MATH2022 Groups and vector spaces

EXERCISES VII Vector spaces

Do not submit solutions

1. For the following subsets of vector spaces, state whether or not the indicated subset is a subspace. Justify your answers by giving a proof or a counter-example in each case.

(i) The subset $U = \left\{ \begin{pmatrix} a+b \\ b \\ 2a+3b \end{pmatrix} : a, b \in \mathbb{R} \right\}$ of the vector space $\mathbb{R}^3$.

(ii) The subset $V = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 : a+b+c = 1 \right\}$ of $\mathbb{R}^3$.

(iii) The set $D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of matrices with determinant zero, in the vector space $M_{2\times 2}(\mathbb{R})$ of all real 2×2 matrices.

(iv) The set G of polynomials $p(x)$ with $p(1) = p(0)$, in the vector space P_3 of real polynomials of degree ≤ 3.

(v) The set of all sequences which are eventually zero,

$$Z = \{v = (v_0, v_1, \ldots) \in F^\infty : \text{there is } n \text{ with } v_i = 0 \text{ for all } i \geq n\},$$

in the vector space F^∞ of infinite sequences $v = (v_0, v_1, v_2 \ldots)$ with $v_i \in F$.

2. Let V be a vector space over F, and let U and W be subspaces of V. The *sum* of U and W, denoted by $U + W$, is the subset $U + W = \{\mathbf{u} + \mathbf{w} : \mathbf{u} \in U, \mathbf{w} \in W\}$. Prove that $U + W$ is a subspace of V.

3. Which of the following mappings are linear transformations? Give a proof (directly using the definition of a linear transformation) or a counterexample in each case. [Recall that P_n is the vector space of all real polynomials $p(x)$ of degree $\leq n$.]

(i) $\theta : \mathbb{R}^3 \to \mathbb{R}^2$ given by $\theta \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ 3y+z \end{pmatrix}$;

(ii) $\phi : P_2 \to P_4$ given by $\phi(p(x)) = p(x^2)$ [so $\phi(ax^2+bx+c) = ax^4+bx^2+c$].

4. For each of the following subspaces, find a basis and state its dimension.

(i) The subspace $U = \left\{ \begin{pmatrix} a \\ 2b \\ a+3b \end{pmatrix} : a, b \in \mathbb{R} \right\}$ of $\mathbb{R}^3$.

(ii) The subspace $W = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{C}^4 : x + y + z = 0 \text{ and } y - iz + w = 0 \right\}$ of $\mathbb{C}^4$.

5. Given a subset $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of a vector space V, show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent if and only if $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_3 + \mathbf{v}_1\}$ is linearly independent. (Note: V is an arbitrary vector space, not necessarily $\mathbb{R}^n$ or F^n, so you can't use the method of writing the vectors as the rows of a matrix.)

6. Prove that the kernel of a linear transformation $\theta : V \to W$ is a subspace of V, and that the image is a subspace of W. (You may use without proof the fact that $\theta(\mathbf{0}) = \mathbf{0}$.)

7. If U and W are vector spaces over F, then the direct product $U \times W$ becomes a vector space with the operations $(\mathbf{u}, \mathbf{w}) + (\mathbf{u}', \mathbf{w}') = (\mathbf{u} + \mathbf{u}', \mathbf{w} + \mathbf{w}')$ and $a(\mathbf{u}, \mathbf{w}) = (a\mathbf{u}, a\mathbf{w})$ for $\mathbf{u}, \mathbf{u}' \in U$, $\mathbf{w}, \mathbf{w}' \in W$ and $a \in F$. Suppose that $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a basis of U and $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ is a basis of W. Show that $\{(\mathbf{u}_1, \mathbf{0}), \ldots, (\mathbf{u}_n, \mathbf{0}), (\mathbf{0}, \mathbf{w}_1), \ldots, (\mathbf{0}, \mathbf{w}_m)\}$ is a basis of $U \times W$. Hence deduce that $\dim(U \times W) = \dim U + \dim W$.