This question paper consists of
4 printed pages, each of which
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School of Mathematics

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MATH202201

Groups and Vector Spaces

Time allowed: 2 hours 30 minutes

You must attempt to answer 4 questions.
If you answer more than 4 questions, only your best 4 answers will be counted towards your final mark for this exam.
All questions carry equal marks.

1. (i) Define the terms *group* and *subgroup* of a group.

Draw up the group table for the quaternion group Q of order 8 having elements $\pm 1, \pm \mathrm{i}, \pm \mathrm{j}, \pm \mathrm{k}$ where $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1$, $\mathrm{ij} = -\mathrm{ji} = \mathrm{k}$, $\mathrm{jk} = -\mathrm{kj} = \mathrm{i}$, $\mathrm{ki} = -\mathrm{ik} = \mathrm{j}$.

Determine which of the following are subgroups of Q, giving reasons:

(a) $\{\mathrm{i}, -\mathrm{i}, \mathrm{j}, -\mathrm{j}\}$, (b) $\{1, -1, \mathrm{k}, -\mathrm{k}\}$, (c) $\{1, \mathrm{i}, \mathrm{j}, -1\}$, (d) $\{1, -1\}$.

(ii) Let $\mathbb{Z}_n^*$ stand for the group of integers in $\{1, 2, \ldots, n-1\}$ coprime with n under multiplication modulo n. Which of (a) $\mathbb{Z}_{13}^*$, (b) $\mathbb{Z}_{14}^*$, (c) $\mathbb{Z}_{15}^*$, (d) $\mathbb{Z}_{16}^*$ are cyclic? Given that two of the groups (a), (b), (c), (d) are isomorphic, find which they are, explaining your reasons.

2. (i) State Lagrange's Theorem, and deduce from it that the order of any element of a finite group divides the order of the group.

(ii) Prove that $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are non-isomorphic groups of order 4, but that any group of order 4 is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(iii) Describe a non-abelian group G of order 12, and state the orders of all its elements. Does G have a subgroup of order 6?

(iv) Determine the conjugacy classes of the dihedral group D_4 whose group table is shown, explaining your reasoning.

	I	R	R^2	R^3	H	V	D	D'
I	I	R	R^2	R^3	H	V	D	D'
R	R	R^2	R^3	I	D'	D	H	V
R^2	R^2	R^3	I	R	V	H	D'	D
R^3	R^3	I	R	R^2	D	D'	V	H
H	H	D	V	D'	I	R^2	R	R^3
V	V	D'	H	D	R^2	I	R^3	R
D	D	V	D'	H	R^3	R	I	R^2
D'	D'	H	D	V	R	R^3	R^2	I

3. (i) Define *odd* and *even* permutations of a finite set X. Show that any permutation is either even or odd, and that the family of even permutations forms a normal subgroup of the group of all permutations of X.

(ii) Express the permutations f and g of $\{1,2,3,4,5,6,7,8,9\}$ given by

$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 9 & 7 & 3 & 5 & 8 & 1 & 6 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 2 & 9 & 6 & 7 & 4 & 3 & 1 & 8 \end{pmatrix}$

as (a) products of disjoint cycles, (b) products of transpositions.

Which of f, g, fg, gf, $f^{-1}gf$ are even?

(iii) Define *normal subgroup* of a group, and state the first isomorphism theorem for groups.

Prove that the map $\theta : (\mathbb{R}, +) \to \mathbb{C}^*$ given by $\theta(x) = \mathrm{e}^{2\pi \mathrm{i} x}$ is a homomorphism. Find the kernel and image of θ. Prove that $\mathbb{R}/\mathbb{Z} \cong \{z \in \mathbb{C} : |z| = 1\}$ (where the operation on this set is multiplication).

4. (i) Let V be a vector space over a field F. Define the terms *linearly independent* subset of V, and *spanning subset* of V.

(ii) Determine with reasons which of the following sets are linearly independent or spanning:

(a) $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \right\}$, in $\mathbb{R}^3$

(b) $\{1 + x, x - x^3, 3 + x + 2x^3\}$, in the space of real polynomials of degree ≤ 3.

(iii) If f is a linear transformation from a vector space V to a vector space W, show that for any $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$, if $\{f\mathbf{v}_1, \ldots, f\mathbf{v}_n\}$ is a linearly independent subset of W, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a linearly independent subset of V.

(iv) If U and W are subspaces of the vector space V, define the *sum* $U + W$ of U and W, and state the circumstances under which this is a *direct sum* (written $U \oplus W$). For which of the following is the sum $U + W$ direct?

(a) $U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : 2x + y - z = 0 \right\}$, $W = \left\{ \begin{pmatrix} a + b \\ 2a \\ a - 3b \end{pmatrix} : a, b \in \mathbb{R} \right\}$ in $\mathbb{R}^3$,

(b) $U = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 : 2x + y = z - 3t = 0 \right\}$, $W = \left\{ \begin{pmatrix} a + 2b \\ a - b \\ 2a - b \\ 3a + b \end{pmatrix} : a, b \in \mathbb{R} \right\}$ in $\mathbb{R}^4$.

5. (i) Consider the linear transformation $\theta : \mathbb{C}^3 \to \mathbb{C}^3$ given by $\theta\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2y \\ y+\mathrm{i}z \\ z-\mathrm{i}x \end{pmatrix}$.

Determine the matrix A of θ with respect to the standard basis $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$.

Find the transition matrix from the standard basis to the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ where $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. Hence or otherwise find the matrix B of θ with respect to $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

(ii) Define the *algebraic multiplicity* and *geometric multiplicity* of an eigenvalue of a square matrix A. Determine whether or not $A = \begin{pmatrix} 3 & 4 & -4 \\ 4 & 3 & -4 \\ 4 & 4 & -5 \end{pmatrix}$ is diagonalizable, and if it is, find a non-singular matrix P such that $P^{-1}AP$ is diagonal.

(iii) Define *orthogonal vectors* and *orthonormal basis*. Use the Gram–Schmidt process to find an orthonormal basis of the subspace of $\mathbb{R}^4$ spanned by $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 2 \\ -1 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$.

End.