

# EHRENFEUCHT-FRAÏSSÉ GAMES ON ORDINALS

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ABSTRACT. Two structures  $A$  and  $B$  are  $n$ -equivalent if player II has a winning strategy in the  $n$ -move Ehrenfeucht-Fraïssé game on  $A$  and  $B$ . Ordinals and  $m$ -coloured ordinals are studied up to  $n$ -equivalence for various values of  $m$  and  $n$ .

## 1. Introduction

Let  $A$  and  $B$  be coloured linear orders. We say that  $A$  is  $n$ -equivalent to  $B$ , written  $A \equiv_n B$ , if player II has a winning strategy in the  $n$ -move Ehrenfeucht-Fraïssé game on  $A$  and  $B$ . In [7] we established bounds on the least representatives of the  $n$ -equivalence classes of coloured linear orders in the special cases in which the ordering is finite, or the number of moves is at most 2. Here our focus is on the case of ordinals, with or without colours. Since the pioneering work on this by Ehrenfeucht and Fraïssé, such games have been extensively used in mathematical logic to analyze questions about the relations between different structures,  $n$ -equivalence being a finer relation than elementary equivalence, and to study decidability issues. Following on Ehrenfeucht's decidability result [3], Läuchli and Leonard also used games in their important paper [5] on the elementary theory of linear order, as did Bissell-Siders in [1] and [2].

We briefly recall the material from [7] on coloured orderings and games that we need. A coloured linear ordering is a triple  $(A, <, F)$  where  $(A, <)$  is a linear order and  $F$  is a mapping from  $A$  onto a set  $C$  which we think of as a set of colours. We just write  $A$  instead of  $(A, <, F)$  provided that the ordering and colouring are clear. In the  $n$ -move Ehrenfeucht-Fraïssé game on coloured linear orders  $A$  and  $B$  (or indeed any relational structures) players I and II play alternately, I moving first. On each move I picks an element of either structure (his choice does not have to be from the same structure on every move), and II responds by choosing an element of the other structure. After  $n$  moves, I and II between them have chosen elements  $x_1, x_2, \dots, x_n$  of  $A$ , and  $y_1, y_2, \dots, y_n$  of  $B$ , and player II *wins* if the map taking  $x_i$  to  $y_i$  for each  $i$  is an isomorphism between induced substructures (that is, it preserves the ordering and colour), and player I wins otherwise. Intuitively, I is trying to demonstrate that there is some difference between the structures, while player II is trying to show that they are at least reasonably similar. We say that  $A$  and  $B$  are  $n$ -equivalent and write  $A \equiv_n B$ , if II has a winning strategy. It is easy to see that  $\equiv_n$  is an equivalence relation, and it is standard that for any  $n$ , there are only finitely many  $n$ -equivalence classes, so it is natural to enquire what their optimal representatives may be. The problem for general orderings seems to be quite hard, but with special conditions on the type of ordering or colouring, or the number of moves, some results can be obtained. If the ordering is an ordinal, then the notion of 'optimality' makes sense: a (coloured) ordinal is *optimal* if it is

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least in its  $n$ -equivalence class. This may still not be unique in the coloured case. If the ordering is finite, then we may take the lexicographically least; in the general case we would hope to make some canonical choice, for instance, exhibiting some eventual periodicity.

Already in [8], some information about the optimal representatives of  $n$ -equivalence classes of (monochromatic) ordinals is given (also see [4]). Rosenstein remarks (as an exercise) that every ordinal is  $2n$ -equivalent to some ordinal in the finite set

$$\{\omega^n \cdot a_n + \omega^{n-1} \cdot a_{n-1} + \dots + \omega \cdot a_1 + a_0 : a_i < 2^{2^n}, a_n \leq 1\}.$$

In section 2 we sharpen this result to give precise lists of all the optimal values for  $n$ -equivalence classes of ordinals, including the case where  $n$  is odd.

In section 3 we move on to consider the coloured case. By [7], we already understand the situation for 2 moves, and we now generalize this to more moves. Here we concentrate on giving some upper bounds for the optimal representatives, which certainly seem unnecessarily large, but at least all lie below the ordinal  $\omega^\omega$ .

Next we recall the notion of ‘character’ from [7], and the main result about characters. Assume that we have found representatives for the  $n$ -equivalence classes of certain  $m$ -coloured linearly ordered sets. We write the representative for  $A$  as  $[A]_n$ . In a coloured linear order  $A$ , the  $n$ -character of  $a \in A$  having colour  $c$  is the ordered pair  $\langle [A^{<a}]_n, [A^{>a}]_n \rangle$  (where  $A^{<a} = \{x \in A : x < a\}$  and  $A^{>a} = \{x \in A : x > a\}$ ). We let  $\rho_n^c(A) = \{\langle [A^{<a}]_n, [A^{>a}]_n \rangle : a \in A \text{ is } c\text{-coloured}\}$ , and if we wish to include the colour as part of the  $n$ -character of  $a$ , we may also write  $\langle [A^{<a}]_n, [A^{>a}]_n \rangle_c$ .

**Theorem 1.1** ([7]).  $A \equiv_{n+1} B$  if and only if  $\rho_n^c(A) = \rho_n^c(B)$  for all  $c \in C$ .

If  $A$  and  $B$  are coloured linear orders, then  $A + B$  stands for the concatenation of  $A$  and  $B$ , that is, we first assume (by replacing by copies if necessary) that  $A$  and  $B$  are disjoint, and we place all members of  $A$  to the left of all members of  $B$ . As a generalization of this, we may write  $\sum\{A_i : i \in I\}$  for the concatenation of a family of (coloured) linear orders  $\{A_i : i \in I\}$  indexed by a linear ordering  $I$ . When forming concatenations we would normally assume that all the orderings have the same colour set. We write  $A \cdot B$  for the anti-lexicographic product,  $B$  ‘copies of’  $A$ , to accord with the customary use for ordinals (and unlike [7], where *lexicographic* products are used). Note that here  $B$  is assumed monochromatic, and colours are assigned to members of  $A \cdot B$  by means of the  $A$ -co-ordinate. The following result will be used without explicit reference.

**Theorem 1.2.** (i) If  $A \equiv_n B$ , then  $X + A + Y \equiv_n X + B + Y$  and  $X \cdot A \cdot Y \equiv_n X \cdot B \cdot Y$ .  
(ii) If  $A_i \equiv_n B_i$  for each  $i \in I$ , then  $\sum\{A_i : i \in I\} \equiv_n \sum\{B_i : i \in I\}$ .

We conclude the introduction by quoting the following results which will be used throughout.

**Lemma 1.3.** Let  $A$  and  $B$  be finite linear orderings. Then  $A \equiv_n B$  if and only if  $(|A| = |B| < 2^n - 1)$  or  $(|A|, |B| \geq 2^n - 1)$ .

This is well known, but may be easily proved using characters.

**Theorem 1.4** (Mostowski-Tarski). For any  $n > 0$  and ordinal  $\beta > 0$ ,

- (i)  $\omega^n \equiv_{2n} \omega^n \cdot \beta$ ,
- (ii)  $\omega^n \not\equiv_{2n+1} \omega^n + \beta$ ,
- (iii)  $\omega^n \not\equiv_{2n+1} \omega^n \cdot \beta$ , for  $\beta > 1$ .

See [8] for a proof. Note that (iii) shows that (i) is the best possible for player II (and (iii) is an immediate consequence of (ii)).

## 2. OPTIMAL REPRESENTATIVES FOR ORDINALS

In this section we present our main result Theorem 2.13 on optimal representatives of ordinals under  $n$ -equivalence. This is accomplished through a series of rather technical lemmas, which enable us to build up equivalences and non-equivalences in the critical cases. The starting point is Theorem 1.4 of Mostowski and Tarski, and it is unsurprising as a consequence that the optimal list has highest power of  $\omega$  at most the integer part  $\lfloor \frac{n}{2} \rfloor$  of  $\frac{n}{2}$ . As here, many of our lemmas come in two versions, one positive (a win for player II) and the other negative (a win for player I). Lemma 2.3 is a generalization of 1.4 where two multiples of a power of  $\omega$  are compared, rather than just the power and one multiple. A corollary immediately allows us to reduce the total number of values needing to be considered to a relatively small finite range where the greatest power of  $\omega$  is at most the integer part of  $\frac{n}{2}$ , and there are also restrictions on the coefficients possible. Lemma 2.6 tells us when we can or cannot reduce the coefficient of  $\omega^{\frac{n}{2}-1}$  ( $n$  even), Lemma 2.7 tells us about this for smaller powers of  $\omega$ , and Lemma 2.8 deals similarly with  $\omega^{\frac{n}{2}}$  ( $n$  even). The optimal form often contains several powers below the greatest, and Lemmas 2.9, 2.10, and 2.11 give information on this situation, Lemmas 2.10 and 2.11 in particular explaining the reason for a long sequence of coefficients 3 in some of the optimal forms. In fact the list of optimal ordinals is so complicated that we feel that, having proved Theorem 2.13, it is worthwhile examining how it works out in three typical and not too small cases (4, 5, and 6). There is a notion of ‘maximality’ under a different ordering  $\succeq$ , and under this ordering, there are two typical ‘starts’ for the Cantor normal form of maximal ordinals in the list, those in which the sequence of 3’s is immediately preceded by a term having coefficient a power of 2 (corresponding to Lemma 2.10), the other where it is preceded by a power of 2 with 4 subtracted (treated in Lemma 2.11). At this stage we can state what the claimed optimal list of representatives is. The role of Lemma 2.12 is to show that these are all optimal in their  $n$ -equivalence class, and in Theorem 2.13 we see that every ordinal is  $n$ -equivalent to a unique element of the given list.

We begin the section by remarking on the situation for  $n = 1$  and 2, which was treated in [7]. For  $n = 1$ , all non-empty linear orders are  $n$ -equivalent. We therefore have two classes which can be represented by linear orders 0 and 1. For  $n = 2$ , it follows from Lemma 3.2 in [7] that a complete family of representatives for ordinals is given by 0, 1, 2, 3 and  $\omega$  (and any infinite ordinal is 2-equivalent to  $\omega$  if it is a limit ordinal, and to 3 if it is a successor). The case  $n = 0$  is degenerate, but still fits into the overall pattern; since there are no moves, all structures are equivalent, so there is one minimal representative, namely 0.

Many of our proofs will be by induction, which means that we shall concentrate on describing the first moves of the two players, and then appeal to the induction hypothesis. We usually write  $A$  and  $B$  for the two structures (or  $\alpha$  and  $\beta$ ), and  $x_1$  and  $y_1$  for the first elements chosen from  $A$ ,  $B$  respectively. Subsequent moves played in  $A$  are  $x_2, \dots, x_n$  and in  $B$  are  $y_2, \dots, y_n$ . Thus on each move, one of player I and player II plays  $x_i$ , and the other plays  $y_i$ , but which one plays which may vary during the game.

First we give the following two lemmas which throughout the paper will reduce the number of cases to be considered.

**Lemma 2.1.** *If  $A = \omega^i \cdot \gamma_0$  and  $B = \gamma_1 + \omega^j$  where  $j < i \leq \frac{n}{2}$ , then  $A \not\equiv_n B$ .*

*Proof.* Player I chooses  $y_1 = \gamma_1 \in B$  so that  $B^{>y_1} \cong \omega^j$  (or  $B^{>y_1} = \emptyset$  if  $j = 0$ ). Whatever  $x_1 \in A$  player II plays,  $A^{>x_1} \cong \omega^i \cdot \gamma_2$  for some  $\gamma_2 > 0$ . If  $j = 0$ ,  $A^{>x_1} \not\equiv_{n-1} B^{>y_1}$  is immediate. Otherwise, by Theorem 1.4(iii),  $\omega^j \not\equiv_{2j+1} \omega^i \cdot \gamma_2$ , and since  $2j + 1 \leq n - 1$ , I can therefore win in the remaining  $n - 1$  moves. ■

**Lemma 2.2.** *Let  $A = \omega^i \cdot a_i + \omega^{i-1} \cdot a_{i-1} + \dots + \omega \cdot a_1 + a_0$  and  $B = \omega^i \cdot b_i + \omega^{i-1} \cdot b_{i-1} + \dots + \omega \cdot b_1 + b_0$ . Then in any play of the  $n$ -move game on  $A$  and  $B$  in which player I starts by playing  $x_1 = \omega^j \cdot \gamma_0$  for some ordinal  $\gamma_0 > 0$  where  $j < \frac{n}{2}$ , unless player II plays  $y_1 = \omega^j \cdot \gamma_1$  for some  $\gamma_1 > 0$  then I can win the game in the remaining  $n - 1$  moves.*

*Proof.* Supposing on the contrary that  $B^{<y_1}$  has a final segment of order-type  $\omega^r$  for some  $r < j$  (possibly 0), we may write  $A^{<x_1} \cong \omega^j \cdot \gamma_0$ ,  $B^{<y_1} \cong \gamma_1 + \omega^r$ , where  $r < j \leq \frac{n-1}{2}$ , and so by Lemma 2.1,  $A^{<x_1} \not\equiv_{n-1} B^{<y_1}$ , and player I wins. ■

**Lemma 2.3.** *Let  $n, m, i, k$  be natural numbers such that  $0 < i \leq \frac{n}{2}$ .*

- (i) *If  $k \geq m = 2^{n-2i}$  then  $\omega^i \cdot k \equiv_n \omega^i \cdot m$ .*
- (ii) *If  $k > m$  and  $m < 2^{n-2i}$  then  $\omega^i \cdot k \not\equiv_n \omega^i \cdot m$ .*

*Proof.* (i) We use induction on  $n$ . Since  $0 < i \leq \frac{n}{2}$ ,  $n \geq 2$ . If  $n = 2$ , then  $i = 1$  and  $m = 1$ . Now  $\omega \cdot 1 \equiv_2 \omega \cdot k$  for any  $k \geq 1$ , giving the result.

So we assume the result for  $n \geq 2$ , and prove it for  $n + 1$ . Let  $0 < i \leq \frac{n+1}{2}$ , and  $k \geq m = 2^{n+1-2i}$ , with the object of showing that  $\omega^i \cdot k \equiv_{n+1} \omega^i \cdot m$ . If  $i = \frac{n+1}{2}$ , then  $n$  is odd, so by Theorem 1.4(i),  $\omega^{\frac{n+1}{2}} \cdot k \equiv_{n+1} \omega^{\frac{n+1}{2}} \cdot m$ . So from now on we assume that  $0 < i < \frac{n+1}{2}$ .

Let  $A = \omega^i \cdot k$  and  $B = \omega^i \cdot m$ . The play may take place on the left, right, or in the middle. First consider the play on the left. On his first move, player II may play so that if  $A^{<x_1}$  has the form  $\omega^i \cdot q_0 + \gamma$  where  $\gamma < \omega^i$  and  $q_0 < 2^{n-2i}$ , then  $A^{<x_1} \cong B^{<y_1}$ . (In other words, if  $x_1$  is I's move, which satisfies this condition, then II can choose a corresponding  $y_1$ , and if  $y_1$  is I's move, which satisfies this condition, then II can choose a corresponding  $x_1$ .) It follows that  $A^{<x_1} \equiv_n B^{<y_1}$ , and in this case,  $A^{>x_1}$  and  $B^{>y_1}$  have the forms  $\omega^i \cdot q_1$  and  $\omega^i \cdot q_2$  respectively, where  $q_1, q_2 \geq 2^{n-2i}$ . By induction hypothesis,  $A^{>x_1} \equiv_n B^{>y_1}$ , so II can win the  $(n + 1)$ -move game by calling on his strategies on the left and right of  $x_1, y_1$  as required in the remaining  $n$  moves.

In a similar way, on the right, player II may play on his first move so that if  $A^{>x_1}$  has the form  $\omega^i \cdot r_0$  where  $1 \leq r_0 \leq 2^{n-2i}$ , then  $A^{>x_1} \cong B^{>y_1}$ . Here  $A^{>x_1} \equiv_n B^{>y_1}$ , and II may also ensure that  $A^{<x_1}$  and  $B^{<y_1}$  have the form  $\omega^i \cdot r_1 + \gamma$  and  $\omega^i \cdot r_2 + \gamma$  respectively, where  $r_1, r_2 \geq 2^{n-2i}$  and therefore by the induction hypothesis, it follows that  $A^{<x_1} \equiv_n B^{<y_1}$ . Once more this provides II with a winning strategy in the  $(n + 1)$ -move game.

Finally, in the middle, player II may play on his first move so that if  $A^{<x_1}$  has the form  $\omega^i \cdot s_0 + \gamma$  where  $2^{n-2i} \leq s_0 < k - 2^{n-2i}$ , and  $\gamma < \omega^i$ , then  $B^{<y_1} \cong \omega^i \cdot 2^{n-2i} + \gamma$ . Again using the induction hypothesis,  $A^{<x_1} \equiv_n B^{<y_1}$ . In this case  $A^{>x_1}$  and  $B^{>y_1}$  have the form  $\omega^i \cdot s_1$  where  $s_1 \geq 2^{n-2i}$ , and  $\omega^i \cdot 2^{n-2i}$ . By the induction hypothesis, we deduce that  $A^{>x_1} \equiv_n B^{>y_1}$ , and again II wins.

(ii) Again using induction, for the basis case,  $n = 2$ , in which case  $i = 1$  and  $m = 0$  so the result is immediate.

Now assume the result for  $n$ , and let  $0 < i \leq \frac{n+1}{2}$ ,  $k > m$ , and  $m < 2^{n+1-2i}$ . Since the result is immediate for  $m = 0$ , we assume that  $m \neq 0$  which means that  $i < \frac{n+1}{2}$ , so  $i \leq \frac{n}{2}$ . On his first move, I plays  $x_1 = \omega^i \cdot r$ , where  $r$  is the integer part  $\lfloor \frac{k}{2} \rfloor$  of  $\frac{k}{2}$ . Then  $A^{<x_1} \cong \omega^i \cdot r$  and  $A^{>x_1} \cong \omega^i(k - r)$ . Suppose that II plays  $y_1$ . By Lemma 2.2 we may suppose that  $y_1 = \omega^i \cdot s$  for some  $s$ . Player I can now win in the remaining  $n$  moves on the left or right, provided that  $A^{<x_1} \not\equiv_n B^{<y_1}$  or  $A^{>x_1} \not\equiv_n B^{>y_1}$ . Note that  $B^{<y_1} \cong \omega^i \cdot s$  and  $B^{>y_1} \cong \omega^i(m - s)$ .

If  $s < r$  and  $s < 2^{n-2i}$  then by induction hypothesis,  $A^{<x_1} \not\equiv_n B^{<y_1}$ .

If however  $s < r$  and  $s \geq 2^{n-2i}$ , then it similarly follows that  $A^{>x_1} \not\equiv_n B^{>y_1}$ .

Now we assume that neither of these holds, so that  $s \geq r$ . Since  $m < k$  it follows that  $m - s < k - r$ . Also  $m - s \geq 2^{n-2i}$  is impossible, since it implies

that  $k - r > 2^{n-2i}$ , so also  $r \geq 2^{n-2i}$ , giving  $s, m - s \geq 2^{n-2i}$  and  $m \geq 2^{n+1-2i}$ , contrary to supposition. The conclusion is that  $m - s < 2^{n-2i}$ , which again gives  $A^{>x_1} \not\equiv_n B^{>y_1}$ . ■

We write  $t$  for the integer part  $[\frac{n}{2}]$  of  $\frac{n}{2}$ .

**Corollary 2.4.** *If  $n > 0$  then every ordinal is  $n$ -equivalent to some ordinal in the finite set  $\Omega_n = \{\omega^t \cdot a_t + \omega^{t-1} \cdot a_{t-1} + \dots + \omega \cdot a_1 + a_0 : a_i \leq 2^{n-2i} \text{ for all } i\}$ .*

*Proof.* First suppose that  $n$  is even. Using Cantor normal form we may write any ordinal  $\alpha$  in the form  $\alpha = \omega^t \cdot \alpha^* + \omega^{t-1} \cdot b_{t-1} + \dots + \omega \cdot b_1 + b_0$  where  $\alpha^*$  is an ordinal, and  $b_i \in \omega$ . By Theorem 1.4(i),  $\omega^t \cdot \alpha^* \equiv_n \omega^t$  if  $\alpha^* \neq 0$ , and by Lemma 2.3(i),  $\omega^i \cdot b_i \equiv_n \omega^i \cdot a_i$  where  $a_i = \min(b_i, 2^{n-2i})$ . Finally letting  $a_t = \min(\alpha^*, 1)$ , we find that  $\alpha \equiv_n \omega^t \cdot a_t + \omega^{t-1} \cdot a_{t-1} + \dots + \omega \cdot a_1 + a_0 \in \Omega_n$ .

The proof for odd  $n$  is similar except that we let  $a_t = \min(\alpha^*, 2)$ . Note that we cannot appeal to Theorem 1.4(i) directly this time to show that  $\omega^t \cdot \alpha^* \equiv_n \omega^t \cdot 2$  for  $\alpha^* \geq 2$ , and instead follow a direct proof. Player II may play so that  $x_1 = \omega^t \cdot q_1 + \gamma$ ,  $y_1 = \omega^t \cdot q_2 + \gamma$ , where  $q_1 < \alpha^*$ ,  $q_2 = 0$  or  $1$  and  $q_1 = 0 \Leftrightarrow q_2 = 0$ . The facts that  $A^{<x_1} \equiv_{n-1} B^{<y_1}$  and  $A^{>x_1} \equiv_{n-1} B^{>y_1}$  follow from Theorem 1.4(i). ■

**Corollary 2.5.** (i) *If  $n$  is even, then any ordinal is  $n$ -equivalent to some ordinal less than or equal to  $\omega^{\frac{n}{2}} \cdot 2$ ,*

(ii) *If  $n$  is odd, then any ordinal is  $n$ -equivalent to some ordinal less than or equal to  $\omega^{\frac{n-1}{2}} \cdot 3$ .*

We now give a list, without proof, of the minimal  $n$ -equivalence class representatives for  $n = 3$  and  $4$ . Proofs that these are the correct lists are given in [6], and they form the basis for the general result we prove in Theorem 2.13, which yields these two lists as special cases.

The minimal 3-equivalence class representatives for all ordinals are

$$\begin{aligned} &0, 1, 2, 3, 4, 5, 6, 7, \\ &\omega, \omega + 1, \omega + 2, \omega + 3, \omega + 4, \\ &\omega \cdot 2, \omega \cdot 2 + 1, \omega \cdot 2 + 2, \omega \cdot 2 + 3. \end{aligned}$$

The minimal 4-equivalence class representatives for all ordinals are

$$\begin{aligned} &0, 1, 2, \dots, 15, \\ &\omega, \omega + 1, \omega + 2, \omega + 3, \omega + 4, \dots, \omega + 12, \\ &\omega \cdot 2, \omega \cdot 2 + 1, \omega \cdot 2 + 2, \omega \cdot 2 + 3, \dots, \omega \cdot 2 + 12, \\ &\omega \cdot 3, \omega \cdot 3 + 1, \omega \cdot 3 + 2, \dots, \omega \cdot 3 + 12, \\ &\omega \cdot 4, \omega \cdot 4 + 1, \omega \cdot 4 + 2, \omega \cdot 4 + 3, \\ &\omega^2, \omega^2 + 1, \omega^2 + 2, \omega^2 + 3. \end{aligned}$$

Rosenstein's list of  $2n$ -equivalence class representatives that we quoted in the introduction includes some redundancies, and indeed we have already illustrated this in Corollary 2.4. We shall show that even this list can be improved, and give explicit lists of representatives of ordinals up to  $n$ -equivalence by making use of the patterns seen in generating the two lists just given. Thus if  $\Omega_n$  is the set of  $n$ -equivalence class representatives provided by Corollary 2.4, we shall find  $\Omega'_n \subseteq \Omega_n$  that contains no redundant elements.

The following result generalizes an exercise in [8] page 106.

**Lemma 2.6.** *For all even  $n \geq 4$  and ordinals  $\alpha \geq 3$ ,*

- (i)  $\omega^{\frac{n}{2}-1}(\alpha + 1) \equiv_n \omega^{\frac{n}{2}-1} \cdot 4$ ,
- (ii)  $\omega^{\frac{n}{2}-1}(\alpha + 1) \not\equiv_n \omega^{\frac{n}{2}-1} \cdot 3$ .

*Proof.* (i) We describe a winning strategy for player II. Let us write  $A = \omega^{\frac{n}{2}-1}(\alpha + 1)$  and  $B = \omega^{\frac{n}{2}-1} \cdot 4$ . Player II may move on his first move so that if  $x_1$  is in the  $0, 1$

or last copy of  $\omega^{\frac{n}{2}-1}$  in  $A$ , then  $y_1$  is in the corresponding copy of  $B$ , and if  $x_1$  is in any other copy of  $\omega^{\frac{n}{2}-1}$  in  $A$ , then  $y_1$  is in the third copy (numbered by 2) of  $\omega^{\frac{n}{2}-1}$  in  $B$ . Furthermore, player II may play so that  $x_1$  and  $y_1$  are the corresponding points in those copies.

The outcomes in these four cases are as follows:

- $A^{<x_1} \cong B^{<y_1}$  and  $A^{>x_1} \cong \omega^{\frac{n}{2}-1}(\alpha + 1)$ ,  $B^{>y_1} \cong \omega^{\frac{n}{2}-1} \cdot 4$ ,
- $A^{<x_1} \cong B^{<y_1}$  and  $A^{>x_1} \cong \omega^{\frac{n}{2}-1}(\alpha_1 + 1)$ ,  $B^{>y_1} \cong \omega^{\frac{n}{2}-1} \cdot 3$ , where  $\alpha = 1 + \alpha_1$ ,
- $A^{<x_1} \cong \omega^{\frac{n}{2}-1} \cdot \alpha + \gamma$ ,  $B^{<y_1} \cong \omega^{\frac{n}{2}-1} \cdot 3 + \gamma$  for some  $\gamma < \omega^{\frac{n}{2}-1}$  and  $A^{>x_1} \cong B^{>y_1}$ ,
- $A^{<x_1} \cong \omega^{\frac{n}{2}-1} \cdot \alpha_2 + \gamma$ ,  $B^{<y_1} \cong \omega^{\frac{n}{2}-1} \cdot 2 + \gamma$ , for some  $\gamma < \omega^{\frac{n}{2}-1}$ ,  $A^{>x_1} \cong \omega^{\frac{n}{2}-1}(\alpha_3 + 1)$ ,  $B^{>y_1} \cong \omega^{\frac{n}{2}-1} \cdot 2$ , where  $\alpha_2 + \alpha_3 = \alpha$ ,  $2 \leq \alpha_2 < \alpha$ .

In each case Player II has a winning strategy in the remaining  $n - 1$  moves, whether player I plays on the left or right of the first moves. When the relevant structures are isomorphic this is immediate. Otherwise, player II may play so that  $x_2$  and  $y_2$  are corresponding points of some copies of  $\omega^{\frac{n}{2}-1}$  (or of a ‘ $\gamma$ ’ part), and if one of the copies is the first one, then so is the other; for the remaining  $n - 2$  moves, player II uses Theorem 1.4(i) to win.

(ii) Let  $A = \omega^{\frac{n}{2}-1}(\alpha + 1)$  and  $B = \omega^{\frac{n}{2}-1} \cdot 3$ . On his first move, player I plays the first point  $x_1$  of the third copy of  $\omega^{\frac{n}{2}-1}$  in  $A$ , (that is, so that  $A^{<x_1} \cong \omega^{\frac{n}{2}-1} \cdot 2$ ), and II responds by playing the first point  $y_1$  of the  $i$ th copy of  $\omega^{\frac{n}{2}-1}$  in  $B$ ,  $0 < i < 3$  (if he chooses 0 or a non-first point, then he loses by Lemma 2.2). If  $i = 1$  then from now on I plays on the left of  $x_1, y_1$ , or if  $i = 2$  he plays on the right of  $x_1, y_1$ , in each case winning using Theorem 1.4(iii).  $\blacksquare$

**Lemma 2.7.** *Let  $m \geq 4$ ,  $0 < i \leq \frac{n-1}{2}$ , and  $k$  be a natural number.*

- (i) *If  $k \geq 2^{n-2i}$ , then  $\omega^i \cdot k + \omega^{i-1} \cdot m \equiv_n \omega^i(2^{n-2i} - 1) + \omega^{i-1} \cdot m$ ,*
- (ii) *If  $l < 2^{n-2i}$  and  $l < k$ , then  $\omega^i \cdot k + \omega^{i-1} \cdot 3 \not\equiv_n \omega^i \cdot l + \omega^{i-1} \cdot 3$ .*

*Proof.* (i) We use induction. Notice that as  $\frac{n-1}{2} \geq 1$ , we have  $n \geq 3$ . If  $n = 3$ , then  $i = 1$ , so we have to check that  $\omega \cdot k + m \equiv_3 \omega + m$  for  $k \geq 2$ . We find that  $\omega \cdot k + m$  and  $\omega + m$  both have 2-character set  $\{(0, 3), \langle 1, 3 \rangle, \langle 2, 3 \rangle, \langle 3, 3 \rangle, \langle \omega, 3 \rangle, \langle 3, 2 \rangle, \langle 3, 1 \rangle, \langle 3, 0 \rangle\}$ , and so they are 3-equivalent.

Now assume the result holds for  $n \geq 3$  and we prove it for  $n + 1$ . So we consider  $A = \omega^i \cdot k + \omega^{i-1} \cdot m$  and  $B = \omega^i(2^{n+1-2i} - 1) + \omega^{i-1} \cdot m$  where  $0 < i \leq \frac{n}{2}$ , and  $k \geq 2^{n+1-2i}$ , and we have to show that player II has a winning strategy in the  $(n + 1)$ -move game. First note that II can play in such a way that if player I plays in the final  $\omega^{i-1} \cdot m$  segment of  $A$  or  $B$ , then II plays the corresponding point of the  $\omega^{i-1} \cdot m$  segment of the other structure. By Lemma 2.3(i),  $\omega^i \cdot k \equiv_n \omega^i \cdot 2^{n-2i} \equiv_n \omega^i(2^{n+1-2i} - 1)$ , and this provides a winning strategy for player II in the remaining  $n$  moves, since this shows that  $A^{<x_1} \equiv_n B^{<y_1}$  (and  $A^{>x_1} \equiv_n B^{>y_1}$  because they are isomorphic).

Now supposing that player I does not play a point of the final part of either structure on his first move, the first case is where  $i = \frac{n}{2}$ . Then  $n$  is even, and  $A = \omega^{\frac{n}{2}} \cdot k + \omega^{\frac{n}{2}-1} \cdot m$ ,  $B = \omega^{\frac{n}{2}} + \omega^{\frac{n}{2}-1} \cdot m$ . Player II can play so that one of the following holds:

$x_1 = y_1 < \omega^{\frac{n}{2}}$ , in which case  $A^{<x_1} \cong B^{<y_1}$  so  $A^{<x_1} \equiv_n B^{<y_1}$ , and  $A^{>x_1} \cong \omega^{\frac{n}{2}} \cdot k + \omega^{\frac{n}{2}-1} \cdot m$ ,  $B^{>y_1} \cong \omega^{\frac{n}{2}} + \omega^{\frac{n}{2}-1} \cdot m$ , which are  $n$ -equivalent by Theorem 1.4(i).

$x_1 = \omega^{\frac{n}{2}} \cdot q_1 + \gamma$  and  $y_1 = \omega^{\frac{n}{2}} + \gamma$  where  $1 \leq q_1 < k$  and  $\gamma < \omega^{\frac{n}{2}-1}$ . Then  $A^{<x_1} \cong \omega^{\frac{n}{2}} \cdot q_1 + \gamma$  and  $B^{<y_1} \cong \omega^{\frac{n}{2}} + \gamma$  so  $A^{<x_1} \equiv_n B^{<y_1}$  by Theorem 1.4(i) and  $A^{>x_1} \cong \omega^{\frac{n}{2}}(k - q_1) + \omega^{\frac{n}{2}-1} \cdot m$  and  $B^{>y_1} \cong \omega^{\frac{n}{2}-1} \cdot m$  so  $A^{>x_1} \equiv_n B^{>y_1}$  by Lemmas 2.6(i) and 2.3(i).

$x_1 = \omega^{\frac{n}{2}} \cdot q_1 + \omega^{\frac{n}{2}-1} \cdot q_2 + \gamma$  and  $y_1 = \omega^{\frac{n}{2}-1} \cdot 4 + \gamma$  where  $1 \leq q_1 < k$ ,  $1 \leq q_2 < \omega$ , and  $\gamma < \omega^{\frac{n}{2}-1}$ . Then  $A^{<x_1} \cong \omega^{\frac{n}{2}} \cdot q_1 + \omega^{\frac{n}{2}-1} \cdot q_2 + \gamma$  and  $B^{<y_1} \cong \omega^{\frac{n}{2}-1} \cdot 4 + \gamma$

so  $A^{<x_1} \equiv_n B^{<y_1}$  by Lemma 2.6(i), and  $A^{>x_1} \cong \omega^{\frac{n}{2}}(k - q_1) + \omega^{\frac{n}{2}-1} \cdot m$ , and  $B^{>y_1} \cong \omega^{\frac{n}{2}} + \omega^{\frac{n}{2}-1} \cdot m$  so  $A^{>x_1} \equiv_n B^{>y_1}$  by Theorem 1.4(i).

Otherwise,  $0 < i < \frac{n}{2}$  and hence  $0 < i \leq \frac{n-1}{2}$ , so by induction hypothesis,  $\omega^i \cdot 2^{n-2i} + \omega^{i-1} \cdot m \equiv_n \omega^i(2^{n-2i} - 1) + \omega^{i-1} \cdot m$ . Player II can play so that one of the following holds:

$x_1 = y_1 < \omega^i(2^{n-2i} + 1)$ , in which case  $A^{<x_1} \cong B^{<y_1}$  and  $A^{>x_1} \cong \omega^i \cdot r_1 + \omega^{i-1} \cdot m$  and  $B^{>y_1} \cong \omega^i \cdot r_2 + \omega^{i-1} \cdot m$  where  $r_1, r_2 \geq 2^{n-2i} - 1$ , so by induction hypothesis,  $A^{>x_1} \equiv_n B^{>y_1}$ , and II wins in the remaining  $n$  moves.

$x_1 = \omega^i \cdot r_3 + \gamma$ ,  $y_1 = \omega^i \cdot r_4 + \gamma$  for some  $\gamma < \omega^i$ , where  $2^{n-2i} + 1 \leq r_3 < k$ ,  $2^{n-2i} + 1 \leq r_4 < 2^{n+1-2i} - 1$  and either  $k - r_3 = 2^{n+1-2i} - 1 - r_4$ , or  $k - r_3 \geq 2^{n-2i}$  and  $r_4 = 2^{n-2i} + 1$ . Then  $A^{<x_1} \equiv_n B^{<y_1}$  by Lemma 2.3(i) and  $A^{>x_1} \equiv_n B^{>y_1}$  follows by the induction hypothesis.

In all cases we deduce that  $A \equiv_{n+1} B$ .

(ii) We use induction. As above,  $n \geq 3$ . If  $n = 3$  then  $i = 1$ , and so we have to show that  $\omega \cdot l + 3 \not\equiv_3 \omega \cdot k + 3$  for  $l \leq 1$  and  $k > l$ . This is verified by consideration of 2-characters. If  $l = 1$  then  $k \geq 2$ , so  $\omega \cdot k + 3$  has  $\langle \omega, 3 \rangle$  as a 2-character, but  $\omega \cdot l + 3$  does not. If  $l = 0$  then  $\omega \cdot k + 3$  has  $\langle \omega, 2 \rangle$  as a 2-character, but  $\omega \cdot l + 3$  does not.

For the induction step we assume the result for  $n \geq 3$  and show that I has a winning strategy in the  $(n+1)$ -move game on  $A = \omega^i \cdot k + \omega^{i-1} \cdot 3$  and  $B = \omega^i \cdot l + \omega^{i-1} \cdot 3$  where  $0 < i \leq \frac{n}{2}$ ,  $l < 2^{n+1-2i}$ , and  $l < k$ . In the first case,  $i = \frac{n}{2}$ , so that  $n$  is even, and we have to show that  $A = \omega^{\frac{n}{2}} \cdot k + \omega^{\frac{n}{2}-1} \cdot 3 \not\equiv_{n+1} B = \omega^{\frac{n}{2}} \cdot l + \omega^{\frac{n}{2}-1} \cdot 3$  where  $l = 1$  or  $0$ , and  $k > l$ . Let I play  $x_1 = \omega^{\frac{n}{2}} \in A$  on his first move. By Lemma 2.2, noting that  $\frac{n}{2} < \frac{n+1}{2}$ , we may suppose that II's reply  $y_1$  is a non-zero multiple of  $\omega^{\frac{n}{2}}$ . Since  $l \leq 1$ , this implies that  $l = 1$  (and so  $k \geq 2$ ) and  $y_1 = \omega^{\frac{n}{2}} \in B$ , and I now plays  $x_2 = \omega^{\frac{n}{2}} \cdot 2 \in A$ . By Lemma 2.2 again, II plays  $y_2 = \omega^{\frac{n}{2}} + \omega^{\frac{n}{2}-1}$  or  $\omega^{\frac{n}{2}} + \omega^{\frac{n}{2}-1} \cdot 2$  in  $B$ . If  $y_2 = \omega^{\frac{n}{2}} + \omega^{\frac{n}{2}-1}$ , player I wins on the intervals  $(x_1, x_2)$  and  $(y_1, y_2)$  using  $\omega^{\frac{n}{2}-1} \not\equiv_{n-1} \omega^{\frac{n}{2}}$  and if  $y_2 = \omega^{\frac{n}{2}} + \omega^{\frac{n}{2}-1} \cdot 2$ , he wins to the right of  $x_2$  and  $y_2$  using  $\omega^{\frac{n}{2}}(k-2) + \omega^{\frac{n}{2}-1} \cdot 3 \not\equiv_{n-1} \omega^{\frac{n}{2}-1}$ , in each case appealing to Theorem 1.4(iii).

Now we suppose that  $i < \frac{n}{2}$ , and let  $q_1 = \min(2^{n-2i}, \lfloor \frac{k}{2} \rfloor)$ . Player I plays  $x_1 = \omega^i \cdot q_1 \in A$ , and by again appealing to Lemma 2.2, we may assume that II's response is of the form  $y_1 = \omega^i \cdot q_2$  for some  $q_2$  with  $1 \leq q_2 \leq l$ . Then  $A^{<x_1} = \omega^i \cdot q_1$ ,  $B^{<y_1} = \omega^i \cdot q_2$ ,  $A^{>x_1} \cong \omega^i(k - q_1) + \omega^{i-1} \cdot 3$ , and  $B^{>y_1} \cong \omega^i(l - q_2) + \omega^{i-1} \cdot 3$ . If  $q_2 < q_1$  then by Lemma 2.3(ii),  $\omega^i \cdot q_1 \not\equiv_n \omega^i \cdot q_2$ , so player I can win by playing on the left of  $x_1$  and  $y_1$ , and if  $q_2 \geq q_1$ , then  $k - q_1 > l - q_2$  and he can play on the right of  $x_1$  and  $y_1$  using  $\omega^i(k - q_1) + \omega^{i-1} \cdot 3 \not\equiv_n \omega^i(l - q_2) + \omega^{i-1} \cdot 3$ , which follows by the induction hypothesis, since  $l - q_2 < 2^{n-2i}$ . For if  $k \geq 2^{n+1-2i}$ , then  $q_2 \geq q_1 = 2^{n-2i}$ , so  $l - q_2 < 2^{n+1-2i} - 2^{n-2i} = 2^{n-2i}$  and if  $k < 2^{n+1-2i}$ , then  $q_1 = \lfloor \frac{k}{2} \rfloor$ , so  $l - q_2 < k - q_1 \leq \frac{k+1}{2} \leq 2^{n-2i}$ . ■

**Lemma 2.8.** *If  $n \geq 4$  is even,  $t = \frac{n}{2}$ , and  $l < 4 \leq m$ , then*

- (i)  $\omega^t + \omega^{t-2} \cdot m \equiv_n \omega^{t-1} \cdot 3 + \omega^{t-2} \cdot m$ ,
- (ii)  $\omega^t + \omega^{t-2} \cdot 3 \not\equiv_n \omega^{t-1} \cdot l + \omega^{t-2} \cdot 3$ .

*Proof.* (i) Let  $A = \omega^t + \omega^{t-2} \cdot m$  and  $B = \omega^{t-1} \cdot 3 + \omega^{t-2} \cdot m$ . First treating the case  $n = 4$ , we have  $A = \omega^2 + m$  and  $B = \omega \cdot 3 + m$ . Player II may play on his first move so that  $x_1 = y_1 < \omega \cdot 2$ , or for some finite  $q, r$  with  $q \geq 2$ ,  $x_1 = \omega \cdot q + r$ ,  $y_1 = \omega \cdot 2 + r$ , or  $x_1, y_1$  are corresponding points of the final  $m$  sections of  $A, B$ . To conclude it suffices to note that  $\omega^2 + m, \omega \cdot 3 + m, \omega \cdot 2 + m$  and  $\omega + m$  all exhibit the same 2-characters, namely  $\langle 0, 3 \rangle, \langle 1, 3 \rangle, \langle 2, 3 \rangle, \langle 3, 3 \rangle, \langle \omega, 3 \rangle, \langle 3, 2 \rangle, \langle 3, 1 \rangle, \langle 3, 0 \rangle$ , so are 3-equivalent.

In general we write  $A = \omega^{t-2}(\omega^2 + m)$  and  $B = \omega^{t-2}(\omega \cdot 3 + m)$ . On the first 4 moves, II employs a winning strategy in  $\omega^2 + m \equiv_4 \omega \cdot 3 + m$  on the copies of  $\omega^{t-2}$ , always playing the point in a copy of  $\omega^{t-2}$  corresponding to that played by I. Thereafter, II continues similarly as long as possible in ‘untouched’ copies of  $\omega^{t-2}$ , until he can do so no longer. He is then forced to play between two previous moves lying in consecutive copies of  $\omega^{t-2}$ . This means that he has to win a game between  $\omega^{t-2} + \gamma$  and  $\omega^{t-2} \cdot \beta + \gamma$  for some ordinals  $\beta \geq 1$  and  $\gamma < \omega^{t-2}$  in at most  $n - 4$  moves. Since  $\omega^{t-2} \cdot \beta \equiv_{n-4} \omega^{t-2}$  by Theorem 1.4(i), he can achieve this.

(ii) On his first four moves, player I plays  $x_1 = \omega^{t-1} \cdot 2$ ,  $x_2 = \omega^{t-1} \cdot 3$ ,  $x_3 = \omega^{t-1} \cdot 4$ ,  $x_4 = \omega^{t-1} \cdot 5$ . Since  $t - 1 < \frac{n}{2}, \frac{n-1}{2}$ , and  $t - 2 < \frac{n-2}{2}, \frac{n-3}{2}$ , by Lemma 2.2, player II must play multiples of  $\omega^{t-1}$  on his first two moves and of  $\omega^{t-2}$  on the next two. If  $y_1 = \omega^{t-1}$  then by Theorem 1.4(iii),  $A^{<x_1} \not\equiv_{n-1} B^{<y_1}$  so player I can win on the left. Hence we suppose that  $y_1 = \omega^{t-1} \cdot 2$  and  $y_2 = \omega^{t-1} \cdot 3$ . The only options for  $y_3$  and  $y_4$  are then  $\omega^{t-1} \cdot 3 + \omega^{t-2}$  and  $\omega^{t-1} \cdot 3 + \omega^{t-2} \cdot 2$ . Hence  $(x_2, x_3) \cong \omega^{t-1}$  and  $(y_2, y_3) \cong \omega^{t-2}$ , and by Theorem 1.4(iii),  $(x_2, x_3) \not\equiv_{n-3} (y_2, y_3)$  and again I wins. ■

**Lemma 2.9.** *Let  $\alpha$  and  $\beta$  be  $n$ -equivalent ordinals such that  $\alpha$  is a non-zero multiple of  $\omega^j$ . If  $i \leq \frac{n-3}{2}$ ,  $i < j \leq \frac{n}{2}$  and  $k \geq 2^{n-2i} - 4 > m$  is finite, and  $l > m$ , then*

- (i)  $\alpha + \omega^i \cdot k \equiv_n \beta + \omega^i(2^{n-2i} - 4)$ ,
- (ii)  $\alpha + \omega^i \cdot l \not\equiv_n \beta + \omega^i \cdot m$ .

*Proof.* We observe that by Lemma 2.1 it follows from the fact that  $\alpha$  is a multiple of  $\omega^j$  that  $\beta$  is too.

- (i) Note that as  $i \leq \frac{n-3}{2}$ ,  $n - 2i \geq 3$ , so  $2^{n-2i} - 4 \geq 4$ .

We use induction. When  $n = 3$ ,  $i$  must be 0, and we have to show that  $\alpha + k \equiv_3 \beta + 4$  for  $k \geq 4$ , which holds since these two ordinals have the same 2-characters (as  $\alpha$  and  $\beta$  are limit ordinals).

For the induction step, assume the result for  $n \geq 3$ , and let  $A = \alpha + \omega^i \cdot k$ ,  $B = \beta + \omega^i(2^{n+1-2i} - 4)$ , where  $\alpha \equiv_{n+1} \beta$ ,  $i \leq \frac{(n+1)-3}{2} = \frac{n}{2} - 1$ , and  $k \geq 2^{n+1-2i} - 4$ , and we show that  $A \equiv_{n+1} B$ .

**Case 1:** If  $i = \frac{n}{2} - 1$  then  $n$  is even and  $A = \alpha + \omega^i \cdot k$ ,  $B = \beta + \omega^i \cdot 4$ , where  $k \geq 4$ . Then player II can play so that for some  $\gamma < \omega^i$ ,

1.  $x_1 \in \alpha$  and  $y_1 \in \beta$  and II has used his winning strategy in the  $(n + 1)$ -move game on  $\alpha$  and  $\beta$ , or
2.  $x_1 = \alpha + \omega^i \cdot r + \gamma$ ,  $y_1 = \beta + \omega^i \cdot s + \gamma$ , where  $\gamma < \omega^i$  and  $r = s = 0$  or
3.  $k - r = 4 - s < 4$ , or
4.  $x_1 = \alpha + \omega^i \cdot r + \gamma$  and  $y_1 = \omega^i \cdot 4 + \gamma$  where  $0 < r \leq k - 4$ .

In these four cases we find that for some  $\alpha_1 \equiv_n \beta_1$ ,

1.  $A^{>x_1} \cong \alpha_1 + \omega^i \cdot k$ ,  $B^{>y_1} \cong \beta_1 + \omega^i \cdot 4$ , and  $A^{>x_1} \equiv_n B^{>y_1}$  by Lemma 2.3(i),
2.  $A^{>x_1} \cong \omega^i \cdot k$ ,  $B^{>y_1} \cong \omega^i \cdot 4$ , so again  $A^{>x_1} \equiv_n B^{>y_1}$  by Lemma 2.3(i),
3.  $A^{>x_1} \cong B^{>y_1}$ , and as  $r, s > 0$ ,  $A^{<x_1} \equiv_n B^{<y_1}$  by Lemma 2.6(i), since for some infinite ordinals  $\delta_1$  and  $\delta_2$ ,  $\alpha = \omega^i \cdot \delta_1$  and  $\beta = \omega^i \cdot \delta_2$ , which gives  $\alpha + \omega^i \cdot r = \omega^i(\delta_1 + r) \equiv_{i+1} \omega^i \cdot 4 \equiv_{i+1} \omega^i(\delta_2 + s) = \beta + \omega^i \cdot s$ ,
4. Since  $0 < r \leq k - 4$ , then  $A^{<x_1} \cong \alpha + \omega^i \cdot r + \gamma$ ,  $B^{<y_1} \cong \omega^i \cdot 4 + \gamma$ , and  $A^{>x_1} \cong \omega^i(k - r)$ ,  $B^{>y_1} \cong \beta + \omega^i \cdot 4$ , so  $A^{<x_1} \equiv_n B^{<y_1}$  and  $A^{>x_1} \equiv_n B^{>y_1}$  both follow from Lemmas 2.6(i) and 2.3(i).

**Case 2:** If  $i < \frac{n}{2} - 1$  then  $i \leq \frac{n-3}{2}$ . We observe that we may write

$$\begin{aligned} A &= \alpha + \omega^i(2^{n-2i} - 4) + \omega^i(k - (2^{n+1-2i} - 4)) + \omega^i \cdot 2^{n-2i} \text{ and} \\ B &= \beta + \omega^i(2^{n-2i} - 4) + \omega^i \cdot 2^{n-2i}. \end{aligned}$$

Player II can play so that



1.  $x_1 \in \alpha$  and  $y_1 \in \beta$  and he has used his winning strategy in the  $(n+1)$ -move game on  $\alpha$  and  $\beta$ , or
2.  $x_1 = \alpha + \gamma$  where  $\gamma < \omega^i(2^{n-2i} - 4)$ , and  $y_1 = \beta + \gamma$ , or
3.  $x_1 = \alpha + \omega^i \cdot q_1 + \gamma$  where  $\gamma < \omega^i$ ,  $2^{n-2i} - 4 \leq q_1 < k - 2^{n-2i}$ , and  $y_1 = \beta + \omega^i(2^{n-2i} - 4) + \gamma$ , or
4.  $x_1 = \alpha + \omega^i \cdot q_2 + \gamma$  and  $y_1 = \beta + \omega^i \cdot q_3 + \gamma$  where  $\gamma < \omega^i$ ,  $k - 2^{n-2i} \leq q_2 < k$ ,  $2^{n-2i} - 4 \leq q_3 < 2^{n+1-2i} - 4$ , and  $k - q_2 = 2^{n+1-2i} - 4 - q_3$ .

1. This case is handled as in Case 1.
2.  $A^{<x_1} \equiv_n B^{<y_1}$ . Also, we see that  $A^{>x_1} \cong \omega^i \cdot r_1$  and  $B^{>y_1} \cong \omega^i \cdot r_2$  where  $r_1 \geq k - (2^{n-2i} - 4)$  and  $r_2 \geq 2^{n-2i}$ . We note that  $k - (2^{n-2i} - 4) \geq 2^{n-2i}$  since  $k \geq 2^{n+1-2i} - 4$ . Therefore  $A^{>x_1} \equiv_n B^{>y_1}$  by Lemma 2.3(i).
3. By the induction hypothesis,  $A^{<x_1} \equiv_n B^{<y_1}$ . In addition,  $A^{>x_1} \cong \omega^i \cdot s$  where  $s \geq 2^{n-2i}$  and  $B^{>y_1} \cong \omega^i \cdot 2^{n-2i}$ . By Lemma 2.3(i),  $A^{>x_1} \equiv_n B^{>y_1}$ .
4.  $A^{<x_1} \equiv_n B^{<y_1}$  by the induction hypothesis, and  $A^{>x_1} \equiv_n B^{>y_1}$  is immediate, since  $A^{>x_1} \cong B^{>y_1}$ .

(ii) First we prove the result for  $k$  in place of  $l$  (for which the additional assumption is that  $k \geq 2^{n-2i} - 4$ ). If  $m = 0$ , then player I plays  $\alpha + \omega^i(k-1)$  on his first move. If  $y_1$  is II's reply, then  $A^{>x_1} \cong \omega^i$  (or  $\emptyset$  if  $i = 0$ ) and  $B^{>y_1} \cong \omega^j \cdot \gamma$  for some  $\gamma > 0$ , so by Theorem 1.4(iii),  $A^{>x_1} \not\equiv_{2i+1} B^{>y_1}$ , and as  $2i+1 \leq n-1$ , I wins. So we now assume that  $m > 0$ , and use induction, with  $n = 3$  as the basis case. Here,  $i = 0$  and  $m < 4 \leq k$ , and we have to show that  $\alpha + k \not\equiv_3 \beta + m$ . Since  $\alpha$  and  $\beta$  are limit ordinals,  $\alpha + k$  exhibits the 2-character  $\langle \omega, 3 \rangle$ , but  $\beta + m$  does not, so they are 3-inequivalent.

Now assume the result for  $n \geq 3$  and we prove it for  $n+1$ . Let  $A = \alpha + \omega^i \cdot k$  and  $B = \beta + \omega^i \cdot m$  where  $m < 2^{n+1-2i} - 4 \leq k$ ,  $\alpha \equiv_{n+1} \beta$ , and  $i \leq \frac{(n+1)-3}{2} = \frac{n}{2} - 1$ . Then  $A$  can be written in the form  $\alpha + \omega^i(2^{n-2i} - 4) + \omega^i \cdot q$  where  $q \geq 2^{n-2i}$ .

**Case 1:**  $i = 0$ .

Player I plays  $x_1 = \alpha + 2^n - 4$ . Then  $|A^{>x_1}| \geq 2^n - 1$  so if II plays  $y_1$  and  $|B^{>y_1}| < 2^n - 1$  then I wins (playing on the right) by the finite case. If  $|B^{>y_1}| \geq 2^n - 1$  and  $y_1 \geq \beta$ , then  $A^{<x_1} \cong \alpha + 2^n - 4$  and  $B^{<y_1} \cong \beta + r$  where  $r \leq m - 2^n < 2^n - 4$ , so I wins in the remaining  $n$  moves playing on the left using the induction hypothesis. If II plays a member  $y_1$  of  $\beta$ , then I plays  $y_2 = \beta$  (possible since  $m > 0$ ) and  $(x_1, x_2) \not\equiv_2 (y_1, y_2)$ , so I wins quickly.

**Case 2:**  $0 < i < \frac{n}{2} - 1$ .

Player I plays  $x_1 = \alpha + \omega^i(2^{n-2i} - 4)$ . By Lemma 2.2 we may assume that II plays a non-trivial multiple  $y_1$  of  $\omega^i$ .

1. If  $y_1 \geq \beta$  and  $B^{>y_1} < \omega^i \cdot 2^{n-2i}$ , then  $A^{>x_1} \not\equiv_n B^{>y_1}$  by Lemma 2.3(ii), and I wins.
2. If  $y_1 \geq \beta$ , and  $B^{>y_1} \geq \omega^i \cdot 2^{n-2i}$ , then  $B^{<y_1} \cong \beta + \omega^i \cdot s_1$  where  $s_1 < 2^{n-2i} - 4$  and  $A^{<x_1} \not\equiv_n B^{<y_1}$  by induction hypothesis.
3. If however  $y_1 < \beta$ , then player I plays  $y_2 = \beta$  on his second move. By Lemma 2.2 we may suppose that II plays a multiple  $x_2 = x_1 + \omega^i \cdot s_2$  of  $\omega^i$ , with  $s_2 > 0$ . Now I plays  $x_3 = x_1 + \omega^i(s_2 - 1)$ , and whatever  $y_3$  II plays,  $(x_3, x_2) \cong \omega^i$  and  $(y_3, y_2) \cong \omega^j \cdot s_3$  for some  $s_3 > 0$ . By Theorem 1.4(iii),  $\omega^i \not\equiv_{2i+1} \omega^j \cdot s_3$ , and since  $2i+1 \leq n-2$ , I wins.

**Case 3:**  $i = \frac{n}{2} - 1$ . Then  $2^{n+1-2i} - 4 = 4$  so  $m < 4 \leq k$ . Player I chooses  $x_1 = \alpha + \omega^i(k-3) \in A$ , so that  $A^{>x_1} \cong \omega^i \cdot 3$ . By Lemma 2.2 we may assume that II picks a multiple  $y_1$  of  $\omega^i$ .

1. If  $y_1 > \beta$  then  $A^{>x_1} \not\equiv_n B^{>y_1}$  follows from Lemma 2.3(ii).
2. If  $y_1 = \beta$ , I plays  $x_2 < x_1$  so that  $(x_2, x_1) \cong \omega^i$ , and  $(y_2, y_1)$  must be a multiple of  $\omega^j$ , so  $(x_2, x_1) \not\equiv_{2i+1} (y_2, y_1)$  by Theorem 1.4(iii), and as  $2i + 1 = n - 1$ , I wins.
3. Finally, if  $y_1 < \beta$ ,  $A^{>x_1} \cong \omega^{\frac{n}{2}-1} \cdot 3$  and  $B^{>y_1}$  is isomorphic to  $\omega^{\frac{n}{2}} \cdot \delta + \omega^{\frac{n}{2}-1} \cdot m$  for some  $\delta > 0$ , and so  $A^{>x_1} \not\equiv_n B^{>y_1}$  by Lemma 2.6(ii).

To deduce the result for  $l$ , if already  $l \geq 2^{n-2i} - 4$ , the result follows from what we have just done. Otherwise, let  $k' = 2^{n-2i} - 4$  and  $m' = m + (2^{n-2i} - 4 - l)$ . Then  $k' \geq 2^{n-2i} - 4 > m'$ , and from what we already have,  $\alpha + \omega^i \cdot k' \not\equiv_n \beta + \omega^i \cdot m'$ . Subtracting  $\omega^i(2^{n-2i} - 4 - l)$  from both sides, it follows that  $\alpha + \omega^i \cdot l \not\equiv_n \beta + \omega^i \cdot m$ .  $\blacksquare$

We next give an inductive generalization of Lemma 2.7.

**Lemma 2.10.** *If  $0 < r \leq i < \frac{n}{2}$ , then*

- (i)  $\omega^i \cdot 2^{n-2i} + \omega^{i-1} \cdot 3 + \dots + \omega^{i-r+1} \cdot 3 + \omega^{i-r} \cdot 4 \equiv_n \omega^i(2^{n-2i} - 1) + \omega^{i-1} \cdot 3 + \dots + \omega^{i-r+1} \cdot 3 + \omega^{i-r} \cdot 4$ ,
- (ii) *if  $b_j \leq 3$  for all  $j < i$ , or for some  $k < i$ ,  $b_j \leq 3$  for all  $j$  such that  $k \leq j < i$  and  $b_k \leq 2$ , then  $\omega^i \cdot 2^{n-2i} + \omega^{i-1} \cdot a_{i-1} + \dots + \omega \cdot a_1 + a_0 \not\equiv_n \omega^i(2^{n-2i} - 1) + \omega^{i-1} \cdot b_{i-1} + \dots + \omega \cdot b_1 + b_0$  for any integers  $a_{i-1}, \dots, a_0$ .*

*Proof.* (i) We use induction on  $n$ . The case  $r = 1$  is covered by Lemma 2.7(i), so we suppose that  $r \geq 2$ , which means that there is at least one term in the sum having a coefficient of 3. Let  $A = \omega^i \cdot 2^{n-2i} + \omega^{i-1} \cdot 3 + \dots + \omega^{i-r} \cdot 4$  and  $B = \omega^i(2^{n-2i} - 1) + \omega^{i-1} \cdot 3 + \dots + \omega^{i-r} \cdot 4$ .

**Case 1:**  $i = \frac{n-1}{2}$ . Then  $n$  is odd, and  $A = \omega^i \cdot 2 + \omega^{i-1} \cdot 3 + \dots + \omega^{i-r} \cdot 4$ , and  $B = \omega^i + \omega^{i-1} \cdot 3 + \dots + \omega^{i-r} \cdot 4$ . We use induction on  $r$ . Player II can play so that one of the following holds:

1.  $x_1 = y_1 < \omega^i$ . Then  $A^{<x_1} \equiv_{n-1} B^{<y_1}$ ,  $A^{>x_1} \cong \omega^i \cdot 2 + \omega^{i-1} \cdot 3 + \dots + \omega^{i-r} \cdot 4$ , and  $B^{>y_1} \cong \omega^i + \omega^{i-1} \cdot 3 + \dots + \omega^{i-r} \cdot 4$ , and so  $A^{>x_1} \equiv_{n-1} B^{>y_1}$  by Theorem 1.4(i).

2.  $\omega^i \leq x_1 = y_1 < \omega^i + \omega^{i-1}$ . This gives  $A^{<x_1} \equiv_{n-1} B^{<y_1}$ , and in addition  $A^{>x_1} \cong \omega^i + \omega^{i-1} \cdot 3 + \dots + \omega^{i-r} \cdot 4$  and  $B^{>y_1} \cong \omega^{i-1} \cdot 3 + \dots + \omega^{i-r} \cdot 4$ .

By Lemma 2.6(i),  $\omega^i + \omega^{i-1} \cdot 3 + \dots + \omega^{i-r} \cdot 4 \equiv_{n-1} \omega^{i-1} \cdot 4 + \omega^{i-2} \cdot 3 + \dots + \omega^{i-r} \cdot 4$ , so we just have to check that  $\omega^{i-1} \cdot 4 + \omega^{i-2} \cdot 3 + \dots + \omega^{i-r} \cdot 4 \equiv_{n-1} \omega^{i-1} \cdot 3 + \omega^{i-2} \cdot 3 + \dots + \omega^{i-r} \cdot 4$ . Player II may play so that  $x_2 = y_2 < \omega^{i-1} \cdot 3$ , or  $x_2 = \omega^{i-1} + y_2 \geq \omega^{i-1} \cdot 3$ . To make sure that this works, we have to check in the first case that  $A^{>x_2} \equiv_{n-2} B^{>y_2}$ , and in the second case, that  $(x_1, x_2) \equiv_{n-2} (y_1, y_2)$ . The former requires that  $\omega^{i-1} \cdot 2 + \omega^{i-2} \cdot 3 + \dots + \omega^{i-r} \cdot 4 \equiv_{n-2} \omega^{i-1} + \omega^{i-2} \cdot 3 + \dots + \omega^{i-r} \cdot 4$ , which follows from the induction hypothesis (on  $r$ ), as  $i - r = (i - 1) - (r - 1)$ , and the second says that  $\omega^{i-1} \cdot 3 \equiv_{n-2} \omega^{i-1} \cdot 2$ , which follows from Lemma 2.3(i).

3.  $x_1 = \omega^i + \omega^{i-1} + \gamma$  and  $y_1 = \omega^{i-1} \cdot 4 + \gamma$  for some  $\gamma < \omega^i$ . Then  $A^{<x_1} \equiv_{n-1} B^{<y_1}$  by Lemma 2.6(i). Furthermore,  $A^{>x_1} \cong B^{>y_1} \cong \omega^i + \omega^{i-1} \cdot 3 + \dots + \omega^{i-r} \cdot 4$  and hence  $A^{>x_1} \equiv_n B^{>y_1}$ .

4.  $x_1$  is an element of the last segment  $\omega^{i-1} \cdot 3 + \omega^{i-2} \cdot 3 + \dots + \omega^{i-r} \cdot 4$  of  $A$  and  $y_1$  is the corresponding point in the last segment of  $B$ . Then  $A^{>x_1} \cong B^{>y_1}$ , so  $A^{>x_1} \equiv_{n-1} B^{>y_1}$ . In addition,  $A^{<x_1} \cong \omega^i \cdot 2 + \gamma$  and  $B^{<y_1} \cong \omega^i + \gamma$ , where  $\gamma < \omega^i$ . By Theorem 1.4(i),  $A^{<x_1} \equiv_{n-1} B^{<y_1}$ .

**Case 2:**  $i < \frac{n-1}{2}$ . Player II can play so that one of the following occurs:

1.  $x_1 = y_1 < \omega^i(2^{n-1-2i} + 1)$ , giving  $A^{<x_1} \equiv_{n-1} B^{<y_1}$ , and  $A^{>x_1} \cong \omega^i \cdot q_1 + \omega^{i-1} \cdot 3 + \dots + \omega^{i-r} \cdot 4$  and  $B^{>y_1} \cong \omega^i \cdot q_2 + \omega^{i-1} \cdot 3 + \dots + \omega^{i-r} \cdot 4$  where  $q_1 \geq 2^{n-1-2i}$  and  $q_2 \geq 2^{n-1-2i} - 1$ , in which case  $A^{>x_1} \equiv_{n-1} B^{>y_1}$  by induction hypothesis.
2.  $x_1 = \omega^i(2^{n-1-2i} + 1) + \gamma$  and  $y_1 = \omega^i \cdot 2^{n-1-2i} + \gamma$  for some  $\gamma$ , and by Lemma 2.3(i),  $A^{<x_1} \equiv_{n-1} B^{<y_1}$ , and  $A^{>x_1} \cong B^{>y_1}$  so  $A^{>x_1} \equiv_{n-1} B^{>y_1}$ .

(ii) Let  $A = \omega^i \cdot 2^{n-2i} + \omega^{i-1} \cdot a_{i-1} + \dots + \omega \cdot a_1 + a_0$  and  $B = \omega^i(2^{n-2i} - 1) + \omega^{i-1} \cdot b_{i-1} + \dots + \omega \cdot b_1 + b_0$ . This idea is that the first term of  $A$  is powerful enough for player I to defeat player II, even without assistance from the remaining terms. Player I chooses  $x_1 < x_2 < \dots < x_{n-2i}$  where  $x_j = \omega^i(2^{n-2i} - 2^{n-j-2i})$ . By Lemma 2.2 applied successively to  $n, n-1, \dots, 2i+1$  we may suppose that II's moves are  $y_1 < y_2 < \dots < y_{n-2i}$  where  $y_j = \omega^i \cdot t_j$  for some  $t_j$ . If  $t_1 < 2^{n-1-2i}$  then by Lemma 2.3(ii) I wins by playing to the left of  $x_1, y_1$  in the remaining  $n-1$  moves, since  $A^{<x_1} \cong \omega^i \cdot 2^{n-1-2i} \not\equiv_{n-1} \omega^i \cdot l \cong B^{<y_1}$ , where  $l < 2^{n-1-2i}$ . Similarly, if any interval  $(y_j, y_{j+1})$  is shorter than the corresponding interval  $(x_j, x_{j+1})$ , then I wins there in the remaining  $n-j-1$  moves. So we may suppose that  $t_1 \geq 2^{n-1-2i}$  and  $t_{j+1} - t_j \geq 2^{n-j-1-2i}$ , so as  $\sum_{j=1}^{n-2i} 2^{n-j-2i} = 2^{n-2i} - 1$ , in fact  $y_j = x_j$  for each  $j$ .

In the remaining  $2i$  moves, I plays on the right, and as  $A^{>x_{n-2i}} \cong \omega^i + \omega^{i-1} \cdot a_{i-1} + \dots + \omega \cdot a_1 + a_0$  and  $B^{>y_{n-2i}} \cong \omega^{i-1} \cdot b_{i-1} + \dots + \omega \cdot b_1 + b_0$  (except that if  $i=1$ ,  $B^{>y_{n-2i}} \cong b_0 - 1$ ) we just have to see that  $\omega^i + \omega^{i-1} \cdot a_{i-1} + \dots + \omega \cdot a_1 + a_0 \not\equiv_{2i} \omega^{i-1} \cdot b_{i-1} + \dots + \omega \cdot b_1 + b'_0$  (where  $b'_0 = b_0$  if  $i > 1$  and it equals  $b_0 - 1$  if  $i=1$ ), and to make the induction work, we actually show that for any ordinal  $\alpha > 0$ ,  $\omega^i \cdot \alpha + \omega^{i-1} \cdot a_{i-1} + \dots + \omega \cdot a_1 + a_0 \not\equiv_{2i} \omega^{i-1} \cdot b_{i-1} + \dots + \omega \cdot b_1 + b'_0$ . The basis case  $i=1$  says that  $\omega \cdot \alpha + a_0 \not\equiv_2 b_0 - 1$  which holds since  $b_0 \leq 3$ : if  $a_0 \neq 0$  then  $\omega \cdot \alpha + a_0 \equiv_2 3 \not\equiv_2 b_0 - 1$  as  $b_0 \leq 3$ , and if  $a_0 = 0$ ,  $\omega \cdot \alpha + a_0 \not\equiv_2 b_0 - 1$  is immediate.

Assuming the result for  $i$ , we describe a winning strategy for player I in the  $(2i+2)$ -move game on  $A_{i+1} = \omega^{i+1} \cdot \alpha + \omega^i \cdot a_i + \dots + \omega \cdot a_1 + a_0$  and  $B_{i+1} = \omega^i \cdot b_i + \dots + \omega \cdot b_1 + b'_0$ . On his first two moves, player I plays  $x_1 = \omega^i \cdot 2$  and  $x_2 = \omega^i \cdot 3$ . Let  $y_1, y_2$  be player II's responses. By Lemma 2.2 we suppose that  $y_1 = \omega^i \cdot t_1$  and  $y_2 = \omega^i \cdot t_2$ . Then  $0 < t_1 < t_2 \leq 3$  so  $t_1 \leq 2$ . If  $t_1 = 1$  then  $A_{i+1}^{<x_1} \cong \omega^i \cdot 2$  and  $B_{i+1}^{<y_1} \cong \omega^i$ . By Theorem 1.4(iii),  $A_{i+1}^{<x_1} \not\equiv_{2i+1} B_{i+1}^{<y_1}$  so I wins in the remaining  $2i+1$  moves. Otherwise,  $t_1 = 2$  and  $t_2 = 3$  (so actually  $b_i = 3$ ). Therefore  $A_i = A_{i+1}^{>x_2} \cong \omega^{i+1} \cdot \alpha + \omega^i \cdot a_i + \omega^{i-1} \cdot a_{i-1} + \dots + \omega \cdot a_1 + a_0$  and  $B_i = B_{i+1}^{>y_2} \cong \omega^{i-1} \cdot b_{i-1} + \dots + \omega \cdot b_1 + b'_0$ , noting that in the special case where  $i=1$ , since  $y_2$  is the first point of the finite block at the end, the previous value of  $b'_0$  which equalled  $b_0$  has decreased by 1, to the new (and correct) value of  $b'_0$ . Since  $A_i$  may be written in the form  $\omega^i(\omega \cdot \alpha + a_i) + \omega^{i-1} \cdot a_{i-1} + \dots + \omega \cdot a_1 + a_0$ , we may appeal to the induction hypothesis to see that this is not  $2i$ -equivalent to  $B_i$ , giving the induction step.  $\blacksquare$

The next lemma is similar to the previous one, though proved without using induction.

**Lemma 2.11.** *If  $\alpha$  is an ordinal of the form  $\omega^k \cdot \gamma$  where  $\gamma \geq 1$ , and  $\alpha \equiv_n \beta$ ,  $1 \leq i < \frac{n}{2} - 1$ ,  $0 < r \leq i < k \leq \frac{n}{2}$ , and  $m \geq 4$ , then*

- (i)  $\alpha + \omega^i(2^{n-2i} - 4) + \omega^{i-1} \cdot 3 + \dots + \omega^{i-r+1} \cdot 3 + \omega^{i-r} \cdot m$   
 $\equiv_n \beta + \omega^i(2^{n-2i} - 5) + \omega^{i-1} \cdot 3 + \dots + \omega^{i-r+1} \cdot 3 + \omega^{i-r} \cdot m,$
- (ii) *if  $b_j \leq 3$  for all  $j < i$ , or for some  $l < i$ ,  $b_j \leq 3$  for all  $j$  such that  $l \leq j < i$  and  $b_l \leq 2$ , then  $\alpha + \omega^i(2^{n-2i} - 4) + \omega^{i-1} \cdot a_{i-1} + \dots + \omega \cdot a_1 + a_0$   
 $\not\equiv_n \beta + \omega^i(2^{n-2i} - 5) + \omega^{i-1} \cdot b_{i-1} + \dots + \omega \cdot b_1 + b_0$ , provided that  $\gamma = 1$  in the case where  $i = \frac{n-3}{2}$ .*

*Proof.* (i) Let  $A = \alpha + \omega^i(2^{n-2i} - 4) + \omega^{i-1} \cdot 3 + \dots + \omega^{i-r+1} \cdot 3 + \omega^{i-r} \cdot m$  and  $B = \beta + \omega^i(2^{n-2i} - 5) + \omega^{i-1} \cdot 3 + \dots + \omega^{i-r+1} \cdot 3 + \omega^{i-r} \cdot m$ .

**Case 1:**  $i = \frac{n-3}{2}$  (so that  $n$  is odd). Here  $2^{n-2i} = 8$ , so  $A = \alpha + \omega^i \cdot 4 + \omega^{i-1} \cdot 3 + \dots + \omega^{i-r} \cdot m$  and  $B = \beta + \omega^i \cdot 3 + \omega^{i-1} \cdot 3 + \dots + \omega^{i-r} \cdot m$ . Player II may play so that one of the following holds:

1.  $x_1 \in \alpha$  and  $y_1 \in \beta$  correspond under a winning strategy for II in  $\alpha \equiv_n \beta$ ,
2.  $x_1$  and  $y_1$  are corresponding points of the  $r_0$ th and  $r_1$ th copies of  $\omega^i$  where  $r_0 = r_1 = 0$  or  $r_0 = r_1 + 1 \geq 2$ ,
3.  $x_1$  and  $y_1$  are corresponding points of the  $r_0$ th copy of  $\omega^i$  in the  $\omega^i \cdot 4$  section of  $A$  and of the  $r_1$ th copy of  $\omega^i$  in  $\beta$  where  $r_0 = 1$  and  $r_1 = 4$ ,
4.  $x_1$  and  $y_1$  are corresponding points of  $\omega^{i-1} \cdot 3 + \dots + \omega^{i-r+1} \cdot 3 + \omega^{i-r} \cdot m$ .

To show that this works, we need to verify the following  $(n-1)$ -equivalences:

1.  $\omega^i \cdot 4 + \omega^{i-1} \cdot 3 + \dots + \omega^{i-r} \cdot m \equiv_{n-1} \omega^i \cdot 3 + \omega^{i-1} \cdot 3 + \dots + \omega^{i-r} \cdot m$ , which follows from Lemma 2.10(i), since  $0 < r \leq i < \frac{n-1}{2}$ , and since  $n-1-2i = 2$ ,
2.  $\alpha + \omega^i \cdot 2 \equiv_{n-1} \beta + \omega^i$ , which follows from Lemma 2.6(i) thus:  $\alpha + \omega^i \cdot 2 \equiv_{n-1} \omega^i \cdot 4 \equiv_{n-1} \beta + \omega^i$ ,
3.  $\alpha + \omega^i \equiv_{n-1} \omega^i \cdot 4$ , which follows from Lemma 2.6(i), and  
 $\omega^i \cdot 3 + \omega^{i-1} \cdot 3 + \dots + \omega^{i-r} \cdot m \equiv_{n-1} \beta + \omega^i \cdot 3 + \omega^{i-1} \cdot 3 + \dots + \omega^{i-r} \cdot m$ , which holds since  $\beta + \omega^i \cdot 3 + \omega^{i-1} \cdot 3 + \dots + \omega^{i-r} \cdot m$   
 $\equiv_{n-1} \omega^i \cdot 4 + \omega^{i-1} \cdot 3 + \dots + \omega^{i-r} \cdot m$  by Lemma 2.6(i)  
 $\equiv_{n-1} \omega^i \cdot 3 + \omega^{i-1} \cdot 3 + \dots + \omega^{i-r} \cdot m$  by Lemma 2.10(i),
4.  $\alpha + \omega^i \cdot 4 \equiv_{n-1} \beta + \omega^i \cdot 3$ , which follows from Lemma 2.6(i).

**Case 2:**  $i < \frac{n-3}{2}$ . Then  $n-1-2i > 2$  so  $2^{n-1-2i} - 4 \geq 4$ . We subdivide  $A$  and  $B$  as follows:

$$\begin{aligned} A &= \alpha + \omega^i(2^{n-1-2i} - 4) + \omega^i + \omega^i(2^{n-1-2i} - 1) + \omega^{i-1} \cdot 3 + \dots + \omega^{i-r+1} \cdot 3 + \omega^{i-r} \cdot m, \\ B &= \beta + \omega^i(2^{n-1-2i} - 4) + \omega^i(2^{n-1-2i} - 1) + \omega^{i-1} \cdot 3 + \dots + \omega^{i-r+1} \cdot 3 + \omega^{i-r} \cdot m. \end{aligned}$$

Then II can play so that one of the following holds:

1.  $x_1 \in \alpha$  and  $y_1 \in \beta$  correspond under a winning strategy for II in  $\alpha \equiv_n \beta$ ,
2.  $x_1$  and  $y_1$  are corresponding points of the  $\omega^i(2^{n-1-2i} - 4)$ ,  $\omega^i(2^{n-1-2i} - 1)$ , or  $\omega^{i-1} \cdot 3 + \dots + \omega^{i-r+1} \cdot 3 + \omega^{i-r} \cdot m$  sections,
3.  $x_1$  lies in the middle  $\omega^i$  section of  $A$ , and  $y_1$  is the corresponding point of the first  $\omega^i$  of the  $\omega^i(2^{n-1-2i} - 1)$  section of  $B$ .

To see that this works, we need to verify the following  $(n-1)$ -equivalences:

1.  $\omega^i \cdot 2^{n-1-2i} + \omega^{i-1} \cdot 3 + \dots + \omega^{i-r+1} \cdot 3 + \omega^{i-r} \cdot m \equiv_{n-1} \omega^i(2^{n-1-2i} - 1) + \omega^{i-1} \cdot 3 + \dots + \omega^{i-r+1} \cdot 3 + \omega^{i-r} \cdot m$ , which follows from Lemma 2.10(i),
2.  $\alpha + \omega^i(2^{n-1-2i} - 3) \equiv_{n-1} \beta + \omega^i(2^{n-1-2i} - 4)$ , which follows from Lemma 2.9(i).
- (ii) Let  $A = \alpha + \omega^i(2^{n-2i} - 4) + \omega^{i-1} \cdot a_{i-1} + \dots + \omega \cdot a_1 + a_0$  and  $B = \beta + \omega^i(2^{n-2i} - 5) + \omega^{i-1} \cdot b_{i-1} + \dots + \omega \cdot b_1 + b_0$ . On his first move, player I plays  $x_1 = \alpha + \omega^i(2^{n-1-2i} - 4)$ . By Lemma 2.2 we may suppose that II plays a multiple  $y_1$  of  $\omega^i$ . If  $y_1 \geq \beta + \omega^i(2^{n-1-2i} - 4)$  then  $A^{>x_1} \cong \omega^i \cdot 2^{n-1-2i} + \omega^{i-1} \cdot a_{i-1} + \dots + \omega \cdot a_1 + a_0$  and  $B^{>y_1} \cong \omega^i \cdot q_1 + \omega^{i-1} \cdot b_{i-1} + \dots + \omega \cdot b_1 + b_0$  for some  $q_1 < 2^{n-1-2i}$ , and I wins by Lemma 2.10(ii).

We suppose therefore that  $y_1 < \beta + \omega^i(2^{n-1-2i} - 4)$ . If  $y_1 = \beta + \omega^i \cdot q_2$  for some  $q_2 < 2^{n-1-2i} - 4$ , then I wins by Lemma 2.9(ii). Otherwise,  $y_1 < \beta$ . Next I plays  $y_2 > y_1$  so that  $(y_1, y_2) \cong \omega^{i+1}$ , and whatever  $x_2$  II plays,  $(x_1, x_2) \cong \gamma_1 + \omega^r$  for some ordinal  $\gamma_1$ , where  $r \leq i$ . Thus, provided  $i+1 \leq \frac{n-2}{2}$ , I wins by using Lemma 2.1. If this fails, then  $i+1 > \frac{n-2}{2}$ , and since  $i < \frac{n}{2} - 1$ , the only possibility remaining that we need to cover is where  $i = \frac{n-3}{2}$  (so in particular,  $n$  is odd).

Since  $n - 2i = 3$ , in this case,  $A = \alpha + \omega^i \cdot 4 + \omega^{i-1} \cdot a_{i-1} + \dots + \omega \cdot a_1 + a_0$  and  $B = \beta + \omega^i \cdot 3 + \omega^{i-1} \cdot b_{i-1} + \dots + \omega \cdot b_1 + b_0$ . Also, by assumption,  $\alpha = \beta = \omega^{i+1}$ .

In this instance, player I plays  $x_1 = \alpha$  and unless II plays in  $\beta$ , as we have seen, I can win as before, so suppose that  $y_1 < \beta$ . If  $y_1 = 0$ , I wins by playing  $x_2 < x_1$ . Otherwise he plays  $y_2 < y_1$  so that  $[y_2, y_1] \cong \omega^r$  for some  $r \leq i$ , and whatever  $x_2 < x_1$  II plays,  $(x_2, x_1) \cong \omega^{i+1}$  and I wins by Theorem 1.4(iii) since  $(y_2, y_1) \not\cong_{2i+1} (x_2, x_1)$  and  $2i + 1 = n - 2$ . ■

We are now ready to state the conditions which feature in the main result of this section. Recall that  $t$  stands for the integer part of  $\frac{n}{2}$ .

One of the clauses refers to the notion of ‘ $n$ -optimality’ of a coefficient; this is only required when this coefficient is 3, and we find it easiest to give an explicit definition. Namely, 3 is an  $n$ -optimal coefficient for  $\omega^k$  in  $\alpha = \omega^i \cdot a_i + \dots + a_0 \in \Omega'_n$  if  $a_i \neq 0$ ,  $k < i$ ,  $a_k = 3$  and either  $i < \frac{n}{2}$  and  $a_i = 2^{n-2i}$ ,  $a_{i-1} = \dots = a_k = 3$ , or for some  $j < i$  such that  $k < j$ ,  $a_j = 2^{n-2j} - 4$ ,  $a_{j-1} = \dots = a_k = 3$ . (The intuition, and an alternative definition as in [6], is that increasing the coefficient of  $\omega^k$  from 3 to 4 results in an ordinal which does *not* lie in  $\Omega'_n$ —see Lemmas 2.10 and 2.11, but this seems harder to work with in practice.)

For any  $n \geq 0$ , we let  $\Omega'_n$  be the set of ordinals of the form

$$\alpha = \omega^t \cdot a_t + \omega^{t-1} \cdot a_{t-1} + \omega^{t-2} \cdot a_{t-2} + \dots + \omega \cdot a_1 + a_0$$

such that

- (1)  $a_i \leq 2^{n-2i}$
- (2)  $a_i \leq 2^{n-2i} - 4$  if  $a_j \neq 0$  for some  $j > i$
- (3)  $a_i \leq 3$  if  $i < \frac{n}{2} - 1$  and  $a_{i+1} = 2^{n-2(i+1)}$ , or
- (4) if  $a_{i+1} = 2^{n-2(i+1)} - 4$  and  $a_j \neq 0$  for some  $j > i + 1$ , or
- (5) if  $a_{i+1} = 3$  and 3 is an  $n$ -optimal coefficient for  $\omega^{i+1}$
- (6)  $a_0 \leq 2^n - 1$  if  $\alpha$  is finite.

In Lemma 2.12 we shall establish ‘optimality’.

**Lemma 2.12.** *Let  $\alpha = \omega^i \cdot a_i + \omega^{i-1} \cdot a_{i-1} + \dots + \omega \cdot a_1 + a_0$  and  $\beta = \omega^i \cdot b_i + \omega^{i-1} \cdot b_{i-1} + \dots + \omega \cdot b_1 + b_0$  where  $\alpha \in \Omega'_n$ . If  $\alpha > \beta$ , then  $\alpha \not\cong_n \beta$ .*

*Proof.* We must show that player I has a winning strategy in the  $n$ -move game on  $\alpha$  and  $\beta$ . Since  $\alpha \in \Omega'_n$ ,  $a_j \leq 2^{n-2j}$  for each  $j$  by clause (1).

If  $a_i = 0$  then as  $\beta < \alpha$ , also  $b_i = 0$ , so these terms could be omitted. We may therefore assume that  $a_i \neq 0$ . If  $\alpha$  is finite, then the result follows from Lemma 1.3, so from now on we assume that  $\alpha$  is infinite, that is,  $i > 0$ .

Let  $j$  be the largest number such that  $a_0 = a_1 = \dots = a_{j-1} = 0$  (so that  $j \leq i$ ). Unless also  $b_0 = b_1 = \dots = b_{j-1} = 0$ , I can win, as otherwise if  $r < j$  is the least such that  $b_r \neq 0$  then we can write  $\alpha$  as  $\omega^j \cdot \gamma_0$  and  $\beta$  as  $\gamma_1 + \omega^r$ , so by Lemma 2.1,  $\alpha \not\cong_n \beta$ . Similarly in reverse. Hence we may suppose that the last non-zero terms of the expressions for  $\alpha$  and  $\beta$  occur at the same point. If  $j = i$  then  $\alpha = \omega^i \cdot a_i$  and  $\beta = \omega^i \cdot b_i$ , and by Lemma 2.3(ii) and since  $b_i < a_i \leq 2^{n-2i}$ , we at once deduce that  $\alpha \not\cong_n \beta$ , so from now on suppose that  $j < i$ , i.e. the expressions for  $\alpha$  and  $\beta$  each contains at least two terms.

We now fix  $k \leq i$  as the first point (i.e. having largest index) for which  $a_k \neq b_k$ . Since  $\alpha > \beta$ ,  $a_k > b_k$ .

**Case 1:**  $j = k$ . Since  $j < i$ , also  $k < i$ , so by clause (2),  $a_k \leq 2^{n-2k} - 4$ . Since  $a_k > 0$ , it follows that  $n - 2k > 2$ , so that  $\frac{n-3}{2} \geq k$ . We may therefore appeal to Lemma 2.9(ii) to see that  $\alpha \not\equiv_n \beta$ .

**Case 2:** For some  $r \leq \frac{n}{2} - 2$ ,  $r < i$ ,  $b_r < a_r$  and  $b_r < 2^{n-2-2r}$  or  $a_r < b_r$  and  $a_r < 2^{n-2-2r}$ .

We suppose that  $b_r < a_r$  and  $b_r < 2^{n-2-2r}$ .

On his first move, player I chooses  $x_1 = \omega^i \cdot a_i + \dots + \omega^{r+1} \cdot a_{r+1} \in \alpha$ . Since  $r+1 < \frac{n}{2}$ , by Lemma 2.2 we may suppose that  $y_1$  is a multiple of  $\omega^{r+1}$ .

If  $y_1 < \omega^i \cdot b_i + \dots + \omega^{r+1} \cdot b_{r+1}$  then I plays  $y_2 = \omega^i \cdot b_i + \dots + \omega^{r+1} \cdot b_{r+1}$ , and now  $(y_1, y_2)$  is a multiple of  $\omega^{r+1}$ . Whatever  $x_2$  is played by II,  $(x_1, x_2) \cong \gamma + \omega^s$  for some  $s \leq r$ , and so as  $s < r+1 \leq \frac{n-2}{2}$ , I wins by appeal to Lemma 2.1. If however  $y_1 = \omega^i \cdot b_i + \dots + \omega^{r+1} \cdot b_{r+1}$ , I instead plays  $x_2 = \omega^i \cdot a_i + \dots + \omega^{r+1} \cdot a_{r+1} + \omega^r(b_r + 1)$ , provided this is less than  $\alpha$  (and if it equals  $\alpha$ , he plays  $x_2 = \omega^i \cdot a_i + \dots + \omega^{r+1} \cdot a_{r+1} + \omega^r \cdot b_r$ ; this happens if  $a_r = b_r + 1$  and all  $a_s$  for  $s < r$  are zero). Since  $r < \frac{n-1}{2}$ , Lemma 2.2 allows us to assume that II plays a multiple  $y_2$  of  $\omega^r$ . Then  $(x_1, x_2) \cong \omega^r(b_r + 1)$  (or  $\omega^r \cdot b_r$  in the second case) and  $(y_1, y_2) \cong \omega^r \cdot t_0$  where  $t_0 \leq b_r$  (or  $t_0 < b_r$  in the second case). It follows by Lemma 2.3(ii) that  $(x_1, x_2) \not\equiv_{n-2} (y_1, y_2)$ , and so I wins in the remaining  $n-2$  moves.

Cases 3, 4, 5, and 6 cover all instances in which  $k = i$ .

**Case 3:**  $k = i < \frac{n}{2}$  and  $b_i < 2^{n-1-2i}$ .

Player I chooses  $x_1 = \omega^i(b_i + 1)$ . Let  $y_1 \in \beta$  be II's reply. By Lemma 2.2 we may suppose that  $y_1 = \omega^i \cdot t_1$  for some  $t_1$  and then  $\beta^{<y_1} \not\equiv_{n-1} \alpha^{<x_1}$  by Lemma 2.3(ii).

**Case 4:**  $k = i < \frac{n}{2}$  and  $2^{n-1-2i} \leq b_i \leq 2^{n-2i} - 2$ .

Player I plays  $x_1 < x_2 < \dots$  as far as possible so that  $x_1 = \omega^i \cdot 2^{n-1-2i}$ , and  $(x_q, x_{q+1}) \cong \omega^i \cdot 2^{n-q-1-2i}$ . Since  $a_i > 2^{n-1-2i}$ ,  $x_1$  exists. Now assume that  $x_1, x_2, \dots, x_r$  have been chosen fulfilling these conditions. Then  $x_r = \omega^i(2^{n-1-2i} + 2^{n-2-2i} + \dots + 2^{n-r-2i}) = \omega^i(2^{n-2i} - 2^{n-r-2i})$ . Thus  $a_i \geq 2^{n-2i} - 2^{n-r-2i}$ , and if  $a_i \geq 2^{n-2i} - 2^{n-r-1-2i}$  then  $x_{r+1}$  can be chosen as desired. Otherwise, if  $x_r = \omega^i \cdot a_i$ , then we stop with  $s = r$ , and if  $x_r < \omega^i \cdot a_i$  we let  $x_{r+1} = \omega^i \cdot a_i$  and  $s = r+1$ . Then  $(x_r, x_{r+1}) \cong \omega^i(a_i - (2^{n-2i} - 2^{n-r-2i})) < \omega^i((2^{n-2i} - 2^{n-r-1-2i}) - (2^{n-2i} - 2^{n-r-2i})) = \omega^i(2^{n-r-2i} - 2^{n-r-1-2i})$ . Note that  $s \leq n-2i$ , and there are  $n-s \geq 2i$  moves remaining.

Let II's moves in  $\beta$  be  $y_1 < y_2 < \dots < y_s$ . By Lemma 2.2 applied to  $n, n-1, \dots, n-(s-1)$  we may suppose that  $y_r = \omega^i \cdot t_r$  for some  $t_r$ . If  $t_1 < 2^{n-1-2i}$  then I wins by Lemma 2.3(ii), and similarly if any of the intervals between successive  $y_r$ 's is less than the corresponding intervals between the  $x_r$ 's. So we suppose that  $t_1 \geq 2^{n-1-2i}$ ,  $t_2 \geq 2^{n-2-2i}$ , and so on, which shows that  $b_i \geq a_i$  after all, contrary to supposition.

**Case 5:**  $k = i < \frac{n}{2}$  and  $b_i = 2^{n-2i} - 1$ .

It follows from the hypotheses that  $a_i = 2^{n-2i}$ .

To streamline consideration of the cases which can arise, we note that by Lemma 2.10(ii), if  $b_l \leq 3$  for all  $l \leq i-1$ , or if there is  $r \leq i-1$  such that  $b_l \leq 3$  for all  $l$  with  $r \leq l \leq i-1$  and  $b_r \leq 2$ , then  $\omega^i \cdot 2^{n-2i} + \omega^{i-1} \cdot a_{i-1} + \dots + \omega \cdot a_1 + a_0 \not\equiv_n \omega^i(2^{n-2i} - 1) + \omega^{i-1} \cdot b_{i-1} + \dots + \omega \cdot b_1 + b_0$ . Thus we may assume that for some  $l \leq i-1$ ,  $b_{i-1} = b_{i-2} = \dots = b_{l+1} = 3$  and  $b_l > 3$ . Let  $r$  be greatest such that  $l \leq r \leq i-1$  and  $a_r \neq b_r$ . This exists, since if not, then  $a_r = b_r$  for all  $r$  in this range, which implies that  $a_l > 3$ , violating clause (5) (or clause (3) if  $l = i-1$ ). Hence, if  $r < m \leq i-1$ ,  $a_m = b_m = 3$ , so by the definition of 'n-optimal coefficient', and clause (5),  $a_r \leq 3$ , and hence  $a_r < b_r$ .

**Case 5A:** For the  $r$  just defined,  $r \leq \frac{n}{2} - 2$ .

Since  $a_r \leq 3 < 4 = 2^{n-2-n+4} \leq 2^{n-2-2r}$ , the result follows from Case 2.

**Case 5B:**  $r > \frac{n}{2} - 2$ .

It follows that  $r \geq \frac{n-3}{2}$ , so as  $r < i < \frac{n}{2}$ ,  $n$  must be odd,  $i = \frac{n-1}{2}$ , and  $r = i - 1$ .

Then  $\alpha = \omega^i \cdot 2 + \omega^{i-1} \cdot a_{i-1} + \dots + a_0$  and  $\beta = \omega^i + \omega^{i-1} \cdot b_{i-1} + \dots + b_0$ ,  $a_{i-1} \leq 3$  and  $a_{i-1} < b_{i-1}$ . Now I plays  $x_1 = \omega^i \cdot 2$ , and by appealing to Lemma 2.2, II must play  $y_1 = \omega^i$ . If  $\alpha = \omega^i \cdot 2 + \omega^{i-1} \cdot a_{i-1}$  and  $\beta = \omega^i + \omega^{i-1} \cdot b_{i-1}$  (i. e. there are no terms beyond this point) then  $\alpha^{>x_1} \cong \omega^{i-1} \cdot a_{i-1}$  and  $\beta^{>y_1} \cong \omega^{i-1} \cdot b_{i-1}$ , and as  $0 < a_{i-1} < b_{i-1}$  and  $a_{i-1} < 4$ , we may use Lemma 2.3(ii) to deduce that  $\alpha^{>x_1} \not\cong_{n-1} \beta^{>y_1}$ , and so I wins in the remaining moves. Otherwise, noting that  $b_{i-1} > a_{i-1} > 0$  so that  $b_{i-1} \geq 2$ , player I may play  $y_2 = \omega^i + \omega^{i-1} \cdot 2$  and by Lemma 2.2 we see that player II must play  $x_2 = \omega^i \cdot 2 + \omega^{i-1} \cdot t_0$ , where  $0 < t_0 \leq a_{i-1}$ . If  $t_0 = 1$ , then  $(x_1, x_2) \cong \omega^{i-1} \not\cong_{2i-1} \omega^{i-1} \cdot 2 \cong (y_1, y_2)$  by Theorem 1.4(iii), and  $2i - 1 = n - 2$ , so I wins. Hence we suppose that  $t_0 \geq 2$ . It follows that  $b_{i-1} > a_{i-1} \geq 2$ , so  $\beta > \omega^i + \omega^{i-1} \cdot 3$ . Hence I can play  $y_3 = \omega^i + \omega^{i-1} \cdot 3$ . By Lemma 2.2, II must respond with  $x_3 = \omega^i \cdot 2 + \omega^{i-1} \cdot t_1$  where  $t_0 < t_1 \leq a_{i-1}$ . Thus  $t_1 = 3$  (so that  $t_0$  must have equalled 2, and  $a_{i-1} = 3$ , and  $b_{i-1} > 3$ ). On subsequent moves, we play on the right, so that we have an  $(n - 3)$ -move game on  $\alpha^{>x_3} \cong \omega^{i-2} \cdot a_{i-2} + \dots + a_0$  and  $\beta^{>y_3} \cong \omega^{i-1}(b_{i-1} - 3) + \omega^{i-2} \cdot b_{i-2} + \dots + b_0$ , and we use the same method as in the proof of Lemma 2.10(ii) to show that I wins. Let player I play in the  $\omega^{i-1}(b_{i-1} - 3)$  section of  $\beta^{>y_3}$  (which we now know is non-empty) at intervals of  $\omega^{i-2} \cdot 2$ . More precisely, we let I play  $y_m = \omega^i + \omega^{i-1} \cdot 3 + \omega^{i-2} \cdot 2(m - 3)$  for  $m \geq 3$ . Appealing to clause (5), we see that either  $a_m = 3$  for all  $m \leq i - 1$ , or the least  $m$  for which this fails satisfies  $a_m \leq 2$ . Let Player II's moves be  $x_3 < x_4 < x_5 < \dots$  where  $x_3 = \omega^i \cdot 2 + \omega^{i-1} \cdot 3$ . By Lemma 2.2(ii), player II's next two moves  $x_4$  and  $x_5$  must both be multiples of  $\omega^{i-2}$ , so are of the form  $x_4 = \omega^i \cdot 2 + \omega^{i-1} \cdot 3 + \omega^{i-2} \cdot t_4$  and  $x_5 = \omega^i \cdot 2 + \omega^{i-1} \cdot 3 + \omega^{i-2} \cdot t_5$  where  $0 < t_4 < t_5 \leq a_{i-2}$ . If  $t_4 = 1$  then  $(x_3, x_4) \cong \omega^{i-2}$  and  $(y_3, y_4) \cong \omega^{i-2} \cdot 2$ , so these are not  $(n - 4)$ -equivalent by Theorem 1.4(iii), and I can win. Otherwise,  $t_4 = 2$  so,  $t_5 = 3$ , and  $a_{i-2} = 3$ . Continuing in this fashion, we see that each  $\omega^m \cdot a_m$  section of the expansion of  $\alpha$  contains exactly two of II's moves, and each  $a_m$  equals 3. This cannot continue for ever, so I's strategy wins.

**Case 6:**  $k = i = \frac{n}{2}$ .

Thus  $n$  is even. By clause (2),  $a_{i-1} \leq 2^2 - 4 = 0$ , so  $\alpha = \omega^i + \omega^{i-2} \cdot a_{i-2} + \dots + a_0$  and  $\beta = \omega^{i-1} \cdot b_{i-1} + \dots + b_0$ . By clause (4),  $a_{i-2} \leq 3$ .

If  $b_{i-1} = 0$  or 1, player I plays  $x_1 = \omega^i$  and by Lemma 2.2, player II must play a multiple of  $\omega^{i-1}$ . If  $b_{i-1} = 0$  this is impossible, so he loses. If  $b_{i-1} = 1$  he must play  $y_1 = \omega^{i-1}$  and now by Theorem 1.4(iii),  $\alpha^{<x_1} \cong \omega^i \not\cong_{n-1} \omega^{i-1} \cong \beta^{<y_1}$ . From now on we therefore suppose that  $b_{i-1} > 1$ .

Look at the first  $l \leq i - 2$ , if any, such that  $a_l \neq b_l$  or  $a_l = b_l \neq 3$ . By clause (4) or (5),  $a_l \leq 3$ . Hence if  $a_l \neq b_l$ , we may use Case 2 to deduce that  $\alpha \not\cong_n \beta$ . If  $a_l = b_l \neq 3$ , then as  $a_l \leq 3$ , actually  $a_l = b_l < 3$ , and by maximality of  $l$ ,  $b_{i-2} = b_{i-3} = \dots = b_{l+1} = 3$ . Player I plays  $y_1 = \omega^{i-1}(b_{i-1} - 1)$ . Then by Lemma 2.2, II must play a multiple of  $\omega^{i-1}$ . If this is  $x_1 = \omega^i$ , then I plays  $y_2 = \omega^{i-1} \cdot b_{i-1}$  and  $(x_1, x_2) \cong \gamma + \omega^r$  for some ordinal  $\gamma$ , and  $r \leq i - 2$ , so as  $r < i - 1 \leq \frac{n-2}{2}$ , I wins in the remaining  $n - 2$  moves by Lemma 2.1. Now assume that II plays  $x_1 = \omega^{i-1} \cdot t_0$  for some finite  $t_0$ . From now on, I plays  $x_2, x_3, \dots$  as long as necessary so that  $(x_s, x_{s+1}) \cong \omega^{i-1}$ . Let  $y_2, y_3, \dots$  be II's replies. If  $y_2 = \gamma + \omega^r$  for some  $r \leq i - 2$  then I wins by Lemma 2.1. So we assume that  $y_2 = \omega^{i-1} \cdot b_{i-1}$ . If for some  $s \geq 2$ ,  $(y_s, y_{s+1}) \not\cong_{n-s-1} \omega^{i-1}$ , then I can win on  $(x_s, x_{s+1})$  and  $(y_s, y_{s+1})$  in the remaining  $n - s - 1$  moves, so we show that this

happens for some  $s \leq n-1$ . Suppose for a contradiction therefore that for each  $s \leq n-1$ ,  $(y_s, y_{s+1}) \equiv_{n-s-1} \omega^{i-1}$ .

Let  $(y_s, y_{s+1}) \cong \gamma + \omega^r$ . Then if  $r \leq \frac{n-s-3}{2}$ , by Lemma 2.1,  $(y_s, y_{s+1}) \not\equiv_{n-s-1} \omega^{i-1}$ . Hence  $r > \frac{n-s-3}{2}$ . If  $s$  is even this tells us that  $r \geq \frac{n-s-2}{2}$ , and if  $s$  is odd, that  $r \geq \frac{n-s-1}{2}$ . We deduce that  $y_3 = y_2 + \omega^{i-2} \cdot t_0$  and  $y_4 = y_2 + \omega^{i-2} \cdot t_1$  for some  $t_1 > t_0 > 0$ . By Theorem 1.4(iii),  $\omega^{i-1} \not\equiv_{n-3} \omega^{i-2}$ , so as  $(x_2, x_3) \cong \omega^{i-1}$  and  $(x_2, x_3) \equiv_{n-3} (y_2, y_3)$ , it follows that  $(y_2, y_3) \not\cong \omega^{i-2}$  and hence  $t_0 > 1$ . Since  $y_4 \leq \omega^{i-1} \cdot b_{i-1} + \omega^{i-2} \cdot b_{i-2}$  and  $b_{i-2} = 3$  we deduce that  $t_0 = 2$ ,  $t_1 = 3$ , and  $y_4 = \omega^{i-1} \cdot b_{i-1} + \omega^{i-2} \cdot b_{i-2}$ . Repeating this argument, we find that  $y_6 = \omega^{i-1} \cdot b_{i-1} + \omega^{i-2} \cdot b_{i-2} + \omega^{i-3} \cdot b_{i-3}, \dots$ , and when we reach the term in  $\omega^l$ , the corresponding  $t_0$  is forced to be 1 since  $b_l \leq 2$ , giving a contradiction.

Finally, suppose that there is no such  $l$ . It follows from clause (5) that for every  $l \leq i-2$ ,  $a_l = b_l = 3$ . In the play described above, player II's moves must continue to the constant term. He has now played  $2(i-1) + 1 = n-1$  moves and ends by playing the final point of  $\beta$ , so I wins on the last move.

We now consider cases in which  $k < i$ , and subdivide in a similar way to Cases 3-6. Since we suppose that Case 1 does not apply,  $a_l \neq 0$  for some  $l < k$ . Note further that if  $k = \frac{n}{2} - 1$  then  $i = \frac{n}{2}$ , so that  $a_k = 0$ , contrary to  $b_k < a_k$ . Hence  $k < \frac{n}{2} - 1$ . By clause (2),  $a_k \leq 2^{n-2k} - 4$ .

**Case 7:**  $k < i$ ,  $a_l \neq 0$  for some  $l < k$ , and  $b_k < 2^{n-1-2k} - 4$ .

Note that it follows from this that  $n-1-2k > 2$ , so  $k < \frac{n-3}{2}$ .

Player I chooses  $x_1 = \omega^i \cdot a_i + \dots + \omega^{k+1} \cdot a_{k+1} + \omega^k(b_k + 1)$ . Then II's response  $y_1$  must be a multiple of  $\omega^k$  (using Lemma 2.2, since  $k < \frac{n}{2}$ ). If  $y_1 < \omega^i \cdot a_i + \dots + \omega^{k+1} \cdot a_{k+1}$  then I plays  $y_2 = \omega^i \cdot a_i + \dots + \omega^{k+1} \cdot a_{k+1}$ . Whatever II's reply  $x_2$  is,  $x_2 = \gamma + \omega^r$  for some  $r \leq k$ , so as  $r < k+1 \leq \frac{n-2}{2}$ , I wins using Lemma 2.1. If  $y_1 = \omega^i \cdot a_i + \dots + \omega^{k+1} \cdot a_{k+1}$  then I plays  $x_2 < x_1$  so that  $(x_2, x_1) \cong \omega^k$ . Whatever  $y_2$  is chosen by II,  $(y_2, y_1)$  is a multiple of  $\omega^{k+1}$ , and so by Theorem 1.4(iii),  $(x_2, x_1) \not\equiv_{2k+1} (y_2, y_1)$ , so as  $2k+1 \leq n-2$ , I wins. Otherwise,  $y_1 = \omega^i \cdot a_i + \dots + \omega^{k+1} \cdot a_{k+1} + \omega^k \cdot t_1$  for some  $t_1$  with  $0 < t_1 \leq b_k$ . By Lemma 2.9(ii),  $\alpha^{<x_1} \not\equiv_{n-1} \beta^{<y_1}$ , so I wins.

**Case 8A:**  $k < i$ ,  $a_l \neq 0$  for some  $l < k$ ,  $k \geq \frac{n-3}{2}$ , and  $2^{n-1-2k} - 4 \leq b_k < 2^{n-2k} - 5$ .

We follow a similar strategy to Case 4.

Player I chooses  $x_1 < x_2 < \dots$  as far as possible so that  $x_l = \omega^i \cdot a_i + \dots + \omega^{k+1} \cdot a_{k+1} + \omega^k \cdot t_l$ , where  $t_l = 2^{n-2k} - 2^{n-l-2k} - 4$ , and at the first point where this is impossible, that is,  $2^{n-2k} - 2^{n-(l-1)-2k} - 4 \leq a_k < 2^{n-2k} - 2^{n-l-2k} - 4$ , we stop at  $r = l-1$  if  $t_{l-1} = a_k$  and let  $t_l = a_k$  otherwise (and then stop with  $r = l$ ). In all cases,  $t_r - t_{r-1} \leq 2^{n-r-2k}$ . Let  $y_1 < y_2 \dots$  be II's responses. As in Case 7, we may appeal to Lemma 2.2 to see that each  $y_r$  may be assumed to be a multiple of  $\omega^k$ . Furthermore, we may suppose that there are  $t'_1 < t'_2 < \dots$  such that  $t'_1 \geq 1$  and  $y_r = \omega^i \cdot a_i + \dots + \omega^{k+1} \cdot a_{k+1} + \omega^k \cdot t'_r$ . We mainly have to justify this for  $y_1$ . The only other options are that  $y'_1 \leq \omega^i \cdot a_i + \dots + \omega^{k+1} \cdot a_{k+1}$ , when the argument of Case 7 applies, in the case of strict inequality appealing to  $k < \frac{n-3}{2}$ .

Since  $b_k < a_k$ ,  $t'_1 < t_1$ , or there is  $l$  such that  $t'_{l+1} - t'_l < t_{l+1} - t_l$ . In the first case we appeal to Lemma 2.9(ii) to deduce that  $\alpha^{<x_1} \not\equiv_{n-1} \beta^{<y_1}$ , and in the latter to Lemma 2.3(ii) to deduce that  $(x_l, x_{l+1}) \not\equiv_{n-l-1} (y_l, y_{l+1})$ , so in each case, player I wins in the remaining moves.

**Case 8B:**  $k < i$ ,  $a_l \neq 0$  for some  $l < k$ ,  $k \geq \frac{n-3}{2}$ , and  $2^{n-1-2k} - 4 \leq b_k < 2^{n-2k} - 5$ .

Since  $k < \frac{n}{2} - 1$ , in fact  $k = \frac{n-3}{2}$  and  $i = \frac{n-1}{2}$ . By clauses (1) and (3),  $a_i \leq 2$ , and if  $a_i = 2$ , then  $a_{i-1} \leq 3$ , in each case  $b_i = a_i$  and  $b_{i-1} < a_{i-1}$ . If  $a_i = 1$ , then



we follow the proof of Case 8A, noting that  $t_1 = 2^{n-2k-1} - 4 = 0$ , so that  $x_1 = \omega^i$ , and Lemma 2.2 ensures that  $y_1$  may be assumed to be a multiple of  $\omega^i$  too, hence equal to  $\omega^i$ .

So we concentrate on the case  $a_i = 2$ ,  $b_{i-1} < a_{i-1} \leq 3$ . Player I plays  $x_1 = \omega^i \cdot 2$  and by Lemma 2.2, II must play  $y_1 = \omega^i$  or  $\omega^i \cdot 2$ . If  $y_1 = \omega^i$ , player I now plays  $y_r = \omega^i + \omega^{i-1}(r-1)$  in  $\beta$  as long as necessary, and as in previous proofs, player II is unable to respond for all of the remaining  $n-1$  moves. If however  $y_1 = \omega^i \cdot 2$ , then player I plays  $x_2 = \omega^i \cdot 2 + \omega^{i-1}(b_{i-1} + 1)$ . By Lemma 2.2, II must play a multiple  $y_2$  of  $\omega^{i-1}$  which is at most  $\omega^i \cdot 2 + \omega^{i-1} \cdot b_{i-1}$ . If  $y_2 = \omega^i \cdot 2 + \omega^{i-1}$  then I wins on  $(x_1, x_2)$  and  $(y_1, y_2)$  by Theorem 1.4(iii) using  $\omega^{i-1} \not\equiv_{2(i-1)+1} \omega^{i-1} \cdot 2$ . Otherwise  $y_2 = \omega^i \cdot 2 + \omega^{i-1} \cdot 2$  so  $b_{i-1} = 2$  and  $a_{i-1} = 3$ . Now player I plays  $x_3 = \omega^i \cdot 2 + \omega^{i-1} \cdot 3$ , and since  $i-1 < \frac{n-2}{2}$ , II must play a multiple of  $\omega^{i-1}$ , which is impossible.

**Case 9:**  $k < i$ ,  $a_l \neq 0$  for some  $l < k$ , and  $2^{n-2k} - 5 \leq b_k$ .

Since  $b_k < a_k \leq 2^{n-2k} - 4$ , it follows that  $a_k = 2^{n-2k} - 4$  and  $b_k = 2^{n-2k} - 5$ . We follow the method of Case 6. Let  $l \leq k-1$  be the greatest (if any) such that  $a_l \neq b_l$  or  $a_l = b_l \neq 3$ . As before, it follows that  $a_l \leq 3$ , and so in the second case  $a_l = b_l \leq 2$ , and for  $l < m \leq k-1$ ,  $a_m = b_m = 3$ , so by Lemma 2.11(ii),  $\alpha \not\equiv_n \beta$ . If  $a_l \neq b_l$ , we can conclude by appealing to Case 2, since  $l < \frac{n}{2} - 2$  and  $2^{n-2-2l} \geq 4$ , so that  $a_l < 2^{n-2-2l}$  (and if  $b_l < a_l$ , also  $b_l < 2^{n-2-2l}$ ). Thus the fact that the coefficients which control the situation belong to smaller powers of  $\omega$  than in the previous cases, means that we avoid the two extra cases, corresponding to Cases 5B and 6. ■

**Theorem 2.13.** *The members of  $\Omega'_n$  are the minimal representatives of the  $n$ -equivalence classes of monochromatic ordinals.*

*Proof.* We have to show that for any ordinal  $\alpha$ , the least ordinal  $\alpha'$  which is  $n$ -equivalent to  $\alpha$  lies in  $\Omega'_n$ , and also that no two members of  $\Omega'_n$  are  $n$ -equivalent. By Corollary 2.4,  $\alpha'$  may be written in the form  $\omega^t \cdot a_t + \omega^{t-1} \cdot a_{t-1} + \dots + \omega \cdot a_1 + a_0$  where  $a_i \leq 2^{n-2i}$ . Since the truth of the result for  $n = 0, 1, 2$  is verified from the lists explicitly given earlier, we assume that  $n \geq 3$ .

We first establish the numbered properties of  $\alpha'$ , (1) having already been done.

- (2) If there is  $j > i$  such that  $a_j \neq 0$  and  $i \leq \frac{n-3}{2}$ , then by Lemma 2.9(i),  $a_i \leq 2^{n-2i} - 4$ . Since  $j \leq \frac{n}{2}$ , the only other possibility is that  $n$  is even and  $i = \frac{n}{2} - 1$ , in which case  $2^{n-2i} - 4 = 0$ , and we appeal to Lemma 2.6(i), since it implies that if  $a_{t-1} > 0$  then  $\omega^{t-1}(\omega + a_{t-1}) \equiv_n \omega^{t-1} \cdot 4$ , which would contradict the minimality of  $\alpha'$ .
- (3) If  $a_{i+1} = 2^{n-2(i+1)}$ , then  $a_i \leq 3$  because if  $a_i \geq 4$  then by Lemma 2.7(i), which may be written in the form  $\omega^{i+1} \cdot k + \omega^i \cdot m \equiv_n \omega^{i+1}(2^{n-2(i+1)} - 1) + \omega^i \cdot m$  (assuming that  $k \geq 2^{n-2(i+1)}$  and  $m \geq 4$ ), we could reduce the coefficient of  $\omega^{i+1}$ , contrary to minimality of  $\alpha'$ .
- (4) If  $2^{n-2(i+1)} - 4 = 0$  then  $n = 2(i+2)$  so  $n$  is even and  $i = \frac{n}{2} - 2$ . We may use Lemma 2.8(i), since if  $a_i \geq 4$  then we could remove the term in  $\omega^i$ . Otherwise we use Lemma 2.11(i), (replacing  $i$  by  $i+1$ ), which tells us that  $\omega^j + \omega^{i+1}(2^{n-2(i+1)} - 4) + \omega^i \cdot 4 \equiv_n \omega^j + \omega^{i+1}(2^{n-2(i+1)} - 5) + \omega^i \cdot 4$ , so if  $a_i \geq 4$ , we could reduce  $\alpha'$  by decreasing the coefficient of  $\omega^{i+1}$ .
- (5) If  $a_{i+1} = 3$  and 3 is an  $n$ -optimal coefficient for  $\omega^{i+1}$ , then by Lemmas 2.10(i) and 2.11(i), and the definition of ' $n$ -optimality',  $a_i \leq 3$ .
- (6)  $a_0 \leq 2^n - 1$  by Lemma 1.3 if  $\alpha$  is finite.

The converse statement, that any  $\alpha = \omega^i \cdot a_i + \omega^{i-1} \cdot a_{i-1} + \dots + \omega \cdot a_1 + a_0$  fulfilling all clauses is optimal, where  $a_i \neq 0$ , follows from Lemma 2.12. ■

To illustrate the above rather complicated proof, we consider the following cases:  $n = 4, 5$  and  $6$ . We can say that each of these is generated by a finite list of ‘maximal’ polynomials in  $\omega$ . (Here ‘maximality’ is with respect to the partial ordering given by  $\sum \omega^i \cdot a_i \succeq \sum \omega^i \cdot b_i$  if for all  $i$ ,  $a_i \geq b_i$ .) For  $n = 4$  this list is

$$15, \omega \cdot 3 + 12, \omega \cdot 4 + 3, \omega^2 + 3,$$

for  $n = 5$  the list is

$$31, \omega \cdot 7 + 28, \omega \cdot 8 + 3, \omega^2 \cdot 2 + \omega \cdot 2 + 28, \omega^2 \cdot 2 + \omega \cdot 3 + 3, \omega^2 + \omega \cdot 3 + 28, \omega^2 + \omega \cdot 4 + 3,$$

and for  $n = 6$  it is

$$63, \omega \cdot 15 + 60, \omega \cdot 16 + 3, \omega^2 \cdot 3 + \omega \cdot 11 + 60, \omega^2 \cdot 3 + \omega \cdot 12 + 3, \omega^2 \cdot 4 + \omega \cdot 3 + 3, \omega^2 \cdot 4 + \omega \cdot 2 + 60, \omega^3 + \omega \cdot 2 + 60, \omega^3 + \omega \cdot 3 + 3.$$

Examining the case of  $n = 6$  in detail, we see that  $t = 3$ , and by Corollary 2.4, every ordinal is 6-equivalent to one of the form  $\omega^3 \cdot a_3 + \omega^2 \cdot a_2 + \omega \cdot a_1 + a_0$  where  $a_3 \leq 1$ ,  $a_2 \leq 4$ ,  $a_1 \leq 16$  and  $a_0 \leq 64$ . In the first case,  $a_3 = 1$ , in which case, by (2),  $a_2 = 0$ , and by applying (4) to  $i = 1$ ,  $a_1 \leq 3$ . In all infinite cases  $a_0 \leq 60$  by (4), and if  $a_1 = 3$  then  $a_0 \leq 3$  by (5). This is because, by definition, 3 is a 6-optimal coefficient for  $\omega$  in  $\omega^3 + \omega \cdot 3 + a_0$ . Now considering the case in which  $a_3 = 0$ , we look at the various possibilities for  $a_2$ . If  $a_2 = 4$  then  $a_1 \leq 3$  by (3) and if  $a_1 = 3$  then  $a_0 \leq 3$  by (5). For by definition, 3 is a 6-optimal coefficient for  $\omega$  in  $\omega^2 \cdot 4 + \omega \cdot 3 + a_0$ . If however  $0 < a_2 \leq 3$  then  $a_1 \leq 12$  by clause (2). If  $a_1 = 12$  then  $a_0 \leq 3$  by clause (4), and if  $a_1 \leq 11$  then  $a_0 \leq 60$  by (2). Next suppose that  $a_3 = a_2 = 0$ . Then if  $a_1 = 16$ , it follows by (3) that  $a_0 \leq 3$ , and if  $0 < a_1 \leq 15$  then  $a_0 \leq 60$  by (2). Finally, if  $a_3 = a_2 = a_1 = 0$ , then  $a_0 \leq 63$  by (6). The other cases can be similarly treated.

We conclude this section by remarking that there is a computable function  $f$  such that for each  $n$ ,  $f(n)$  lists minimal representatives of the  $n$ -equivalence classes of ordinals. To make sense of this, we should encode the ordinals in some standard way; in this case we can just regard the ordinal  $\alpha = \omega^k \cdot a_k + \omega^{k-1} \cdot a_{k-1} + \omega^{k-2} \cdot a_{k-2} + \dots + \omega \cdot a_1 + a_0$  as represented by the finite sequence  $(a_k, a_{k-1}, \dots, a_1, a_0)$  which in turn may be prime power encoded if desired. The function  $f$  is then obtained by letting  $f(n)$  list (codes for) the members of  $\Omega'_n$  in increasing order. The fact that  $f$  is computable follows from the very explicit definition given of  $\Omega'_n$ .

### 3. $m$ -coloured ordinals up to $n$ -equivalence

In this section, we give an analysis of  $m$ -coloured ordinals up to  $n$ -equivalence. It is a triviality that there is a countable ordinal  $\alpha$  such that every  $m$ -coloured ordinal  $(X, <, F)$  is  $n$ -equivalent to some  $m$ -coloured ordinal less than  $\alpha$ . Namely, for each  $(X, <, F)$  we find a suitable countable ordinal by the downward Löwenheim–Skolem–Tarski Theorem (which is even *elementarily* equivalent to  $X$ , and is still well-ordered as it is a substructure), and as there are only finitely many  $\equiv_n$ -classes, we can just take the maximum of these ordinals. The point however is to find a much smaller, and explicit bound, in the style of [7]. We would like to find a complete and explicit set of representatives as in the monochromatic case in the previous section, but this seems too ambitious at present. Some precise information was given in [7] for 2 moves, but for larger values of  $n$ , things get considerably more complicated. We are able to obtain the same *overall* bound as in the monochromatic case, namely,  $\omega^\omega$ . However, this is approached much more rapidly by the individual upper bounds provided by our main theorem as the number of moves  $n$  increases, and it seems to us likely that the true value will be considerably lower.

A key tool will be the ‘cutting lemma’ as given in [7], which applies also in the infinite case, by the same proof as there, and this says the following.

**Lemma 3.1.** *Let  $A$  be an  $m$ -coloured linear order and let  $a$  and  $b$  be elements of  $A$  such that  $a < b$  satisfying the following conditions:*

- (i)  $F(a) = F(b)$ ,
- (ii)  $a$  and  $b$  determine the same  $n$ -character, that is,  $\langle [A^{<a}]_n, [A^{>a}]_n \rangle = \langle [A^{<b}]_n, [A^{>b}]_n \rangle$ ,
- (iii) for every  $x \in A$  with  $a < x \leq b$ , there is  $y \leq a$  of the same colour as  $x$  and such that  $\langle [A^{<x}]_n, [A^{>x}]_n \rangle = \langle [A^{<y}]_n, [A^{>y}]_n \rangle$ .

Then  $A$  is  $(n + 1)$ -equivalent to  $B = A - (a, b]$ .

We have another ‘cutting lemma’, relevant just for the case of limit ordinals, which is our main new tool over the finite case. Before we can prove this, an auxiliary result is required, which actually applies to all coloured linear orders, not just ordinals.

**Lemma 3.2.** *Let  $A$  be an  $m$ -coloured linear order and let  $a_1 < a_2$  and  $b_1 < b_2$  be elements of  $A$  such that  $a_1$  and  $b_1$  have the same  $n$ -characters, and so do  $a_2$  and  $b_2$ , where  $n \geq 1$ , and such that the families of  $n$ -characters of members of  $(a_1, a_2)$  and  $(b_1, b_2)$  are equal to the same set  $C_n$ , and there are at least  $2^n - 1$  blocks of occurrences of members of  $C_n$  in each of  $(a_1, a_2)$  and  $(b_1, b_2)$ ; where this means that  $(a_1, a_2)$  and  $(b_1, b_2)$  may each be written as the disjoint union of this number of convex subsets on each of which all members of  $C_n$  are represented. Then  $(a_1, a_2) \equiv_n (b_1, b_2)$ .*

*Proof.* We use induction. For the basis case,  $n = 1$ , so there is at least one block of occurrences of  $C_1$  (which in this case is given anyhow by the definition of  $C_1$ ). This implies that the sets of colours of points occurring in  $(a_1, a_2)$  and  $(b_1, b_2)$  are the same, so player II can therefore win in one move by playing a point of the same colour as player I did.

Now assume the result for  $n$ , and we indicate how player II can play to win the  $(n + 1)$ -move game between  $(a_1, a_2)$  and  $(b_1, b_2)$ , assuming that  $a_1$  and  $b_1$  realize the same  $(n + 1)$ -character, and so do  $a_2$  and  $b_2$ , and that there are at least  $2^{n+1} - 1$  blocks of occurrences of members of  $C_{n+1}$  in each of  $(a_1, a_2)$  and  $(b_1, b_2)$ . Without loss of generality player I starts by playing  $x_1 \in (a_1, a_2)$ .

First suppose that  $(a_1, x_1)$  and  $(x_1, a_2)$  each have at least  $2^n - 1$  blocks of occurrences of members of  $C_{n+1}$ . Since  $2^{n+1} - 1 = (2^n - 1) + 1 + (2^n - 1)$ , player II can play a point  $y_1$  (of the ‘middle’ block) having the same  $(n + 1)$ -character as  $x_1$ . Now from the fact that  $(a_1, x_1)$  and  $(x_1, a_2)$  exhibit precisely the same  $(n + 1)$ -characters, and so do  $(b_1, y_1)$  and  $(y_1, b_2)$ , it follows that they also exhibit the same  $n$ -characters, so the induction hypothesis assures us that  $(a_1, x_1) \equiv_n (b_1, y_1)$  and  $(x_1, a_2) \equiv_n (y_1, b_2)$ , and so II can win.

Next suppose that  $(a_1, x_1)$  does not have  $2^n - 1$  blocks of occurrences of members of  $C_{n+1}$  (and a similar argument applies if  $(x_1, a_2)$  does not have  $2^n - 1$  blocks). Since  $a_1$  and  $b_1$  have the same  $(n + 1)$ -character,  $(a_1, \infty) \equiv_{n+1} (b_1, \infty)$ , so there is  $y \in (b_1, \infty)$  of the same colour as  $x_1$  such that  $(a_1, x_1) \equiv_n (b_1, y)$  and  $(x_1, \infty) \equiv_n (y, \infty)$ . Also  $(-\infty, x_1) = (-\infty, a_1) \cup \{a_1\} \cup (a_1, x_1) \equiv_n (-\infty, b_1) \cup \{b_1\} \cup (b_1, y) = (-\infty, y)$ , which shows that  $x_1$  and  $y$  have the same  $n$ -character.

If  $(b_1, y)$  does not contain  $2^n - 1$  blocks of occurrences of members of  $C_{n+1}$ , then  $y < b_2$ , and furthermore, each of  $(x_1, a_2)$  and  $(y, b_2)$  contains at least  $2^n - 1$  blocks of occurrences of members of  $C_{n+1}$ . Clearly, points having the same  $(n + 1)$ -character also have the same  $n$ -character (though not conversely), so these intervals also contain at least  $2^n - 1$  blocks of occurrences of members of  $C_n$ . Since  $x_1$  and  $y$

have the same  $n$ -character, we may apply the induction hypothesis to deduce that  $(x_1, a_2) \equiv_n (y, b_2)$ , and so II wins by playing  $y_1 = y$  on his first move.

The difficulty comes about in the remaining case, when  $(b_1, y)$  contains at least  $2^n - 1$  blocks of occurrences of members of  $C_{n+1}$ , so  $y$  may not even lie in  $(b_1, b_2)$ , and we have to substitute a point nearer to the left which player II is able to play. For this we select a point  $y_1$  in the middle block of  $(b_1, b_2)$  having the same  $n$ -character as  $y$  (which lies in  $C_n$  since the  $n$ -characters of  $x_1$  and  $y$  are equal), and this is what II plays. By the induction hypothesis,  $(b_1, y) \equiv_n (b_1, y_1)$ , and it follows that  $(a_1, x_1) \equiv_n (b_1, y_1)$ . Also both  $(x_1, a_2)$  and  $(y_1, b_2)$  contain at least  $2^n - 1$  blocks of occurrences of members of  $C_{n+1}$ , hence also of  $C_n$ , so by induction hypothesis, they are  $n$ -equivalent. By applying the same argument as in the previous paragraph,  $x_1$  and  $y_1$  have the same  $n$ -character, so player II wins. ■

We can now present our main new ‘cutting’ lemma. Note that we need to (and can) cut many intervals from our set simultaneously, though conceptually it seems easiest to consider one of them at a time.

**Lemma 3.3.** *Let  $A$  be an  $m$ -coloured ordinal,  $\Lambda$  an ordinal, and for each  $\alpha \in \Lambda$  let  $a_\alpha$  and  $b_\alpha$  be elements of  $A$  which are limit ordinals such that  $\alpha < \beta \Rightarrow a_\alpha < b_\alpha < a_\beta$  and for each limit ordinal  $\lambda \in \Lambda$ ,  $\sup_{\alpha < \lambda} b_\alpha < a_\lambda$ . Suppose further that the sets of  $n$ -characters which occur cofinally in  $(-\infty, a_\alpha)$  and  $(-\infty, b_\alpha)$  are equal to the same set  $C_\alpha$ , and the  $n$ -characters of all points of  $[a_\alpha, b_\alpha)$  also lie in  $C_\alpha$ . Then  $A$  is  $(n+1)$ -equivalent to  $B = A - \bigcup_{\alpha \in \Lambda} [a_\alpha, b_\alpha)$ .*

*Proof.* First we choose  $c_\alpha < a_\alpha$  so that all  $n$ -characters arising in  $[c_\alpha, a_\alpha)$  (and hence also in  $[c_\alpha, b_\alpha)$ ) lie in  $C_\alpha$ . This is possible since there are only finitely many characters in all, so there is some point less than  $a_\alpha$  beyond which any  $n$ -characters which do *not* occur cofinally in  $(-\infty, a_\alpha)$  no longer arise. Furthermore, the hypothesis allows us to suppose that  $\sup_{\alpha < \beta} b_\alpha < c_\beta$ .

We describe a winning strategy for player II in the  $(n+1)$ -move game on  $A$  and  $B$ . We write  $x_i$  and  $y_i$  for the  $i$ th moves played in  $A, B$  respectively. The map taking  $x_i$  to  $y_i$  will be order-preserving (and all  $x_i$  will be distinct). Furthermore, if  $x_i, y_i$  have been chosen for  $i \leq k$ , and  $I$  is an open interval determined by adjacent  $x_i, x_{i'}$  or between  $x_i$  and  $\pm\infty$ , and  $J$  is the corresponding interval determined by the  $y_i$ , then  $I \equiv_{n+1-k} J$ . Also,  $x_i, y_i$  will have the same colour.

Player II can clearly play on his first move so that  $x_1$  and  $y_1$  have the same  $n$ -characters, and either  $x_1 = y_1$ , or  $x_1 \in [a_\alpha, b_\alpha)$  and  $y_1 \in (c_\alpha, a_\alpha)$ . The fact that this is possible follows from the choice of  $c_\alpha$ , and the cofinality hypotheses.

Now suppose that  $x_i, y_i$  have been chosen for  $i \leq k$ , and we have to say how player II can respond to any possible move by player I on his  $(k+1)$ th move.

Let  $I$  or  $J$  be the interval that I decides to play in (if he plays in  $A$  or  $B$  respectively). By assumption,  $I \equiv_{n+1-k} J$ , and we consider the response to I’s play made by player II using a strategy thereby given. Let  $x_{k+1}$  and  $y$  be the moves thus played. If  $y \in B$  we just let  $y_{k+1} = y$ , and all hypotheses carry through to the next step. If however  $y \notin B$  (in which case player I must have played  $x_{k+1}$ ) then for some  $\alpha$ ,  $y \in [a_\alpha, b_\alpha)$ . By cofinality of the occurrences of points of  $n$ -character lying in  $C_\alpha$  in  $(a_\alpha, b_\alpha)$  we may find a point  $y_{k+1}$  of  $(\max(y_i, c_\alpha), a_\alpha)$  having the same  $n$ -character as  $y$ , and with sufficiently many blocks of occurrences of  $C_\alpha$  in  $(\max(y_i, c_\alpha), y_{k+1})$  (where ‘sufficiently many’ means as are needed to appeal to Lemma 3.2, noting that since these occur cofinally, any finite number can certainly be achieved). II plays this  $y_{k+1}$  on his  $(k+1)$ th move. Since  $x_{k+1}$  and  $y$  have the same  $(n-k)$ -character and  $y$  and  $y_{k+1}$  have the same  $n$ -character, it follows that  $x_{k+1}$  and  $y_{k+1}$  have the same  $(n-k)$ -character. The fact that  $(x_i, x_{k+1}) \equiv_{n-k} (y_i, y_{k+1})$  and

$(x_{k+1}, x_{i'}) \equiv_{n-k} (y_{k+1}, y_{i'})$  follows from Lemma 3.2, so the induction goes through.

■

The main theorem is now as follows:

**Theorem 3.4.** *For any positive integers  $m$  and  $n$ , and for any  $m$ -coloured ordinal, there is an  $n$ -equivalent  $m$ -coloured ordinal less than some finite power of  $\omega$ .*

*Proof.* The case  $n = 1$  is easy and completely described in [7]. In fact, two coloured linear orders are 1-equivalent if and only if they exhibit precisely the same sets of colours, so we get (finite) optimal representatives of size at most  $m$ .

Moving on to  $n > 1$ , let  $(A, <, F)$  be an  $m$ -coloured ordinal of minimal order-type in its  $\equiv_n$ -class. We start by considering the occurrences of the  $(n - 1)$ -characters appearing in  $A$ . There are just finitely many, which from now on we refer to just as ‘characters’ and so we may find the first occurrence of each, and let these be  $x_0 < x_1 < x_2 < \dots < x_{k-1}$  (where clearly  $x_0$  and  $x_1$  are the first two members of  $A$ ). For ease we also let  $x_k = +\infty$ , so that we can refer to the intervals  $I_i = [x_i, x_{i+1})$  for all  $i < k$ .

Now by choice of the  $x_i$  as the *first* occurrences of the characters, any character arising in  $(x_i, x_{i+1})$  already occurs in  $(-\infty, x_i]$ . Hence if any character arises more than once in  $I_i$ , then we may use Lemma 3.1 to cut out the section in between. Unlike in the finite case, this may however not reduce the order-type, but it does enable us to make some deductions about the form that  $A$  has, or may be assumed to have. Let us assume then that all characters of  $I_i$  appear with minimal order-type. For a particular character, write the order-type of its occurrences in  $I_i$  in Cantor normal form as  $\omega^{\alpha_0} \cdot a_0 + \omega^{\alpha_1} \cdot a_1 + \dots + \omega^{\alpha_l} \cdot a_l$  where  $\alpha_0 > \alpha_1 > \dots > \alpha_l$  and  $a_i \in \omega$ . If  $l > 0$  then we may cut a section from the first point of the  $\omega^{\alpha_0} \cdot a_0$ -block to the first point of the  $\omega^{\alpha_l} \cdot a_l$ -block and achieve a strictly smaller order-type, contrary to assumption. Hence  $l = 0$ . A similar argument applies if  $a_0 > 1$ . We therefore deduce that the character appears with order-type of the form  $\omega^\alpha$  for some ordinal  $\alpha$  (which is 1 if  $\alpha = 0$ ).

We note that it also follows that  $I_i$  is expressible as a finite union of intervals of the form  $J_j = [y_j, y_{j+1})$  where no character appears in more than one  $J_j$ , and each character of  $J_j$  appears cofinally (the case  $|J_j| = 1$  is allowed). To achieve this, the points  $y_{j+1}$  are taken to be the suprema of the occurrences of characters. Given this, to see that no character appears in more than one  $J_j$ , observe that no character which appears in  $[y_{j+1}, x_{i+1})$  can also appear in  $(x_i, y_{j+1})$ . For if it did, we could apply Lemma 3.1 and reduce the order-type of the occurrences of the characters having supremum  $y_{j+1}$ , contrary to the assumption that the order-types of occurrences of all characters have been minimized.

We emphasize that the same character can (and will) occur in more than one of the intervals  $I_i$ , but for *fixed*  $i$ , no character will occur in more than one  $J_j$ .

To conclude the proof, we show by induction on  $r \geq 1$  that  $J_j$  has a subset  $B_r$  such that  $A \equiv_n A - B_r$  and for any non-empty set  $X$  of  $r$  characters, all convex subsets of  $J_j - B_r$  exhibiting only members of  $X$  have order-type  $< \omega^r \cdot 2$ .

For the basis case,  $r = 1$ , and let  $c$  be a single character. Define  $\sim$  on  $J_j$  by  $x \sim y$  if  $x = y$ , or if all points of  $[x, y]$  (or  $[y, x]$  if  $y < x$ ) have character  $c$ . Then the  $\sim$ -classes are convex subsets of  $J_j$ . Let  $[\beta, \gamma)$  be a  $\sim$ -class, and  $\lambda_1$  its least limit ordinal, if any, and  $\gamma = \lambda_2 + s$  for finite  $s$ , where  $\lambda_2$  is a limit ordinal (or 0). We may apply Lemma 3.3 to cut out  $[\lambda_1, \lambda_2)$  from all  $\sim$ -classes containing a limit ordinal, giving a subset in which all  $\sim$ -classes have order-type  $< \omega \cdot 2$ . Note that the requirement that the supremum of the right hand endpoints of the cut out intervals is strictly less than the next one is automatically fulfilled, since each  $\lambda_1$  is immediately preceded by a non-empty block of points all having character  $c$ . Now

repeat this for each of the remaining characters and let  $B_1$  be the union of all the sets cut out.

For the induction step, assume that we have found  $B_r$  and that  $1 \leq r < k$ . Let  $X$  be a set of characters of size  $r + 1$ , and define  $\sim$  on  $J_j - B_r$  by letting  $x \sim y$  if  $x = y$ , or if all points of  $[x, y]$  (or  $[y, x]$  if  $y < x$ ) have character in  $X$ . By appropriately cutting segments from each  $\sim$ -class, they will have the form  $[\beta, \gamma]$  where all members of  $X$  are cofinal in at most one limit ordinal in  $[\beta, \gamma]$ . This means that for all other limit ordinals  $\lambda$  in this interval, the set of characters which occur cofinally in  $[\beta, \lambda]$  is a proper subset of  $X$ . We show that the order-type of  $[\beta, \gamma]$  is less than  $\omega^{r+1} \cdot 2$ . If not, then there is a limit ordinal  $\lambda$  in  $[\beta, \gamma]$  such that  $[\delta, \lambda] \cong \omega^{r+1}$  for some  $\delta \in [\beta, \lambda]$  and such that the set  $Y$  of characters which occur in  $[\delta, \lambda]$  is a proper subset of  $X$ , but this contradicts the induction hypothesis. Now repeat this argument finitely many times for all sets of characters of size  $r + 1$ , and this gives the induction step by taking for  $B_{r+1}$  the union of all the sets removed at these finitely many steps.

Finally we look at the case where  $r = k$  which has now been established. We have a set of order-type less than  $\omega^k \cdot 2$  which is  $n$ -equivalent to  $J_j$ . This gives a bound  $\omega^k \cdot 2k^2$  for the order-type of  $A$ , where  $k$  is the number of all  $(n - 1)$ -characters. This is an explicit bound, but since the number of  $(n - 1)$ -characters grows very fast, we believe that it is much greater than the optimum. ■

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