MATH2022 Groups and vector spaces

EXERCISES III Cyclic groups and isomorphisms

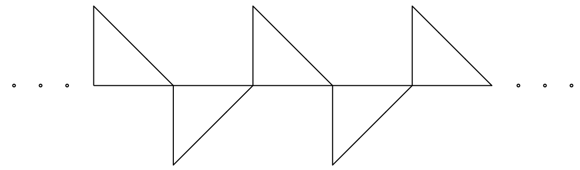
Do not submit solutions

1. Which of the following groups are cyclic?

(i) The group G of positive rational numbers which can be written as a/b with a and b odd, under multiplication.

(ii) $\mathbb{Z}_9^* = \{1, 2, 4, 5, 7, 8\}$ under $\times$ modulo 9.

(iii) The group G of isometries of the plane preserving the following frieze:



(iv) The group H of matrices of the form $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, with $n \in \mathbb{Z}$, under matrix multiplication.

2. Find which pairs of the following groups are isomorphic:

(i) The dihedral group D_3 of symmetries of an equilateral triangle.

(ii) The group of (complex) matrices

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}\right\}$$

under matrix multiplication, where $\omega^3 = 1$, $\omega \neq 1$.

(iii) $\mathbb{Z}_9^*$ as in question 1.

(iv) $\mathbb{Z}_2 \times \mathbb{Z}_3$.

3. Find the (right) cosets for the following subgroups of the stated groups:

(i) The subgroup $H = \{1, 13\}$ of $\mathbb{Z}_{14}^*$.

(ii) The subgroup $K = \{0, 4, 8\}$ of $\mathbb{Z}_{12}$.

4. Let H and K be subgroups of a group G. Show that their intersection $H \cap K$ is also a subgroup of G. If G is the additive group $\mathbb{Z}$ of integers and H and K are the subgroups $6\mathbb{Z}$ and $10\mathbb{Z}$, identify the subgroup $6\mathbb{Z} \cap 10\mathbb{Z}$. What about $m\mathbb{Z} \cap n\mathbb{Z}$ more generally (a proof is not required)?

5. Let H and K be finite subgroups of a group G, with orders h and k respectively. Show that if the numbers h and k are coprime, then $H \cap K = \{1\}$.

6. Find the orders of $\mathbb{Z}_{16}^*$ and $\mathbb{Z}_{15}^*$, and of all their elements. Are $\mathbb{Z}_{16}^*$, $\mathbb{Z}_{15}^*$ abelian? Are they isomorphic? (If you think they are isomorphic, you should try to give a 1–1 map from $\mathbb{Z}_{16}^*$ onto $\mathbb{Z}_{15}^*$ which is an isomorphism.)

7. Show that $\mathbb{Z}_{10}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_5$, but that $\mathbb{Z}_8$ is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$.

8. Let G and H be the groups of all reals under $+$, and of all non-zero reals under $\times$, respectively. Show that H has an element of order 2, but that G does not, and deduce that G and H are not isomorphic.

9. Let G be a non-cyclic group of order 10. Following the strategy used in lectures for groups of order 6, show that G is isomorphic to the dihedral group D_5 of symmetries of a regular pentagon as follows.

(i) Show that G contains an element r of order 5.

(ii) Suppose that x is an element in G but not in $H = \langle r \rangle$. Show that $G = H \cup Hx$.

(iii) Show that x has order 2. [Hint: if it has order 5, show that $x^2 \in H$, and consider $(x^2)^3$.]

(iv) Show that $xr = r^4 x$.

(v) Explain why this determines the multiplication table for G. (You don't need to find it.)