MATH2022 Groups and vector spaces

EXERCISES VIII Subspaces and linear transformations

Hand in your solutions on November 30th

1. For the following subsets of vector spaces, state whether or not the indicated subset is a subspace. Justify your answers by giving a proof or a counter-example in each case.

(i) The subset $W = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 : a+b+c=0 \text{ and } b+2c=0 \right\}$ of the vector space $\mathbb{R}^3$.

(ii) The subset $L = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 : a=0 \text{ or } b=c \right\}$ of the vector space $\mathbb{R}^3$.

(iii) The set T of matrices with zero trace, in the vector space $M_{2\times 2}(\mathbb{R})$ of all real 2×2 matrices. (The *trace* is the sum of the diagonal elements.)

(iv) The set Q of all odd polynomials $p(x)$ in the vector space P_3 of real polynomials of degree ≤ 3. (An *odd* polynomial is one with $p(-x) = -p(x)$).

(v) The set C of all sequences which are eventually constant,

$$C = \{v = (v_0, v_1, v_2, \ldots) \in F^\infty : \text{there is } n \text{ such that } v_i = v_n \text{ for all } i \geq n\},$$

in the vector space F^∞ of infinite sequences $v = (v_0, v_1, v_2, \ldots)$ with $v_i \in F$.

2. Which of the following mappings are linear transformations? Give a proof (directly using the definition of a linear transformation) or a counterexample in each case.

(i) $\theta : \mathbb{R}^3 \to \mathbb{R}^2$ given by $\theta \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} |x| \\ y+z \end{pmatrix}$;

(ii) $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ given by $\phi \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} z+2w \\ -z-w \end{pmatrix}$;

(iii) $\psi : P_3 \to P_6$ given by $\psi(p(x)) = p(x)^2$ [so $\psi(ax^2+bx+c) = (ax^2+bx+c)^2$].

3. For each of the following subspaces, find a basis, and state the dimension.

(i) The subspace $U = \left\{ \begin{pmatrix} a-b \\ b-c \\ c-a \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$ of $\mathbb{R}^3$.

(ii) The subspace $W = \{p(x) : p(x) = p(-x) \text{ and } p(3) = 0\}$ of $P_4(x)$.

4. Given a subset $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of a vector space V, show that

$$Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = Span\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_3 + \mathbf{v}_1\}.$$

[Note: V is an arbitrary vector space, not necessarily $\mathbb{R}^n$ or F^n, so you can't use the method of writing the vectors as the rows of a matrix.]

5. Let $S : U \to V$ and $T : V \to W$ be linear transformations between finite-dimensional vector spaces, and let $TS : U \to W$ be their composition.

(i) Show that TS is a linear transformation.

(ii) Show that $\mathrm{im}(TS) \subseteq \mathrm{im}T$, and hence deduce that $rank(TS) \leq rank(T)$.

(iii) Similarly, by considering $\ker S$ and $\ker(TS)$, show that $nullity(S) \leq nullity(TS)$.

(iv) Hence show that $r(TS) \leq \min(r(T), r(S))$.

6. Let U and W be subspaces of a vector space V whose intersection is the zero subspace $\{\mathbf{0}\}$.

(i) Prove that each member of $U + W$ may be *uniquely* written in the form $\mathbf{u} + \mathbf{w}$ for $\mathbf{u} \in U$ and $\mathbf{w} \in W$.

(ii) As U and W are vector spaces in their own right, we may form their direct sum $U \times W = \{(\mathbf{u}, \mathbf{w}) : \mathbf{u} \in U, \mathbf{w} \in W\}$ under the co-ordinatewise operations of addition and multiplication by scalars. Prove that the map $\theta : U \times W \to U + W$ given by $\theta(\mathbf{u}, \mathbf{w}) = \mathbf{u} + \mathbf{w}$ is an isomorphism.

(iii) Deduce that $\dim(U + W) = \dim U + \dim W$.