural numbers.

We now show that in the presence of the choice operator and of full induction on the natural numbers, we can derive the existence of an ω–iterator.

Lemma 4.3 (**OST**). **OST** *proves* ω*-iteration.*

Proof. Similarly as in Theorem 3.16 we here show that for any set–theoretic function F with domain a and range $\subseteq a$, for $x \in a$

$$\forall m \in \omega \exists G\, [Iter^*(G, F, \mathsf{Suc}\, m, a, x)].$$

Note, however, that in the present case, where exponentiation is not available, the existential quantifier is unbounded. The claim is hence proved by unbounded induction on the natural numbers, which is available in **OST**. We can now note that by proposition 3.3 there is a term, say t_{Iter^*}, representing the Δ_0 formula $Iter^*(G, F, \mathsf{Suc}\, m, a, x)$, that is

$$\forall m \in \omega \exists G [t_{Iter^*}(G, F, \mathsf{Suc}\, m, a, x) \simeq \top].$$

[7] The constants pair, un, IT, $\emptyset, \omega$ of $\mathcal{L}^O$ are not needed for defining **OST**. Note also that in [21] and subsequent papers, Jäger introduces a constant for the bounded existential quantifier, with corresponding axiom, instead of **all**. In [22] Jäger investigates the proof–theoretic strength of extensions of **OST** by operators for Powerset and unbounded Existential quantifier.

We can now apply uniform choice (**C**) to obtain

$$\forall m \in \omega\, [\mathbf{C}(\lambda y.t_{Iter^*}(y, F, \mathsf{Suc}\, m, a, x)) \downarrow$$
$$\wedge\, t_{Iter^*}(\mathbf{C}(\lambda y.t_{Iter^*}(y, F, \mathsf{Suc}\, m, a, x)), F, \mathsf{Suc}\, m, a, x) \simeq \top].$$

Thus

$$\forall m \in \omega\, [\mathbf{C}(\lambda y.t_{Iter^*}(y, F, \mathsf{Suc}\, m, a, x)) \downarrow$$
$$\wedge\, Iter^*(\mathbf{C}(\lambda y.t_{Iter^*}(y, F, \mathsf{Suc}\, m, a, x)), F, \mathsf{Suc}\, m, a, x)].$$

We deduce that $\lambda m.\mathbf{C}(\lambda y.t_{Iter^*}(y, F, \mathsf{Suc}\, m, a, x)) : \omega \to \mathbf{V}$. We can now apply im and un to obtain the iterator:

$$\lambda F \lambda a \lambda x.\mathsf{un}\, (\mathsf{im}\, (\lambda m.\mathbf{C}(\lambda y.t_{Iter^*}(y, F, \mathsf{Suc}\, m, a, x)))\omega).$$

□

Let (**EM**) denote the principle of Excluded Middle. Let **P** be a new constant for powerset and (**P**) denote uniform powerset (that is the pure, i.e. set only, version of **IZFR**'s (POW)):

$$(\mathbf{P}) \qquad \mathbf{P} : \mathbf{V} \to \mathbf{V} \wedge \forall a \forall x\, (x \in \mathbf{P}a \leftrightarrow \forall z \in x(z \in a)).$$

Proposition 4.4. *(i)* **EST** $+ (\mathbf{C}) + (\in -IND) + (\mathbf{EM}) = \mathbf{OST}$.
(ii) **ESTE** $+ (\mathbf{C}) + (\in -IND) + (\mathbf{EM}) = \mathbf{OST} + \mathbf{P}$.

Proof. (i): Note first of all that in the presence of **EM**, $\Omega = \mathbb{B}$. The applicative axioms, extensionality and the operational axioms of membership, separation and image in **EST** and **OST** are thus equivalent. Showing first of all that **EST** is a subtheory of **OST**, we note that one can show that in the latter theory there are terms representing operations of unordered pair and union (see [17], Corollary 2). The same corollary of Feferman shows that in **OST** we can define constants for the emptyset and for the first infinite ordinal. Thus one can easily derive **EST**'s axioms of emptyset and infinity (where (ω 2) requires set–induction). Finally, by Lemma 4.3 we obtain ω–iteration.

In the opposite direction, showing that **OST** is contained in **EST** $+ (\mathbf{C}) + (\in -IND) + (\mathbf{EM})$, we note first of all that the implicit axioms of pair and union are consequences of their explicit counterparts. Infinity follows from (ω1). As to the logical operations axioms, we can interpret $\bot$ and $\top$ with $\emptyset$ and $\{\emptyset\}$ (i.e. $\mathsf{pair}\,\emptyset\emptyset$), respectively. Finally, by Lemma 3.2, we may let $\mathbf{non} = \lambda x.\mathsf{imp}\, x\emptyset$, $\mathbf{dis} = \mathsf{or}$ and $\mathbf{all} = \mathsf{all}$.

(ii) To see that **ESTE** is contained in $\mathbf{OST} + \mathbf{P}$, note that for set a and b the following is a set

$$D \;:=\; \{F \in \mathbf{P}(\mathbf{prod}\, ab) : Fun(F) \wedge Dom(F) = a \wedge Ran(F) \subseteq b\}.$$

By Proposition 3.3 the set D may be regarded as an application term, too, so that $\lambda a \lambda b.D$ uniformly represents exponentiation.

We now show that in the given extension of **ESTE** there is an application term representing the powerset operation. We note first of all that

$$\forall F \in \mathbb{B}^a \exists u\, (\forall x \in a(\langle x, \top\rangle \in F \leftrightarrow x \in u)).$$

Let's write $t(a, F, u)$ or simply t for the term representing the bounded formula $\forall x \in a(\langle x, \top\rangle \in F \leftrightarrow x \in u)$. We can thus apply **OST**'s choice operator to obtain

$$\forall F \in \mathbb{B}^a[(\mathbf{C}\lambda y.t) \downarrow \wedge \forall x \in a\, (\langle x, \top\rangle \in F \leftrightarrow x \in (\mathbf{C}\lambda y.t))].$$

Thus we have an operation $\lambda F.(\mathbf{C}\lambda y.t(a,F,y)) : \mathbb{B}^a \to \mathbf{V}$. We can thus apply im to obtain $\lambda a.\mathsf{im}\,(\lambda F.(\mathbf{C}\lambda y.t(a,F,y)))\mathbb{B}^a$, which represents the powerset operation. □

4.3. **Relation with Friedman's system B.** The theory **EST** has analogies with Friedman's constructive set theory **B** deprived of the principle of Δ_0–Dependent Choice (also called Limited Dependent Choice, **LDC** in [19]. See also [3]). Let's call $\mathbf{B}^-$ the system obtained from **B** by omitting **LDC**. It is easy to see that $\mathbf{B}^-$ can be interpreted in **EST**.[8] Friedman's system includes a principle of abstraction which takes the place of **ZF**'s replacement. This states:
$\forall x\, \exists z\, (z = \{\{u \in x : \varphi(\vec{y}, u)\} : \vec{y} \in x\})$, for $\varphi(\vec{y}, u)$ a Δ_0 formula.
Abstraction is clearly derivable in **EST** by bounded separation and image.

5. Proof theoretic reduction

In this section we show that the proof–theoretic strength of **EST** is the same as that of **PA**.

Theorem 5.1 (The recursive content of **EST**). *A number theoretic function f is of type $\omega \to \omega$ provably in* **EST** *iff f is provably recursive in* **PA** *(hence in* **HA***).*

The proof is given in two steps, the lower bound and the upper bound.

5.1. Lower bound.

Theorem 5.2. HA *is interpretable in* **EST**.

Proof. The domain of the interpretation is ω; the constant '0' is interpreted as the empty set, while the successor operation is the map $x \mapsto \mathsf{Suc}\, x$. The usual properties of 0 and successor are easily verified. Also **HA**'s induction schema is given by $\Delta_0 - IND_\omega$ (Lemma 3.12). We now verify that we can define two ternary relations SUM and TIMES on ω, which exist as sets and encode the graphs of addition and multiplication on ω.

Existence of SUM

Let S be the set–theoretic function corresponding to Suc; this function exists in **EST** (by uniform union, pairing, (ω1), explicit separation, image constructor and extensionality). Then by ω-iteration there exists an operation f such that, for $m \in \omega$,

$$fm = IT(S, \omega, m).$$

By explicit replacement there exists the set

$$H = \mathsf{im}\,(\lambda m.fm, \omega)$$

of all set–theoretic functions defined by iterating S from m, when $m \in \omega$. Let $\omega^3 = \mathbf{prod}\,(\omega(\mathbf{prod}\,\omega\omega))$. Then by explicit bounded separation there is a set:

$$\begin{aligned}\mathsf{SUM} = \{&u \in \omega^3 : (\exists F \in H)(\exists x, y, z \in \omega)[u = \langle x, y, z\rangle \\ &\wedge Fun(F(x)) \wedge Dom(F(x)) = \omega \wedge Ran(F(x)) \subseteq \omega \wedge \langle y, z\rangle \in F(x)]\}.\end{aligned}$$

We claim that SUM is the graph of number theoretic addition.

First of all

$$\forall x \in \omega\, \forall y \in \omega\, \exists z \in \omega(\langle x, y, z\rangle \in \mathsf{SUM}).$$

[8] The interpretation of $\mathbf{B}^-$ in **EST** can also be seen as another way of obtaining the lower bound for **EST**'s proof–theoretic strength (see section 5.1).

Indeed, given x, $y \in \omega$, there exists a set–theoretic function $F(x) := IT(S, \omega, x)$, which is defined by ω-iteration with initial value x. Hence for every $y \in \omega$ we can find $z \in \omega$ such that $\langle y, z\rangle \in F(x)$. Then we can also verify uniqueness, for $x, y, z \in \omega$:

$$\langle x, y, z\rangle \in \mathsf{SUM} \wedge \langle x, y, w\rangle \in \mathsf{SUM} \to z = w.$$

Indeed, assume $\langle x, y, z\rangle \in \mathsf{SUM}$ and $\langle x, y, w\rangle \in \mathsf{SUM}$. Then there exist elements u_1,u_2,u_3, v_1,v_2,v_3 in ω , and $G, G' \in H$ such that

$$\langle x, y, z\rangle = \langle u_1, u_2, u_3\rangle \wedge\ Fun(G(u_1)) \wedge Dom(G(u_1)) = \omega \wedge Ran(G(u_1)) \subseteq \omega \\ \wedge\langle u_2, u_3\rangle \in G(u_1) \\ \langle x, y, w\rangle = \langle v_1, v_2, v_3\rangle \wedge\ Fun(G'(v_1)) \wedge Dom(G'(v_1)) = \omega \wedge Ran(G'(v_1)) \subseteq \omega \\ \wedge\langle v_2, v_3\rangle \in G'(v_1).$$

By ordered pairing:

$$Fun(G(x)) \wedge Dom(G(x)) = \omega \wedge Ran(G(x)) \subseteq \omega\ \wedge \langle y, z\rangle \in G(x) \\ Fun(G'(x)) \wedge Dom(G'(x)) = \omega \wedge Ran(G'(x)) \subseteq \omega\ \wedge \langle y, w\rangle \in G'(x).$$

Since G and G' are both defined by iterating S from the same initial value x, they coincide by lemma 3.13 and hence $z = w$.

Existence of TIMES.

Let F_m be the set–theoretic function:

$$\{c \in \omega^2 : (\exists u, v \in \omega)(c = \langle u, v\rangle \wedge \langle u, m, v\rangle \in \mathsf{SUM})\}$$

which exists by explicit separation. By ω-iteration, there exists an operation g such that for all $m \in \omega$:

$$gm = IT(F_m, \omega, 0).$$

Clearly $(gm)(n) = m \cdot n$. By explicit replacement there exists the set $G = \mathsf{im}\,(\lambda m.(gm))\omega$. Hence by explicit separation there exists a set:

$$\mathsf{TIMES} = \{u \in \omega^3 : (\exists H \in G)(\exists x, y, z \in \omega)(u = \langle x, y, z\rangle \\ \wedge Dom(H(x)) = \omega \wedge Ran(H(x)) \subseteq \omega \wedge \langle y, z\rangle \in H(x)\}.$$

Now, given x, $y \in \omega$, there exists a function $H(x) := IT(F_m, \omega, 0)$ defined by ω-iteration with initial value 0, and we can choose $\langle y, z\rangle \in H(x)$. Hence

$$(\forall x \in \omega)(\forall y \in \omega)(\exists z \in \omega)(\langle x, y, z\rangle \in \mathsf{TIMES}).$$

The verification of uniqueness, for x, y, z, w in ω

$$\langle x, y, z\rangle \in \mathsf{TIMES} \wedge \langle x, y, w\rangle \in \mathsf{TIMES} \to z = w$$

is similar to the case of addition and follows again by lemma 3.13. □

5.2. **Upper bound.** In this section we introduce two auxiliary theories, $\mathbf{ECST}^*$ and $\mathbf{T_c}$, and show that: (i) (a suitable extension of) **EST** can be interpreted in $\mathbf{ECST}^*$; (ii) $\mathbf{ECST}^*$ can be interpreted in $\mathbf{T_c}$ and thence has the same strength as **HA**.

5.2.1. *Elementary Constructive Set Theory.* In [2] the authors introduce a subsystem of **CZF** called **ECST** (for Elementary Constructive Set Theory). They show that many standard set–theoretic constructions may be carried out already in this fragment of constructive set theory. We shall here be interested in a strengthening of **ECST** by addition of exponentiation.

The language of **ECST** is the same language as that of Zermelo–Fraenkel set theory. In this context, the notion of Δ_0 formula is the standard one, that is, a formula is Δ_0 or bounded if no unbounded quantifier occur in it.

Definition 5.3. The theory **ECST** includes the principles of first order intuitionistic logic plus the following set–theoretic principles.

(1) Extensionality;
(2) Pair;
(3) Union;
(4) Δ_0-Separation;
(5) Replacement;
(6) Strong Infinity.

Here Strong Infinity is the following principle:

$$\exists a\,[Ind(a) \wedge \forall z\,(Ind(z) \rightarrow a \subseteq z)],$$

where we use the following abbreviations:

- $Empty(y)$ for $(\forall z \in y)\perp$,
- $Suc(x,y)$ for $\forall z\,[z \in y \leftrightarrow z \in x \vee z = x]$,
- $Ind(a)$ for $(\exists y \in a)Empty(y) \wedge (\forall x \in a)(\exists y \in a)Suc(x,y)$.

As usual, we write ω also for the set defined by strong infinity (which is unique by extensionality).

Note that **ECST** differs from **CZF** in that it only has Replacement in place of Strong Collection and it omits both Subset Collection and $\in$-Induction. Rathjen ([29]) has shown that **ECST** is very weak, as for example it does not prove the existence of the addition function on ω.

Let exponentiation be the axiom:

$$\forall a,b\,\exists c\,\forall F\,(F \in c \leftrightarrow (Fun(F) \wedge Dom(F) = a \wedge Ran(F) \subseteq b)),$$

where as usual Fun is a bounded formula expressing the fact that F is a set–theoretic function, $Dom(F)$ and $Ran(F)$ are the domain and range of F, respectively.

Definition 5.4. The theory **ECST*** is obtained from **ECST** by adding the axiom of exponentiation.

To establish the upper bound we need to show that (a suitable extension of) **EST** can be interpreted in **ECST*** and that in turn **ECST*** can be reduced to **PA**. We start from the latter problem.

5.2.2. *Reducing* **ECST*** *to* **PA**. We here modify the interpretation of [9] of a system of constructive set theory with urelements in a classical theory, $\mathbf{T_c}$, of abstract self–referential truth. The final result relies on the fact that $\mathbf{T_c}$ is conservative over **PA** ([7]). The main idea of the interpretation in [9] was to rephrase, in the new context, Aczel's interpretation of **CZF** in Constructive Type Theory and combine it with a suitable form of realizability.

First of all, let's recall the theory $\mathbf{T_c}$.

5.2.3. *The theory* $\mathbf{T_c}$. The basic first order language $\mathcal{L}_T$ of $\mathbf{T_c}$ comprises the predicate symbols $=$, $\mathcal{T}$, $\mathcal{N}$, the binary function symbol ap (application), combinators K, S, successor, predecessor, definition by cases on numbers, pairing with projections. Terms are inductively generated from variables and individual constants via application. As usual $ts := ap(t, s)$; missing brackets are restored by associating to the left. Formulas are inductively generated from atoms of the form $t = s$, $\mathcal{T}(t)$, $\mathcal{N}(t)$ by means of sentential operations and quantifiers. We adopt the following conventions:

(i) By $[\varphi]$ we denote a term representing the propositional function associated with φ and such that $\mathbf{FV}([\varphi]) = \mathbf{FV}(\varphi)$. We fix distinct closed *terms* $\hat{\forall}$, $\hat{\exists}$, $\hat{\neg}$ $\hat{\wedge}$, ..., naming the *logical constants*. In addition, $\hat{=}$, $\hat{N}$ name the equality and the number predicates, respectively. Then $[\varphi]$ is inductively defined by stipulating $[t = s] = (\hat{=}\, ts)$, $[\mathcal{N}(s)] = \hat{\mathcal{N}}s$, $[\mathcal{T}(s)] = s$ and closing under application of the "small hat" operations, noting that $[\forall x\varphi] = \hat{\forall}(\lambda x[\varphi])$, $[\exists x\varphi] = \hat{\exists}(\lambda x[\varphi])$.

(ii) Given a formula φ we define abstraction by letting $\{x : \varphi\} := \lambda x.[\varphi]$.

(iii) We define intensional membership, η, as follows:

$$x\,\eta\,a := \mathcal{T}(ax);$$
$$x\,\bar{\eta}\,a := \mathcal{T}(\hat{\neg}(ax)).$$

(iv) The notion of class (or classification) is so specified:

$$\mathbf{Cl}(a) := \forall x\,(x\,\eta\,a \vee x\,\bar{\eta}\,a).$$

(v) A formula φ is *$\mathcal{T}$–positive* iff φ is inductively generated from prime formulas of the form $\mathcal{T}(t)$, $t = s$, $\neg t = s$, $\mathcal{N}(t)$, $\neg\mathcal{N}(t)$ by means of $\vee$, $\wedge$, $\forall$, $\exists$.

(vi) A formula φ is *$\mathcal{T}$–positive operative in* v (in short, a *positive operator*) iff φ belongs to the smallest class of formulas inductively generated from prime formulas of the form $\mathcal{T}(t)$, $s\,\eta\,v$, $t = s$, $\neg t = s$, $\mathcal{N}(t)$, $\neg\mathcal{N}(t)$ by means of $\vee$, $\wedge$, $\forall y$, $\exists y$, where y is distinct from v and v does not occur in t, s.

(vii) For each formula φ, fixed points are defined by letting:

$$\mathbf{I}(\varphi) := \mathbf{Y}(\lambda v.\{x : \varphi(x, v)\})$$

where $\mathbf{Y}$ is Curry's fixed point combinator.

The system $\mathbf{T_c}$ comprises the following prinicples, besides *classical* predicate calculus with equality.

(1) The base theory $\mathbf{TON}^-$ (see e. g. [24]), which formalises the notion of total extensional combinatory algebra expanded with natural numbers. This includes the obvious axioms on combinators, pairing, projections. In addition, closure axioms for the predicate $\mathcal{N}$ defining a copy of the natural numbers, together with number theoretic conditions on the basic operations of successor SUC, predecessor $PRED$, 0, definition by cases on the natural numbers.

(2) A fixed point axiom (**Tr**) for abstract truth

$$\mathbf{Tr}(x, \mathcal{T}) \leftrightarrow \mathcal{T}(x).$$

Here $\mathbf{Tr}(x, \mathcal{T})$ is a formula encoding the closure properties:

$$\frac{a = b}{\mathcal{T}[a = b]} \qquad \frac{\neg(a = b)}{\mathcal{T}[\neg(a = b)]} \qquad \frac{\mathcal{N}(a)}{\mathcal{T}[\mathcal{N}(a)]} \qquad \frac{\neg\mathcal{N}(a)}{\mathcal{T}[\neg\mathcal{N}(a)]}$$

for the basic atomic formulas with $=$ and $\mathcal{N}$. Further, the following additional clauses for the compound formulas:

$$\frac{\mathcal{T}(a)}{\mathcal{T}(\hat{\neg}\hat{\neg}a)} \qquad \frac{\mathcal{T}a \quad \mathcal{T}b}{\mathcal{T}(a\hat{\wedge}b)} \qquad \frac{\mathcal{T}(\hat{\neg}a)\ [\ or\ \mathcal{T}\hat{\neg}b]}{\mathcal{T}(\hat{\neg}(a\hat{\wedge}b))}$$

$$\frac{\forall x\,\mathcal{T}(ax)}{\mathcal{T}(\hat{\forall}a)} \qquad \frac{\exists x\,\mathcal{T}\hat{\neg}ax}{\mathcal{T}(\hat{\neg}\hat{\forall}a)}$$

(3) Consistency axiom: $\neg(\mathcal{T}x \wedge \mathcal{T}\hat{\neg}x)$.

(4) Induction on natural numbers $\mathcal{N}$ for *classes*:

$$\mathbf{Cl}(a) \wedge \mathbf{Clos}_{\mathcal{N}}(a) \to \forall x(\mathcal{N}(x) \to x\,\eta\,a)$$

with $\mathbf{Clos}_{\mathcal{N}}(a) \ := \ 0\eta a \wedge \forall x\,(x\eta a \to (SUCx)\eta a)$.

(5) The principle **GID**, ensuring the minimality of the fixed points: if $\varphi(x,v)$ is a positive operator

$$\mathbf{Clos}_{\varphi}(\psi) \to \forall x\,(x\eta\mathbf{I}(\varphi) \to \psi(x))$$

with $\mathbf{Clos}_{\varphi}(\psi) \ := \ \forall x\,(\varphi(x,\psi) \to \psi(x))$.[9]

$\mathbf{T}^-$ is the theory $\mathbf{T_c}$ without number theoretic induction.

Let **CL** be $\{x : \mathbf{Cl}(x)\}$ (which is provably not a class). Then we can show that **CL** has natural closure conditions which are essential for the interpretation of $\mathbf{ECST}^*$. That is, $\mathbf{T}^-$ is closed under elementary comprehension, generalized disjoint union, generalized disjoint product. It satisfies a form of positive comprehension: if φ is $\mathcal{T}$–positive, then $\mathcal{T}[\varphi] \leftrightarrow \varphi$ and $\forall x\,(x\eta\{u:\varphi\} \leftrightarrow \varphi[u:=x])$. Also a version of the second recursion theorem holds: if φ is positive $\forall x\,(x\eta\mathbf{I}(\varphi) \leftrightarrow \varphi(x,\mathbf{I}(\varphi)))$; for the proofs, see [8], II.9B, II.10A.

Theorem 5.5. $\mathbf{T_c}$ *is proof–theoretically equivalent to* **PA**.

Proof. See [9], Theorem 7.3 or [7]. □

5.2.4. *Reducing* $\mathbf{ECST}^*$ *to* $\mathbf{T_c}$. In the following, unless otherwise stated, we work in the theory $\mathbf{T}^-$. We define a suitable counterpart of a universe $\mathcal{V}_N$ of sets, in a similar vein as in [9] (see also [10], [11]). A point of departure from [9] is however the treatment of infinity, as the subsystem of **COST** utilised there had urelements for natural numbers. For the present purpose it is instead crucial that the set of von Neumann natural numbers is interpreted in our weak theory, so to ensure that strong infinity holds under the given interpretation. For this purpose we add an initial condition to our version of Aczel's universe, adapting to our case a trick of Rathjen ([29]). In particular, in addition to the usual condition which defines sets as elements of the type of iterative sets, we also introduce a separate rule which defines the natural numbers as elements of the same type.

Let (x,y) denote the basic pairing operation which is built-in the axioms of $\mathbf{T}^-$; (x,y,z) stands for $(x,(y,z))$, and, if $u=(x,y,z)$, $u_0=x$, $u_1=y$ and $u_2=z$. Let N be the class $\{x:\mathcal{N}(x)\}$ and

$$N_k := \{m : m\,\eta\,N \wedge m <_N k\},$$

[9] Here $\varphi(x,\psi)$ is the formula obtained by replacing each occurrence of the formula $t\,\eta\,v$ in $\varphi(x,v)$ by means of $\psi(t)$.

where $<_N$ represents the ordering relation on N. Henceforth, we simply write $<$ instead of $<_N$. Note that N_k is a class for every $k\,\eta\,N$. We also write $\sup(a,f)$ for $(1,a,f)$.

Choose by the fixed point theorem an operation ν such that

$$\nu x = sup(N_x,\nu). \tag{10}$$

Informally, the idea is that $sup(N_k,\nu)$ represents the von Neumann ordinal associated to the number k.

The universe of sets $\mathcal{V}_N$ is defined by means of two rules, one for initial finite segments of natural numbers and one for sets:

$$\frac{k\,\eta\,N}{sup(N_k,\nu)\,\eta\,\mathcal{V}_N}$$

and

$$\frac{\mathbf{Cl}(a) \qquad \forall u\,\eta\,a\,(fu\,\eta\,\mathcal{V}_N)}{sup(a,f)\,\eta\,\mathcal{V}_N}.$$

Lemma 5.6. *If $m\,\eta\,N$ and $k\,\eta\,N$ then $N_m = N_k \leftrightarrow m = k$.*

Proof. Obvious from right to left. Conversely, note that, if $N_m = N_k$ and $m \neq k$, we obtain a contradiction. □

Proposition 5.7. *There exists a closed term $\mathcal{V}_N$ such that*

(i)

$$\begin{aligned} a\,\eta\,\mathcal{V}_N \ \leftrightarrow\ & \exists n\,\eta\,N\,(a = \sup(N_n,\nu)) \\ & \vee (a = \sup(a_1,a_2) \wedge \mathbf{Cl}(a_1) \wedge \forall u\,\eta\,a_1\,((a_2u)\,\eta\,\mathcal{V}_N)); \end{aligned}$$

(ii) $\forall x(\mathcal{V}(x,\varphi) \to \varphi(x)) \to \forall x\,(x\eta\mathcal{V}_N \to \varphi(x))$,

where φ is an arbitrary formula and $\mathcal{V}(x,\varphi)$ is an abbreviation for $\exists n\,\eta\,N\,(x = \sup(N_n,\nu)) \vee (x = \sup(x_1,x_2) \wedge \mathbf{Cl}(x_1) \wedge (\forall u\,\eta\,x_1)(\varphi(x_2u)))$.

Proof. See [9], Proposition 8.1. Observe that (ii) is an application of **GID**. □

Note that, as N_i is a class for each $i\,\eta\,N$, and $\nu i = \sup(N_i,\nu)$, we have

$$\sup(N_i,\nu)\,\eta\,\mathcal{V}_N \ \leftrightarrow\ \mathbf{Cl}(N_i) \wedge \forall k\,\eta\,N_i(\nu k\,\eta\,\mathcal{V}_N);$$

hence, by proposition 5.7 (i):

$$a\,\eta\,\mathcal{V}_N \ \leftrightarrow\ a = \sup(a_1,a_2) \wedge \mathbf{Cl}(a_1) \wedge \forall u\,\eta\,a_1\,((a_2u)\,\eta\,\mathcal{V}_N).$$

In the following, applications of proposition 5.7 (ii) will be simply referred to as *proofs by induction on $\mathcal{V}_N$*.

Proposition 5.8. *There are operations assigning $\bar{a}$ and $\tilde{a}$ to each $a\,\eta\,\mathcal{V}_N$ and such that $\mathbf{Cl}(\bar{a})$ and $\tilde{a} : \bar{a} \to \mathcal{V}_N$ (that is $\forall x\eta\bar{a}\,(\tilde{a}x\eta\mathcal{V}_N)$).*

Proof. By induction on $\mathcal{V}_N$, using the recursion theorem. □

We next define recursively an equivalence relation, $\doteq$, on $\mathcal{V}_N$.

If $a \in \mathcal{V}_N$, let

$$Nat(a) := \exists k(k\,\eta\,N \wedge a = \sup(N_k,\nu)).$$

Lemma 5.9. *There exists a term $\doteq$ such that*

$$\begin{aligned} a \doteq b \leftrightarrow\ & a\,\eta\,\mathcal{V}_N \wedge b\,\eta\,\mathcal{V}_N \wedge [\exists k(k\,\eta\,N \wedge N_k = \bar{a} = \bar{b} \wedge \tilde{a} = \tilde{b} = \nu) \vee \\ & \vee (\neg(Nat(a) \wedge Nat(b)) \wedge \forall x\,\eta\,\bar{a}\,\exists y\,\eta\,\bar{b}\,(\tilde{a}x \doteq \tilde{b}y) \wedge \forall y\,\eta\,\bar{b}\,\exists x\,\eta\,\bar{a}\,(\tilde{a}x \doteq \tilde{b}y))]. \end{aligned}$$

Lemma 5.10. *For $a, b, c\,\eta\,\mathcal{V}_N$ the following holds*

(1) $a \doteq a$
(2) $a \doteq b \rightarrow b \doteq a$
(3) $a \doteq b \wedge b \doteq c \rightarrow a \doteq c$.

Definition 5.11. Let $a, b\,\eta\mathcal{V}_N$:

$$a\dot{\in}b \;:=\; \exists x\,\eta\,\bar{b}\,(a \doteq \tilde{b}x).$$

The interpretation proceeds similarly as in [9], section 8. We here present only the most relevant steps of the interpretation.

Lemma 5.12 (Extensionality)**.** *Let $a, b\,\eta\,\mathcal{V}_N$.*

$$\forall x\,\eta\,\mathcal{V}_N\,(x\dot{\in}a \;\leftrightarrow\; x\dot{\in}b) \rightarrow a \doteq b.$$

Proof. **Case 1**: Assume $a = \sup(N_m, \nu)$, $b = \sup(N_k, \nu)$ and

$$\forall x(x\dot{\in}a \;\leftrightarrow\; x\dot{\in}b).$$

This easily implies

$$(\forall i < m)(\exists j < k)(\sup(N_i, \nu) \doteq \sup(N_j, \nu))$$
$$(\forall j < k)(\exists i < m)(\sup(N_j, \nu) \doteq \sup(N_i, \nu)).$$

By lemma 5.9

$$(\forall i < m)(i < k) \wedge (\forall i < k)(i < m),$$

which implies $m = k$, that is by definition $\sup(N_m, \nu) \doteq \sup(N_k, \nu)$.

Case 2: At least one between a, b is generated in $\mathcal{V}_N$ according to the second clause. Suppose $z\,\eta\,\bar{a}$. Then $\tilde{a}z\,\eta\,\mathcal{V}_N$ and $\tilde{a}z\dot{\in}a$, so that by hypothesis, also $\tilde{a}z\dot{\in}b$. Then there exists a y such that $y\,\eta\,\bar{b}$ and $\tilde{a}z \doteq \tilde{b}y$. Similarly one proves the other conjunct in the definition of $a \doteq b$. □

Lemma 5.13. *For $a, b\,\eta\,\mathcal{V}_N$,*

$$\mathcal{T}[a \doteq b] \vee \mathcal{T}[\neg a \doteq b];$$
$$\mathcal{T}[a\dot{\in}b] \vee \mathcal{T}[\neg a\dot{\in}b].$$

Proof. See [9], Lemma 8.12. □

Proposition 5.14. *The structure $\langle\, \mathcal{V}_N, \doteq, \dot{\in}\rangle$ is a model of the theory* **ECST*** *without replacement and exponentiation, provably in* $\mathbf{T_c}$.

Proof. See Proposition 8.1 of [9]. The main differences with that proposition concern extensionality, which is taken care of by Lemma 5.12, and strong infinity, which we address in the following.

Define $\hat{\omega} := sup(N, \mathbf{j})$ where, for $m\,\eta\,N$:

$$\mathbf{j}(m) = sup(N_m, \nu).$$

We need to show that:

(1) $\hat{\omega}\,\eta\,\mathcal{V}_N$ and $\hat{\omega}$ is inductive (i.e. $\hat{\omega}$ contains the empty set and is closed under the set–theoretic successor, as defined within $\mathcal{V}_N$);
(2) if $a\,\eta\,\mathcal{V}_N$ and a is inductive, then $\hat{\omega} \subseteq a$.

The first half of the first claim is obvious by construction. The second half requires class induction. As to the second claim, we assume that a is inductive and by class induction, using lemma 5.13, we show that

$$(\forall i\,\eta\,N)(\exists v\,\eta\,\bar{a})(\tilde{a}v \doteq \mathbf{j}i = \sup(N_i,\nu)).$$

If $i = 0$, we are done by assumption on a. Let $i = SUCm$ and assume by IH that for some $v\,\eta\,\bar{a}$, $\tilde{a}v \doteq \sup(N_m,\nu)$. For $c\,\eta\,\mathcal{V}_N$, let's write $(c \cup \{c\})$ also for the appropriate interpretation of the successor in $\mathcal{V}_N$ (obtained by interpreting pair and union as appropriate). Now $\tilde{a}v\dot{\in}a$; by definition of inductive set, we also know that $(\tilde{a}v \cup \{\tilde{a}v\})\dot{\in}a$ and hence, for some $w\,\eta\,\bar{a}$, $\tilde{a}w\dot{\in}a$ and $\tilde{a}w \doteq (\tilde{a}v \cup \{\tilde{a}v\})$. Then also $(\mathbf{j}m \cup \{\mathbf{j}m\})\dot{\in}a$. Since we can easily verify that

$$(\mathbf{j}m \cup \{\mathbf{j}m\}) \doteq \mathbf{j}(SUCm)$$

we have the expected conclusion $\mathbf{j}(SUCm) \doteq \tilde{a}i$. □

Finally, to give an interpretation of the theory $\mathbf{ECST}^*$ (including replacement and exponentiation) we can define a suitable notion of realisability in the theory $\mathbf{T_c}$. First of all, if φ is a bounded formula of $\mathbf{ECST}^*$, we inductively define a map $\varphi \mapsto \|\varphi\|$, where (roughly) $\|\varphi\|$ collects the proof objects for φ, provided the parameters range over $\mathcal{V}_N$.

Let $\top$ denote the classification which only has the empty classification as element, while $a + b := \{u : u = (u_0, u_1) \wedge ((u_0 = 0 \wedge u_1\,\eta\,a) \vee (u_0 = 1 \wedge u_1\,\eta\,b))\}$ represents the direct sum of a, b.

Definition 5.15.

$$\begin{aligned}
\|\bot\| &= \{e\,\eta\,\top : 0 = 1\};\\
\|a = b\| &= \{e : e = 0 \wedge \exists k(k\,\eta\,N \wedge N_k = \bar{a} = \bar{b} \wedge \tilde{a} = \tilde{b} = \nu)\}\\
&\quad + \{e :\ e = (e_0, e_1) \wedge \neg(Nat(a) \wedge Nat(b)) \wedge\\
&\quad \wedge \forall u\,\eta\,\bar{a}\,(e_0u)_0\,\eta\,\bar{b} \wedge (e_0u)_1\,\eta\,\|\tilde{a}u = \tilde{b}(e_0u)_0\| \wedge\\
&\quad \wedge \forall v\,\eta\,\bar{b}\,(e_1v)_0\,\eta\,\bar{a} \wedge (e_1v)_1\,\eta\,\|\tilde{a}(e_1v)_0 = \tilde{b}v\|\};\\
\|a \in b\| &= \{e : e = (e_0, e_1) \wedge e_0\,\eta\,\bar{b} \wedge e_1\,\eta\,\|a = \tilde{b}e_0\|\};\\
\|\varphi \wedge \psi\| &= \{e : e = (e_0, e_1) \wedge e_0\,\eta\,\|\varphi\| \wedge e_1\,\eta\,\|\psi\|\};\\
\|\varphi \vee \psi\| &= \|\varphi\| + \|\psi\|;\\
\|\varphi \to \psi\| &= \{e : \forall q\,\eta\,\|\varphi\|(eq\,\eta\,\|\psi\|)\};\\
\|\exists x \in a\,\varphi(x)\| &= \{e : e = (e_0, e_1) \wedge e_0\,\eta\,\bar{a} \wedge e_1\,\eta\,\|\varphi(\tilde{a}e_0)\|\};\\
\|\forall x \in a\,\varphi(x)\| &= \{e : \forall u\,\eta\,\bar{a}\,(eu\,\eta\,\|\varphi(\tilde{a}u)\|)\}.
\end{aligned}$$

Formally speaking, the definition of $\|\varphi\|$ above makes sense only after showing by a fixed point argument in $\mathbf{T}^-$ that there exists an operation $H(a, b)$ satisfying the equation for $\|a = b\|$ (hence the definition inductively extends H to arbitrary bounded conditions).

Definition 5.16. Let φ be an arbitrary formula of $\mathbf{ECST}^*$; we inductively define a formula $e \Vdash \varphi$ of $\mathbf{T_c}$ with the same free variables as φ and a fresh variable e:

(1) if φ is a bounded formula of $\mathbf{ECST}^*$, then

$$e \Vdash \varphi \text{ iff } e\,\eta\,\|\varphi\|;$$

else:

(2)

$$\begin{aligned}
e \Vdash \varphi \to \psi &\text{ iff } \forall f (f \Vdash \varphi \to ef \Vdash \psi)\,;\\
e \Vdash \varphi \wedge \psi &\text{ iff } e = (e_0, e_1) \wedge e_0 \Vdash \varphi \wedge e_1 \Vdash \psi\,;\\
e \Vdash \varphi \vee \psi &\text{ iff } (e = (0, e_1) \wedge e_1 \Vdash \varphi) \vee (e = (1, e_1) \wedge e_1 \Vdash \psi)\,;\\
e \Vdash \forall x \in a\, \varphi(x) &\text{ iff } \forall x\, \eta\, \bar{a}\, (ex \Vdash \varphi(\tilde{a}x))\,;\\
e \Vdash \exists x \in a\, \varphi(x) &\text{ iff } e = (e_0, e_1) \wedge e_0\, \eta\, \bar{a} \wedge e_1 \Vdash \varphi(\tilde{a}e_0)\,;\\
e \Vdash \exists x\, \varphi &\text{ iff } e = (e_0, e_1) \wedge e_0\, \eta\, \mathcal{V}_N \wedge e_1 \Vdash \varphi(e_0)\,;\\
e \Vdash \forall x\, \varphi &\text{ iff } \forall x\, \eta\, \mathcal{V}_N\, (ex \Vdash \varphi(x))\,.
\end{aligned}$$

Lemma 5.17. *Let φ be a bounded formula of* $\mathbf{ECST}^*$. *Then* $\mathbf{T}^-$ *proves*

$$\vec{x} \in \mathcal{V}_N \to Cl(\|\varphi(\vec{x})\|); \tag{11}$$

$$e \Vdash \varphi(\vec{x}) \text{ iff } e\, \eta\, \|\varphi(\vec{x})\|. \tag{12}$$

Theorem 5.18. *Every theorem of* $\mathbf{ECST}^*$ *is realized in* $\mathbf{T_c}$*, i.e. if* $\mathbf{ECST}^* \vdash \varphi(\vec{x})$*, then there exists a closed term e such that, provably in* $\mathbf{T_c}$*, for* $\vec{a} \in \mathcal{V}_N$

$$e\vec{a} \Vdash \varphi(\vec{a}).$$

Proof. See Theorem 8.22 of [9]. □

5.3. **Interpreting $\mathbf{\Gamma_{BEST}}$ in $\mathbf{ECST}^*$.** Let **BEST** be **ESTE** + **FO**. We shall prove that **BEST** is conservative over $\mathbf{ECST}^*$ for a suitable class of formulas in the common language. This is achieved through two steps. First we give a sequent style formulation of **BEST**, called $\mathbf{\Gamma_{BEST}}$, so that the active formulas are positive in *App* and a partial cut elimination theorem holds. Then we give an asymmetric interpretation of $\mathbf{\Gamma_{BEST}}$ in $\mathbf{ECST}^*$, which yields the final result.

Step 1. We only give a sketch of the theory $\mathbf{\Gamma_{BEST}}$. As usual, capital Greek letters $\Gamma, \Lambda, \ldots$ denote finite sequences of formulas of $\mathbf{\Gamma_{BEST}}$. Sequents are of the form $\Gamma \Rightarrow \Lambda$. The system $\mathbf{\Gamma_{BEST}}$ is an extension of the intuitionistic Gentzen calculus ([31]). The logical rules consist of the usual rules for intuitionistic logic, including cut and =. In addition, there are the structural rules of weakening, exchange and contraction. In the following we first present the axioms and rules involving application; in particular, we include trivial independence conditions on constants for operations. Then we state the main rules for the set–theoretic constructors of $\mathbf{\Gamma_{BEST}}$.

In order to simplify the statements, we extend the language by adding new terms as follows:

(*) if t, s are terms, so are K_t, S_t, pair_t, im_t, sep_t, el_t, exp_t, S_{ts}.[10]

Finally, note that in the following, separation and explicit replacement are split into distinct rules to ease the asymmetric interpretation of section 5.4.

[10] Formally, the special terms can be eliminated by means of a set–theoretically defined ordered pairing operation $\langle -, - \rangle$ and 8 distinct sets $c_1, \ldots, c_8$, e.g. to be identified with distinct elements of ω. For example, K_t, can be identified with $\langle c_1, t \rangle$.

Gentzen-style presentation of non-logical axioms and rules. $\mathbf{\Gamma_{BEST}}$ includes (the closure under substitution of) the following sequents and rules:

(1) Uniqueness:

$$\Gamma, ts \simeq p, ts \simeq q \Rightarrow p = q$$

(2) let C be a constant among K, S, pair, im, sep, el, exp; then

$$\Gamma \Rightarrow \mathsf{C}t \simeq \mathsf{C}_t$$
$$\Gamma \Rightarrow \mathsf{S}_t s \simeq \mathsf{S}_{ts}$$

(3) Combinatory completeness:

$$\Gamma \Rightarrow K_t s \simeq t$$

$$\frac{\Gamma \Rightarrow tr \simeq u \qquad \Gamma \Rightarrow sr \simeq v \qquad \Gamma \Rightarrow uv \simeq w}{\Gamma \Rightarrow \mathsf{S}_{ts} r \simeq w}$$

(4) Independence:

- let $\mathsf{C}^1, \mathsf{C}^2 \in \{\mathsf{K}, \mathsf{S}, \mathsf{pair}, \mathsf{un}\,, \mathsf{im}, \mathsf{sep}, \mathsf{el}, \mathsf{exp}\}$; then

$$\Gamma\,, \mathsf{C}^1 = \mathsf{C}^2 \Rightarrow$$

- let $\mathsf{C}^1, \mathsf{C}^2 \in \{\mathsf{K}, \mathsf{S}, \mathsf{pair}, \mathsf{im}, \mathsf{sep}, \mathsf{el}, \mathsf{exp}\}$; then

$$\Gamma\,, \mathsf{C}^1_t = \mathsf{C}^2_s \Rightarrow t = s \wedge \mathsf{C}^1 = \mathsf{C}^2$$

- let $\mathsf{C}^1, \mathsf{C}^2 \in \{\mathsf{S}\}$; then

$$\mathsf{C}^1_{ts} = \mathsf{C}^2_{pq} \Rightarrow t = p \wedge s = q \wedge \mathsf{C}^1 = \mathsf{C}^2$$

(5) Extensionality:

$$\Gamma, \forall x\,(x \in p \leftrightarrow x \in q) \Rightarrow p = q$$

(6) Empty-set:

$$\Gamma \Rightarrow \forall x(x \notin \emptyset)$$

(7) Representing elementhood:

$$\Gamma \Rightarrow \exists z[z \subseteq \top \wedge \mathsf{el}_a b \simeq z \wedge \forall u(u \in z \leftrightarrow u = \bot \wedge a \in b)]$$

(8) Union:

$$\Gamma \Rightarrow \exists z[\mathsf{un}a \simeq z \wedge \forall u(u \in z \leftrightarrow \exists y \in a\,(u \in y))]$$

(9) Pairing:

$$\Gamma \Rightarrow \exists z[\mathsf{pair}_a b \simeq z \wedge \forall u(u \in z \leftrightarrow u \in a \vee u \in b)]$$

(10) Strong infinity:

$$\Gamma \Rightarrow \emptyset \in \omega$$
$$\Gamma, t \in \omega \Rightarrow Suct \in \omega$$
$$\Gamma, \emptyset \in t \wedge \forall y(y \in t \rightarrow \mathsf{Suc}\, y \in t) \Rightarrow \omega \subseteq t$$

(11) Separation:

$$\frac{\Gamma \Rightarrow (\forall u \in a)(\exists y \subseteq \top)(fu \simeq y)}{\Gamma \Rightarrow \exists z[(\forall u \in z)(fu \simeq \top \wedge u \in a) \wedge (\forall u \in a)(\forall y(fu \simeq y \rightarrow y = \top) \rightarrow u \in z)]}$$

From the premisses

- $\Gamma \Rightarrow (\forall u \in a)(\exists y \subseteq \top)(fu \simeq y)$
- $\Gamma \Rightarrow (\forall u \in z)(fu \simeq \top \wedge u \in a)$
- $\Gamma \Rightarrow (\forall u \in a)(\forall y(fu \simeq y \to y = \top) \to u \in z)$

infer:

$$\Gamma \Rightarrow \mathsf{sep}_a f \simeq z$$

(12) Explicit replacement:

$$\frac{\Gamma \Rightarrow (\forall x \in a)\exists y(fx \simeq y)}{\Gamma \Rightarrow \exists z[(\forall y \in z)(\exists x \in a)(fx \simeq y) \wedge (\forall x \in a)(\exists y \in z)(fx \simeq y)]}$$

From the premisses

- $\Gamma \Rightarrow (\forall u \in a)\exists y(fu \simeq y)$
- $\Gamma \Rightarrow (\forall y \in z)(\exists x \in a)(fx \simeq y)$
- $\Gamma \Rightarrow (\forall x \in a)(\exists y \in z)(fx \simeq y)$

infer:

$$\Gamma \Rightarrow \mathsf{im}_a f \simeq z$$

(13) Exponentiation:

$$\Gamma \Rightarrow \exists z[\mathsf{exp}_a b \simeq z \wedge \forall F(F \in z \leftrightarrow (Fun(F) \wedge Dom(F) = a \wedge Ran(F) \subseteq b))]$$

(14) Beeson's axiom **FO**: every function is an operation, i.e.

$$\Gamma, Fun(F), \langle x, y\rangle \in F \Rightarrow Fx \simeq y$$
$$\Gamma, Fun(F), Fx \simeq y \Rightarrow \langle x, y\rangle \in F.$$

We stress that *the active formulas of the inferences and axioms are positive in App.*

Theorem 5.19 (Quasi-normal form)**.** *A* $\mathbf{\Gamma_{BEST}}$*-derivation* $\mathcal{D}$ *can be effectively transformed into a* $\mathbf{\Gamma_{BEST}}$*-derivation* $\mathcal{D}^*$ *of the same sequent, such that every cut formula occurring in* $\mathcal{D}^*$ *is positive in* $\simeq$*.*

5.4. **Step 2. The asymmetric interpretation.** We now define an asymmetric interpretation of $\mathbf{\Gamma_{BEST}}$ into $\mathbf{ECST}^*$: the idea is to replace *App* by its finite stages App^n which, for each given n, can be explicitly defined and proved to exist in the pure set–theoretic language of $\mathbf{ECST}^*$. Thus the finite approximations of the rules can be justified in the *App*-free system $\mathbf{ECST}^*$. However, the interpretation is asymmetric in the sense that it depends on a pair of number parameters $m \leq n$; in particular the positive occurrences of *App* are separated from the negative ones (the former being replaced by App^n and the second by App^m).

Let $\mathcal{A}(x, y, z, P)$ be the *App-positive formula*, inductively generating the application predicate. The formula belongs to the language of $\mathbf{ECST}^*$, except (i) for the ternary predicate symbol P and (ii) for the terms of the form C_t, S_{ts} (C being a constant among K, S, im, sep, el, exp, pair). Since these special terms can be readily eliminated (in the sense that we can define a translation thereof in the pure set–theoretic language), we can assume that $\mathcal{A}(x, y, z, P)$ belongs to the language of $\mathbf{ECST}^*$, expanded with P.

Definition 5.20. Let $\perp$ also be an abbreviation for $\neg\mathsf{K} = \mathsf{S}$ and define inductively:

$$\begin{aligned} App^0(x,y,z) &:= \perp \\ App^{k+1}(x,y,z) &:= \mathcal{A}(x,y,z,App^k). \end{aligned}$$

Here above $\mathcal{A}(x,y,z,App^k)$ is obtained from $\mathcal{A}(x,y,z,P)$ by replacing P everywhere with App^k.

Definition 5.21.

(i) We inductively define $A[m,n]$, where A is a formula of $\mathbf{\Gamma_{BEST}}$: uniformly in n, m.

$$\begin{aligned} A[m,n] &:= A \text{ provided } A \text{ has the form } t = s \text{ or } t \in s \\ App(t,s,r)[m,n] &:= App^n(t,s,r) \\ (A \to B)[m,n] &:= (A[n,m] \to B[m,n]); \end{aligned}$$

moreover $A \mapsto A[m,n]$ commutes with $\wedge$, $\vee$, $\forall$, $\exists$.

(ii) If $\Gamma := \{A_1, \ldots, A_p\}$, $\Gamma[m,n] := \{A_1[m,n], \ldots, A_p[m,n]\}$;

(iii) $(\Gamma \Rightarrow \Delta)[m,n] := \Gamma[n,m] \Rightarrow \Delta[m,n]$.

Lemma 5.22.

(i) *For each $k \in \omega$, App^k is a formula of* $\mathbf{ECST}^*$.

(ii) *In addition we have, provably in* $\mathbf{ECST}^*$,

$$k \leq m \Rightarrow App^k(x,y,z) \to App^m(x,y,z);$$

(iii) *if A is App-positive (negative), then $A[m,n] := A^n$ ($A[m,n] := A^m$); if A is App-free, $A[m,n] := A$.*

Lemma 5.23 (Persistence)**.** *Let $m \leq p \leq q \leq n$. Then provably in* $\mathbf{ECST}^*$*:*

$$\begin{aligned} A[p,q] &\to A[m,n]; \\ A[n,m] &\to A[q,p]. \end{aligned}$$

Below we also use the more suggestive notation $xy \simeq^m z$ instead of $App^m(x,y,z)$.

Lemma 5.24 (Uniqueness)**.** *Provably in* $\mathbf{ECST}^*$*: If $Fun(F)$, $Dom(F) = a$, $Ran(F) \subseteq a$ and $x \in a$ then*

$$Iter(z,F,a,x) \wedge Iter(y,F,a,x) \to z = y. \tag{13}$$

Furthermore, for each given $m \in \omega$:

$$xy \simeq^m z \wedge xy \simeq^m w \to z = w. \tag{14}$$

Proof. As to (13), this is analogous to Lemma 3.13.
As to (14), we argue informally by outer induction on $m \in \omega$. If $m = 0$, the conclusion is trivial. As to the verification of the induction step $m = j + 1$, we first apply the independence axioms. This immediately yields uniqueness in all trivial cases where x is among un, pair, exp, K, S.

Assume $xy \simeq^{j+1} z$, $xy \simeq^{j+1} w$, i.e. $\mathcal{A}(x,y,z,App^j)$ and $\mathcal{A}(x,y,w,App^j)$. Then, for some a, b, c, d, we obtain $x = S_{ab}$ and $x = S_{cd}$. By independence, $a = c$, $b = d$ and hence $S_{ab}y \simeq^{j+1} z$, $S_{ab}y \simeq^{j+1} w$, which imply, for some p, q, r, s:

- $ay \simeq^j p$, $by \simeq^j q$, $pq \simeq^j z$
- $ay \simeq^j r$, $by \simeq^j s$, $rs \simeq^j w$.

By IH $p = r$, $q = s$ and hence $pq \simeq^j z$, $pq \simeq^j w$, which yields $z = w$ again by IH.

Consider the case where $\mathsf{im}_a f \simeq^{j+1} z$, $\mathsf{im}_a f \simeq^{j+1} w$ (we implicitly use independence conditions on terms of the form im_a). Then we have

- $(\forall u \in z)(\exists x \in a)(fx \simeq^j u) \wedge (\forall x \in a)(\exists u \in z)(fx \simeq^j u)$;
- $(\forall u \in w)(\exists x \in a)(fx \simeq^j u) \wedge (\forall x \in a)(\exists u \in w)(fx \simeq^j u)$.

We prove $z \subseteq w$. Let $u \in z$: then by the first condition above $fx \simeq^j u$, for some $x \in a$. Then by the second condition, $fx \simeq^j v$, for some $v \in w$. By IH $u = v$ and hence $u \in w$. We also easily verify that $w \subseteq z$ and hence $w = z$ by extensionality.

□

Theorem 5.25. *Let $\mathcal{D}$ be a $\mathbf{\Gamma_{BEST}}$-derivation of $\Gamma \Rightarrow \Delta$. Then there exists a natural number $c \equiv c_{\mathcal{D}}$ such that, for every $m > 0$ and every n such that $n \geq c+m$,*

$$(\Gamma \Rightarrow \Delta)[m, n]$$

is derivable in $\mathbf{ECST}^*$.

Proof. By the preparation lemma we can assume that the given derivation of $\Gamma \Rightarrow \Delta$ is quasi-normal, i.e. cuts occur only on *App*-positive formulas. Furthermore, by the previous lemma 5.23 it is enough to check, for some constant c depending on the given quasi-normal derivation,

$$(\Gamma \Rightarrow \Delta)[m, c+m]. \tag{15}$$

Cut: Assume that our derivation $\mathcal{D}$ ends with a cut on an *App*-positive formula C and that the immediate subderivations of $\mathcal{D}$ end with $\Gamma \Rightarrow C$ and $C, \Gamma \Rightarrow A$. By IH we have, for some c_0, c_1, for each $m > 0$:

$$\begin{aligned} \Gamma[c_0 + m, m] &\Rightarrow C^{c_0+m} \\ C^m, \Gamma[c_1 + m, m] &\Rightarrow A[m, c_1 + m]. \end{aligned}$$

Choose $m := c_0 + m$ in the second sequent. Then, for $c = c_0 + c_1$, we obtain:

$$C^{c_0+m}, \Gamma[c + m, c_0 + m] \Rightarrow A[c_0 + m, c + m].$$

Hence with a cut

$$\Gamma[c + m, c_0 + m], \Gamma[c_0 + m, m] \Rightarrow A[c_0 + m, c + m].$$

But $m \leq c_0 + m \leq c + m$ and hence by persistence:

$$\Gamma[c + m, m], \Gamma[c + m, m] \Rightarrow A[m, c + m].$$

The conclusion follows by contraction.

Explicit replacement: By IH, for some c_0, for every $m > 0$, we have:

$$\Gamma[c_0 + m, m] \Rightarrow (\forall x \in a)(\exists y)(fx \simeq^{c_0+m} y)$$

As y is unique, by replacement, there exists a function F (hence a set), depending on $c_0 + m$, such that

$$(\forall x \in a)(fx \simeq^{c_0+m} F(x)).$$

Hence we can choose a set $z = \{F(x) \mid x \in a\}$, depending on $c_0 + m$; z satisfies the asymmetric translation of the conclusion choosing $c := c_0$, i.e. we can derive in $\mathbf{ECST}^*$ the sequent whose antecedent is $\Gamma[c + m, m]$ and whose succedent is

$$(\forall y \in z)(\exists x \in a)(y \simeq^{c+m} fx) \wedge (\forall x \in a)(\exists y \in z)(fx \simeq^{c+m} y).$$

On the other hand, by IH we have

- $\Gamma[c_0 + m, m] \Rightarrow (\forall u \in a)(\exists y)(fu \simeq^{c_0+m} y)$
- $\Gamma[c_0 + m, m] \Rightarrow (\forall y \in z)(\exists x \in a)(fx \simeq^{c_0+m} y)$
- $\Gamma[c_0 + m, m] \Rightarrow (\forall x \in a)(\exists y \in z)(fx \simeq^{c_0+m} y)$.[11]

Hence by definition of the operator defining $\simeq$ we have, for $c = c_0 + 1$:

$$\Gamma[c + m, m] \Rightarrow \mathsf{im}_a f \simeq^{c+m} z.$$

Separation: By IH, for some c_0, for every $m > 0$, we have:

$$\Gamma[c_0 + m, m] \Rightarrow (\forall x \in a)(\exists y \subseteq \top)(fx \simeq^{c_0+m} y).$$

By replacement, there exists a function F (hence a set), depending on $c_0 + m$, such that

$$(\forall x \in a)(F(x) \subseteq \top \wedge fx \simeq^{c_0+m} F(x)).$$

Hence

$$z = \{x \in a \mid \langle x, \top\rangle \in F\}$$

is a set by bounded separation and it satisfies the asymmetric interpretation of the conclusion choosing $c = c_0$. As in the previous case, we can derive by definition of the operator defining $\simeq$, for $c = c_0 + 1$:

$$\Gamma[c + m, m] \Rightarrow \mathsf{sep}_a f \simeq^{c+m} z$$

provided z satisfies the asymmetric interpretation of the premisses of the second separation rule.

Exp, Union, Pairing, Elementhood: by the appropriate corresponding axioms choosing $c = 0$.

□

Corollary 5.26. *Every* $\mathbf{\Gamma_{BEST}}$*-derivation of an App-free condition can be effectively transformed into a derivation in* $\mathbf{ECST}^*$.

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[11] Strictly speaking, each premiss will be assigned its own bounding constant c_i, where $i = 1, 2, 3$, but by persistence we can replace it by $c_0 = max\{c_1, c_2, c_3\}$.

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