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Categorifications of the 4-term relations via infinitesimal 2-braidings

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References:
arXiv:1106.0042; arXiv:1207.1132; arXiv:1309.4070

Categorification of link invariants

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Context-Categorification of link invariants

- Quantum link invariants can be defined:
  - Combinatorially (via quantum groups and Yang-Baxter operators)
  - Analytically (via the holonomy of the Knizhnik-Zamolodchikov connection)
- Most approaches to categorification of quantum link invariants use combinatorial frameworks.
- It seems however natural to use differential-geometric approaches for categorifying quantum link invariants.
- Main aim of this project: Define a 2-connection *categorifying* the Knizhnik-Zamolodchikov connection.

The Knizhnik-Zamolodchikov connection

- The configuration space $\mathbb{C}(n)$ of n (distinguishable) particles in the complex plane is

$$\mathbb{C}(n) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_i \neq z_j \text{ if } i \neq j\}.$$

- Let $\mathfrak{g}$ be a Lie algebra.
- Let $<,>$ be a $\mathfrak{g}$-invariant, non-degenerate, symmetric, bilinear form in $\mathfrak{g}$.
- Let $\{s_i\}$ be basis of $\mathfrak{g}$.
- Let $\{t^i\}$ be the dual basis of $\mathfrak{g}^* \cong \mathfrak{g}$.
- Let

$$r = \sum_i s_i \otimes t^i \in \mathfrak{g} \otimes \mathfrak{g}$$

- Note that r is symmetric: $r_{12} = r_{21}$.
- Choose a representation of $\mathfrak{g}$ on a vector space V.

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The Knizhnik-Zamolodchikov connection is given by the $\mathrm{Hom}(V^{n\otimes})$-valued 1-form A in the configuration space $\mathbb{C}(n)$,

$$A = \frac{h}{2\pi i} \sum_{1 \leq a < b \leq n} \omega_{ab} \phi_{ab}(r),$$

Where

$$\omega_{ab} = \frac{dz_a - dz_b}{z_a - z_b}$$

and $\phi_{ab}(r) \colon V^{\otimes n} \to V^{\otimes n}$ is the linear map such that:

$$\begin{aligned} \phi_{ab}(r)(v_1 \otimes \ldots \otimes v_a \otimes \ldots \otimes v_b \otimes \ldots \otimes v_n) \\ = \sum_i v_1 \otimes \ldots \otimes s_i \triangleright v_a \otimes \ldots \otimes t_i \triangleright v_b \otimes \ldots \otimes v_n, \end{aligned}$$

where $r = \sum_i s_i \otimes t_i \in \mathfrak{g} \otimes \mathfrak{g}$.

The 4-term relation

Categorifications of the 4-term relations via infinitesimal 2-braidings

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- The Knizhnik-Zamolodchikov connection is flat, that is the curvature 2-form $F_A = dA + \frac{1}{2} A \wedge A$ vanishes.
- This follows from the $\mathfrak{g}$-invariance of $\langle , \rangle$, which implies the following relations (known as the 4-term relations):

$$[r_{12} + r_{13}, r_{23}] = 0 \qquad [r_{12}, r_{13} + r_{23}] = 0$$

in

$$\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \subset \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}).$$

Here:

$$r_{12} = \sum_i s_i \otimes t_i \otimes 1, \quad r_{13} = \sum_i s_i \otimes 1 \otimes t_i, \quad r_{23} = \sum_i 1 \otimes s_i \otimes t_i.$$

- Such a symmetric tensor $r = \sum_i s_i \otimes t_i \in \mathfrak{g} \otimes \mathfrak{g}$ will be called an infinitesimal Yang-Baxter operator in $\mathfrak{g}$).

Infinitesimal braid group relations

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- The 4-term relation and the symmetry of r imply that:

$$\phi_{ab}(r)\phi_{bc}(r) + \phi_{ac}(r)\phi_{bc}(r) = \phi_{bc}(r)\phi_{ab}(r) + \phi_{bc}(r)\phi_{ac}(r),$$
$$\phi_{ab}(r) = \phi_{ba}(r),$$

  for each distinct $a, b, c \in \{1, \dots, n\}$.
- We also have:

$$[\phi_{ab}(r), \phi_{a'b'}(r)] = 0, \text{ if } \{a, b\} \cap \{a', b'\} = \emptyset.$$

- These relations will be called *infinitesimal braid group relations.*
- Compare with the usual braid group relations:

$$X_a X_{a+1} X_a = X_{a+1} X_a X_{a+1}$$

$$X_a X_b = X_b X_a, \text{ if } |a - b| \geq 2.$$

Drinfeld-Kohno Theorem

- Recall that the braid group B_n is isomorphic to the fundamental group of $C(n)/S_n$, where the symmetric group S_n acts on:

$$\mathbb{C}(n) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_i \neq z_j \text{ if } i \neq j\}$$

by permutation of coordinates.
- The Knizhnik-Zamolodchikov connection A is invariant under the action of the symmetric group.
- Therefore we have a quotient Knizhnik-Zamolodchikov connection A in the quotient vector bundle

$$\left(\mathbb{C}(n) \times V^{n\otimes}\right) / S_n.$$

- Given that A is flat, by considering its holonomy, we have a group morphism:

$$\mathrm{Hol}_{\mathrm{KZ}} \colon \pi_1(\mathbb{C}(n)/S_n) \cong B_n \to \mathrm{GL}(V^{n\otimes}).$$

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Drinfeld-Kohno Theorem

Theorem (Drinfeld-Kohno)

If $\mathfrak{g}$ is semisimple, $<,>$ is the Cartan-Killing form, and V is a representation of $\mathfrak{g}$, then the representation of the braid group B_n given by the holonomy of the Knizhnik-Zamolodchikov connection is equivalent to the representation of the braid group B_n coming from the Yang-Baxter operator in $U_h(\mathfrak{g})$ and the action of $U_h(\mathfrak{g})$ on V_h (the quantisation of V).

- The holonomy of the Knizhnik-Zamolodchikov connection cannot be immediately extended to links in S^3, since the forms $\omega_{ab} = \frac{dz_a - dz_b}{z_a - z_b}$ explode at maximal and minimal points.
- There exist regularisation techniques for the holonomy at extreme points, and this leads to the usual quantum group link invariants (and the Kontsevich Integral).

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The Kontsevich integral

- Consider the Lie algebra ch_n, generated by the symbols r_{ab}, where $1 \leq a, b \leq n$, satisfying the infinitesimal braid group relations

$$
\begin{aligned}
r_{ab} &= r_{ba}, \\
[r_{ab}, r_{cd}] &= 0 \;\text{ for } \{a,b\} \cap \{c,d\} = \emptyset\,, \\
[r_{ab} + r_{ac}\,, r_{bc}] &= 0 = [r_{ab}, r_{ac} + r_{bc}]\,.
\end{aligned}
$$

- Call it the *Lie algebra of horizontal chord diagrams in n-strands.*
- Consider the following connection form in $\mathbb{C}(n)$:

$$A = \sum_{1 \leq a < b \leq n} \omega_{ab} r_{ab}.$$

- As before:

$$\omega_{ab} = \frac{dz_a - dz_b}{z_a - z_b}.$$

The Kontsevich integral

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- The holonomy of A takes values in the space of formal power series over the universal enveloping algebra $\mathcal{U}(\mathrm{ch}_n)$ of ch_n.
- This holonomy can be regularised at maximal and minimal points of embedded links, defining a link invariant with values in the space of formal power series in the Hopf algebra of chord diagrams in the circle.
- This is called the (framed) Kontsevich integral, and can be proven to be a universal Vassiliev invariants of links.

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Main aim of this work:

- *Categorify* the Knizhnik-Zamolodchikov connection in order to (possibly) obtain invariants of braid cobordisms.
- *Categorify* the Lie algebra of horizontal chord diagrams.
- Discuss the *infinitesimal relations* for braid cobordisms.
- *Categorify* the notion of an infinitesimal Yang-Baxter operator in a Lie algebra.
- find examples.
- Setting: 2-connections on 2-bundles.
- Particular case considered here: 2-connections on vector bundles with typical fibre being a chain complex of vector spaces.

Lie crossed modules

Definition (Lie crossed module)

A Lie crossed module:

$$\mathcal{G} = (\partial \colon H \to G, \triangleright)$$

is given by a Lie group morphism $\partial \colon H \to G$ together with a smooth left action $\triangleright$ of G on H by automorphisms, such that the following relations, called Peiffer relations, hold:

1. $\partial(g \triangleright h) = g\partial(h)g^{-1}$; for each $g \in G$ and $h \in H$,
2. $\partial(h) \triangleright h' = hh'h^{-1}$; for each $h, h' \in H$.

The category of crossed modules is equivalent to the category of strict 2-groups (Brown and Spencer).

Differential crossed modules

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The differential counterpart of a Lie crossed module is what is called a differential crossed module.

Definition (Differential crossed module)

A differential crossed module:

$$\mathfrak{G} = (\partial\colon \mathfrak{h} \to \mathfrak{g}, \triangleright)$$

is given by a Lie algebra morphism $\partial\colon \mathfrak{h} \to \mathfrak{g}$ together with a left action of $\mathfrak{g}$ on $\mathfrak{h}$ by derivations, such that the following relations, also called Peiffer relations, hold:

1. $\partial(X \triangleright \xi) = [X, \partial(\xi)]; \text{ for each } X \in \mathfrak{g}, \text{ and each } \xi \in \mathfrak{h}\,,$
2. $\partial(\xi) \triangleright \nu = [\xi, \nu]; \text{ for each } \xi, \nu \in \mathfrak{h}\,.$

The category of differential crossed modules is equivalent to the category of strict Lie-2-algebras (Baez and Crans).

Differential crossed modules from complexes of vector spaces

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We can construct a differential crossed module

$$\mathfrak{gl}(\mathcal{V}) = \left(\beta\colon \mathfrak{gl}^1(\mathcal{V}) \to \mathfrak{gl}^0(\mathcal{V}), \triangleright\right)$$

from any complex of vector spaces

$$\mathcal{V} = (\dots \xrightarrow{\partial} V_n \xrightarrow{\partial} V_{n-1} \xrightarrow{\partial} \dots).$$

- Define a Lie algebra $\mathfrak{gl}^0(V)$, given by all chain maps $f\colon \mathcal{V} \to \mathcal{V}$, with the usual commutator of chain-maps.
- There exist two natural Lie algebra structures on the vector space $\mathrm{Hom}^1(\mathcal{V})$ of degree 1 maps $\mathcal{V} \to \mathcal{V}$: where:

$$\{s,t\}_l = s\partial t - t\partial s + st\partial - ts\partial,$$

$$\{s,t\}_r = s\partial t - t\partial s + \partial st - \partial ts.$$

- There exists a Lie algebra map $\beta\colon \mathrm{Hom}^1(\mathcal{V}) \to \mathfrak{gl}^0(\mathcal{V})$ with

$$\beta(s) = \partial s + s\partial.$$

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- There exists an action by derivations of $\mathfrak{gl}^0(\mathcal{V})$ on $\mathrm{Hom}^1(\mathcal{V})$ such that: $f \triangleright s = fs - sf$.
- We do not always have differential crossed modules since the relation $\{s,t\} = \beta(s) \triangleright t$ may fail in general, unless we are considering a chain complex of length two.
- Consider the map $\beta' \colon \mathrm{Hom}^2(\mathcal{V}) \to \mathrm{Hom}^1(\mathcal{V})$ such that

$$\beta'(h) = -h\partial + \partial h.$$

Then $\beta'(\mathrm{Hom}^2(\mathcal{V}))$ is a $\mathfrak{gl}^0(\mathcal{V})$-invariant Lie algebra ideal of $\mathrm{Hom}^1(\mathcal{V})$, for $\{,\}_l$ and $\{,\}_r$, contained in $\ker(\beta)$.

- We have a quotient Lie algebra $\{,\}_l/ = \{,\}_r/$, in

$$\mathfrak{gl}^1(\mathcal{V}) = \frac{\mathrm{Hom}^1(\mathcal{V})}{\beta'(\mathrm{Hom}^2(\mathcal{V}))},$$

provided with a (quotient) map $\beta \colon \mathfrak{gl}^1(\mathcal{V}) \to \mathfrak{gl}^0(\mathcal{V})$.

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Theorem: Given a complex $\mathcal{V}$ of vector spaces there exists a differential crossed module:

$$\mathfrak{gl}(\mathcal{V}) = \left(\beta\colon \mathfrak{gl}^1(\mathcal{V}) \to \mathfrak{gl}^0(\mathcal{V}), \triangleright\right).$$

- Where $\mathfrak{gl}^0(\mathcal{V})$ is the Lie algebra of chain-maps $\mathcal{V} \to \mathcal{V}$, with commutator

$$[f, g] = fg - gf,$$

- $\mathfrak{gl}^1(\mathcal{V}) = \mathrm{Hom}^1(\mathcal{V})/\beta'\left(\mathrm{Hom}^2(\mathcal{V})\right)$ with commutator:

$$\{s, t\} = s\partial t + st\partial - t\partial s - ts\partial,$$

- $\beta(s) = s\partial + \partial s,$
- $\beta'(h) = -\partial h + h\partial,$
- $f \triangleright s = fs - sf.$

Categorical representations of differential crossed modules

- A representation of a differential crossed module $\mathfrak{G} = (\mathfrak{h} \to \mathfrak{g}, \triangleright)$ on a complex of vector spaces $\mathcal{V}$ is a differential crossed module map $\rho\colon \mathfrak{G} \to \mathfrak{gl}(\mathcal{V})$

$$\mathfrak{G} = (\mathfrak{h} \to \mathfrak{g}, \triangleright) \xrightarrow{(\rho^1,\rho^0)} \big(\beta\colon \mathfrak{gl}^1(\mathcal{V}) \to \mathfrak{gl}^0(\mathcal{V}), \triangleright\big) = \mathfrak{gl}(\mathcal{V}).$$

- For any $X \in \mathfrak{g}$ we have a chain map $\rho^0_X\colon \mathcal{V} \to \mathcal{V}$ and for each $v \in \mathfrak{h}$ we have a chain homotopy (up to 2-fold homotopy) $\rho^1_v \in \mathfrak{gl}^1(\mathcal{V})$, such that:
  1. $[\rho^0_X, \rho^0_Y] = \rho^0_{[X,Y]}$ where $X, Y \in \mathfrak{g}$.
  2. $\{\rho^1_v, \rho^1_w\} = \rho^1_{[v,w]}$, where $v, w \in \mathfrak{h}$.
  3. $\beta(\rho^1_v) = \rho^0_{\partial(v)}$, where $v \in \mathfrak{h}$.
- If there are representations ρ and ρ' of $\mathfrak{G}$ on $\mathcal{V}$ and $\mathcal{V}'$ then we have a representation of $\mathfrak{G}$ on $\mathcal{V} \otimes \mathcal{V}'$:
  1. $(\rho \otimes \rho')^0_X = \rho^1_X \otimes \mathrm{id} + \mathrm{id} \otimes {\rho'}^0_X$.
  2. $(\rho \otimes \rho')^1_v = \rho^1_v \otimes \mathrm{id} + \mathrm{id} \otimes {\rho'}^1_v$

Adjoint representation of a differential crossed module

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Example Let $(\partial\colon \mathfrak{h} \to \mathfrak{g}, \triangleright)$ be a differential crossed module. The adjoint representation of $\mathfrak{G}$ on its underlying chain complex $\mathfrak{h} \xrightarrow{\partial} \mathfrak{g}$ is given by the pair $\rho = (\rho_1, \rho_0)$, where:

- If $X \in \mathfrak{g}$ the chain map $\rho_0^X\colon \mathfrak{G} \to \mathfrak{G}$ is such that

$$\rho_0^X(Y) = [X, Y]$$

  and

$$\rho_0^X(\zeta) = X \triangleright \zeta$$

  where $Y \in \mathfrak{g}$ and $\zeta \in \mathfrak{h}$.

- If $\zeta \in \mathfrak{h}$ the homotopy $\rho_1^\zeta\colon \mathfrak{g} \to \mathfrak{h}$ is such that

$$\rho_1^\zeta(X) = -X \triangleright \zeta.$$

Local 2-connections and their 2-dimensional holonomy

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Let

$$\mathcal{V} = (\dots \xrightarrow{\partial} V_n \xrightarrow{\partial} V_{n-1} \xrightarrow{\partial} \dots).$$

be a chain complex of vector spaces, with associated differential crossed module

$$\mathfrak{gl}(\mathcal{V}) = \left(\beta \colon \mathfrak{gl}^1(\mathcal{V}) \to \mathfrak{gl}^0(\mathcal{V}), \triangleright\right).$$

A local 2-connection (A, B) in a manifold M is given by

- A 1-form A with values in $\mathfrak{gl}^0(\mathcal{V})$.
- A 2-form B with values in $\mathfrak{gl}^1(\mathcal{V})$.
- Such that $\beta(B) = F_A = dA + \frac{1}{2} A \wedge A$.
- The 2-curvature of (A, B) is, by definition:

$$\mathcal{M}_{(A,B)} = dB + A \wedge B.$$

- Local 2-connections can be integrated to give a 2-dimensional holonomy.

The 2-category $\mathrm{Aut}(\mathcal{V})$ for a chain complex $\mathcal{V}$

Let

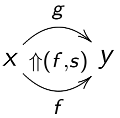
$$\mathcal{V} = (\dots \xrightarrow{\partial} V_n \xrightarrow{\partial} V_{n-1} \xrightarrow{\partial} \dots).$$

be a chain complex of vector spaces.

- Define a 2-category $\mathrm{Aut}(\mathcal{V})$ with a single object.
- 1-morphisms: chain maps $f\colon \mathcal{V} \to \mathcal{V}$.
- Composition is done in the reverse order:

$$(\mathcal{V} \xrightarrow{f} \mathcal{V} \xrightarrow{g} \mathcal{V}) = (\mathcal{V} \xrightarrow{fg} \mathcal{V}).$$

- The 2-morphisms $f \Longrightarrow g$, have the form:

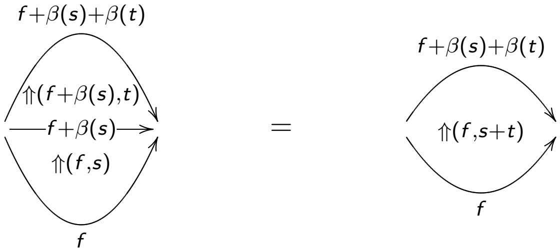


,

where $s \in \mathfrak{gl}^1(\mathcal{V}) = \mathrm{Hom}^1(\mathcal{V})/\beta'(\mathrm{Hom}^2(\mathcal{V}))$, with:
$g = f + \beta(s) = f + \partial s + s\partial$.

The 2-category $\mathrm{Aut}(\mathcal{V})$ for a chain complex $\mathcal{V}$

The vertical composition of 2-morphisms is



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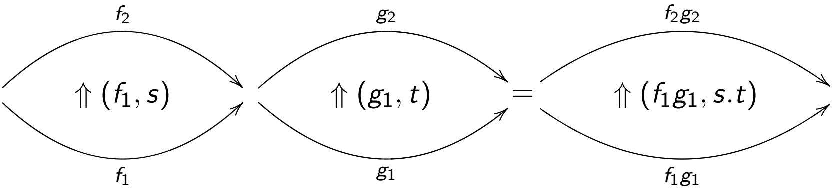
The 2-category $\mathrm{Aut}(\mathcal{V})$ for a chain complex $\mathcal{V}$

The horizontal composition of 2-morphisms is



here

$$f_2 g_2 = f_1 g_1 + \beta(f_1 t + s g_2) = f_1 g_1 + \beta(s g_1 + f_2 t)$$

and

$$s.t = f_1 t + s g_2 = s g_1 + f_2 t.$$

These coincide in

$$\mathfrak{gl}^1(\mathcal{V}) = \frac{\mathrm{Hom}^1(\mathcal{V})}{\beta'(\mathrm{Hom}^2(\mathcal{V}))}.$$

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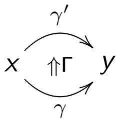
Two dimensional holonomy

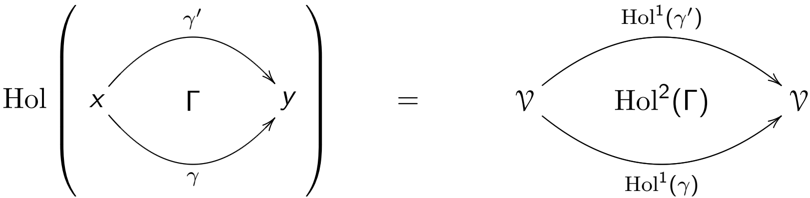
- A path $x \xrightarrow{\gamma} y$ in a manifold M is a piecewise smooth map $\gamma : [0,1] \to M$, connecting x and y.
- Given paths $x \xrightarrow{\gamma} y$ and $x \xrightarrow{\gamma'} y$ a 2-path $\gamma \overset{\Gamma}{\Longrightarrow} \gamma'$, written as:



  is given by piecewise smooth map $\Gamma \colon [0,1]^2 \to M$, defining a homotopy $\gamma \to \gamma'$, relative to the boundary.
- These compose vertically and horizontally in the obvious way.

The 2-dimensional holonomy of a local 2-connection

Let $\mathcal{V}$ be a chain-complex of vector spaces, M be a manifold, and (A, B) a $\mathfrak{gl}(\mathcal{V})$-valued local 2-connection. There exists a 2-dimensional holonomy $\Gamma \mapsto \mathrm{Hol}(\Gamma)$, which for a 2-path $\gamma \xrightarrow{\Gamma} \gamma'$ associates a 2-morphism $\mathrm{Hol}(\Gamma)$ of $\mathrm{Aut}(\mathcal{V})$, say:



which preserves horizontal and vertical composites:

$$\mathrm{Hol}(\Gamma\Gamma') = \mathrm{Hol}(\Gamma)\mathrm{Hol}(\Gamma'))$$

and

$$\mathrm{Hol}\left(\frac{\Gamma}{\Gamma'}\right) = \frac{\mathrm{Hol}(\Gamma)}{\mathrm{Hol}(\Gamma')}.$$

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As is the case of 1-dimensional holonomy, the variation of the holonomy when we vary the 2-paths is ruled by the 2-curvature 3-form:

Theorem

Suppose (A, B) has zero 2-curvature 3-tensor $\mathcal{M}_{(A,B)} = dB + A \wedge B$ and that Γ and Γ' are homotopic, relative to the boundary of D^2. Then $\mathrm{Hol}^2(\Gamma) = \mathrm{Hol}^2(\Gamma')$.

Flatness conditions for the 2-Knizhnik-Zamolodchikov connection

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- Let $\mathcal{V}$ be a chain complex of vector spaces. Recall the construction of the differential crossed module

$$\mathfrak{gl}(\mathcal{V}) = \left(\beta \colon \mathfrak{gl}^1(\mathcal{V}) \to \mathfrak{gl}^0(\mathcal{V}), \triangleright\right).$$

- We are interested in 2-flat local 2-connections (A, B) in $\mathbb{C}(n) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : i \neq j \implies z_i \neq z_j\}$, with values in the differential crossed module $\mathfrak{gl}(\mathcal{V})$.
- We thus want to define a $\mathfrak{gl}^0(\mathcal{V})$-valued 1-form A and a $\mathfrak{gl}^1(\mathcal{V})$-valued 2-form B such that: $\beta(B) = F_A = dA + \frac{1}{2} A \wedge A$.
- The 1-form A should resemble the Knizhnik-Zamolodchikov connection $\sum_{i<j} \omega_{ij} r_{ij}$ with

$$\omega_{ij} = \frac{dz_i - dz_j}{z_i - z_j}.$$

Setting

Consider a family of chain maps $\{r_{ab}\} \in \mathfrak{gl}^0(\mathcal{V})$ ($a, b \in \{1, \dots, n\}$, $a \neq b$) such that:

$$r_{ab} = r_{ba}, \qquad [r_{ab}, r_{cd}] = 0 \ \text{ for } \{a, b\} \cap \{c, d\} = \emptyset.$$

Define a $\mathfrak{gl}^0(\mathcal{V})$-valued connection 1-form A over $\mathbb{C}(n)$ as

$$A = \sum_{1 \leq a < b \leq n} \omega_{ab} r_{ab}, \ \text{where } \omega_{ab} = \frac{dz_a - dz_b}{z_a - z_b}.$$

The curvature $\mathcal{F}_A = dA + A \wedge A$ of A is:

$$\mathcal{F}_A = -2 \sum_{a<b<d} \omega_{bd} \wedge \omega_{da} \, [r_{ab} + r_{ad}, r_{bd}] - 2 \sum_{a<b<d} \omega_{da} \wedge \omega_{ab} \, [r_{ab}, r_{bd} + r_{ad}].$$

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Flatness conditions for the 2-Knizhnik-Zamolodchikov connection

We now need a $\mathfrak{gl}^1(\mathcal{V})$-valued 2-form B such that $\beta(B) = \mathcal{F}_A$. Chose homotopies $P_{abc}, Q_{abc} \in \mathfrak{gl}^1(\mathcal{V})$ such that:

$$\partial(P_{abc}) = [r_{bc}, r_{ab} + r_{ac}] \quad \text{and} \quad \partial(Q_{abc}) = [r_{ab}, r_{ac} + r_{bc}]$$

and moreover:

$$r_{ab} \triangleright P_{ijk} = 0 = r_{ab} \triangleright Q_{ijk}, \text{ if } \{a, b\} \cap \{i, j, k\} = \emptyset$$

Put:

$$B = 2 \sum_{a<b<c} \omega_{bc} \wedge \omega_{ca} \, P_{abc} - 2 \sum_{a<b<c} \omega_{ca} \wedge \omega_{ab} \, Q_{abc} \, .$$

Flatness conditions for 2- Knizhnik-Zamolodchikov connections

Theorem (Cirio, JFM)

The 2-curvature 3-form $\mathcal{M}_{(A,B)} = dB + A \wedge^{\triangleright} B$ *of* (A, B) *vanishes, if and only if, the following conditions are satisfied:*

$$
\begin{aligned}
(r_{14}+r_{24}+r_{34}) \triangleright P_{123} - (r_{12}+r_{13}) \triangleright Q_{234} + r_{23} \triangleright (Q_{124}+Q_{134}) &= 0\,,\\
(r_{12}+r_{13}+r_{14}) \triangleright P_{234} + r_{34} \triangleright (P_{123}+P_{124}) - (r_{23}+r_{24}) \triangleright P_{134} &= 0\,,\\
(r_{14}+r_{24}+r_{34}) \triangleright Q_{123} + r_{12} \triangleright (Q_{134}+Q_{234}) - (r_{13}+r_{23}) \triangleright Q_{124} &= 0\,,\\
(r_{12}+r_{13}+r_{14}) \triangleright Q_{234} + r_{23} \triangleright (P_{124}+P_{134}) - (r_{24}+r_{34}) \triangleright P_{123} &= 0\,,\\
r_{12} \triangleright (P_{134}+P_{234}) - r_{34} \triangleright (Q_{123}+Q_{124}) &= 0\,,\\
r_{13} \triangleright (P_{124}-P_{234}-Q_{234}) + r_{24} \triangleright (Q_{123}+P_{123}-Q_{134}) &= 0\,.
\end{aligned}
$$

with $a < b < c < d \in \{1, \ldots, n\}$.

Observation: These relations are satisfied (in $\mathfrak{gl}^0(\mathcal{V})$) if

$$P_{abc} = [r_{ab} + r_{ac}, r_{bc}] \text{ and } Q_{abc} = [r_{ab}, r_{ac} + r_{bc}].$$

This follows from Bianchi identity $dF_A + A \wedge F_A = 0$.

Flatness and equivariance conditions for the 2-Knizhnik-Zamolodchikov connection

Consider a representation $\sigma \mapsto \rho_\sigma$ of S_n on $\mathcal{V}$ by chain-complex maps. Choose chain complex maps

$$r_{ab} \in \mathfrak{gl}_0(\mathcal{V}),$$

where $a, b \in \{1, \ldots, n\}$, with $a \neq b$,and also chain-homotopies

$$K_{ijk} \in \mathfrak{gl}^1(\mathcal{V}),$$

where i, j, k are distinct indices in $\{1, \ldots, n\}$. Suppose:

- $r_{ab} = r_{ba}$.
- $[r_{ab}, r_{cd}] = 0$ for $\{a, b\} \cap \{c, d\} = \emptyset$
- $\beta(K_{ijk}) = R_{ijk} = [r_{ij} + r_{ik}, r_{jk}]$.
- $r_{ab} \triangleright K_{ijk} = 0$ if $\{a, b\} \cap \{i, j, k\} = \emptyset$.

We want that the two dimensional holonomy of (A, B) descend to a two-dimensional holonomy in $\mathbb{C}(n)/S_n$. We now impose:

$$\rho_\sigma^{-1}(\sigma^*(A)) = A \text{ and } \rho_\sigma^{-1}(\sigma^*(B)) = B, \text{ for each } \sigma \in S_n.$$

Flatness and S_n-equivariance conditions for the 2-Knizhnik-Zamolodchikov connection

Categorifications of the 4-term relations via infinitesimal 2-braidings

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Theorem (Cirio, FM)

The $\mathfrak{gl}(\mathcal{V})$-valued 2-connection (A, B), where

$$A = \sum_{a<b} \omega_{ab} r_{ab} \quad \mathit{and} \quad B = \sum_{a<b<c} K_{bac}\, \omega_{ab} \wedge \omega_{ac} + K_{abc}\, \omega_{ab} \wedge \omega_{bc}$$

is invariant under the action of S_n, if, and only if:

$$K_{abc} + K_{bca} + K_{cab} = 0, \qquad K_{bca} = K_{bac}$$

for each distinct $a, b, c, d \in \{1, \dots, n\}$, and if for each $\sigma \in S_n$:

$$r_{\sigma(a)\sigma(b)} = \rho_\sigma(r_{ab}) \ \mathit{and}\ K_{\sigma(a)\sigma(b)\sigma(c)} = \rho_\sigma(K_{abc}).$$

Moreover, in such a case (A, B) is 2-flat if, and only if,

$$r_{ad} \triangleright (K_{bac} + K_{bcd}) + (r_{ab} + r_{bc} + r_{bd}) \triangleright K_{cad} - (r_{ac} + r_{cd}) \triangleright K_{bad} = 0,$$
$$r_{bc} \triangleright (K_{bad} + K_{cad}) - r_{ad} \triangleright (K_{dbc} + K_{abc}) = 0.$$

Differential crossed module of chord diagrams

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The differential crossed module of totally symmetric horizontal 2-chord diagrams:

$$2\mathfrak{ch}_n = (\beta\colon \mathrm{ch}_n \to \mathrm{ch}_n^+)$$

is the differential crossed module formally generated by the elements:

$$r_{ab} \in \mathrm{ch}_n^+ \qquad \text{and} \qquad K_{abc} \in \mathrm{ch}_n,$$

where $a \neq b, a \neq c, b \neq c$, with relations:

- $r_{ab} = r_{ba}$.
- $[r_{ab}, r_{cd}] = 0$ for $\{a, b\} \cap \{c, d\} = \emptyset$
- $\beta(K_{abc}) = [r_{ab} + r_{ac}, r_{bc}]$
- $r_{ab} \triangleright K_{ijk} = 0$ if $\{a, b\} \cap \{i, j, k\} = \emptyset$.
- $r_{ad} \triangleright (K_{bac} + K_{bcd}) + (r_{ab} + r_{bc} + r_{bd}) \triangleright K_{cad} - (r_{ac} + r_{cd}) \triangleright K_{bad} = 0$
- $r_{bc} \triangleright (K_{bad} + K_{cad}) - r_{ad} \triangleright (K_{dbc} + K_{abc}) = 0,$
- $K_{abc} + K_{bca} + K_{cab} = 0$
- $K_{bca} = K_{bac}$

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Infinitesimal Yang-Baxter operators

An infinitesimal Yang-Baxter operator in a Lie algebra $\mathfrak{g}$ is a symmetric tensor $r = \sum_i s_i \otimes t_i \in \mathfrak{g} \otimes \mathfrak{g}$, with

$$[r_{12} + r_{13}, r_{23}] = 0.$$

Given an infinitesimal Yang-Baxter operator and a representation V of $\mathfrak{g}$, the $\mathrm{Hom}(V^{n\otimes}, V^{n\otimes})$-valued connection 1-form in $\mathbb{C}(n)$:

$$A = \sum_{a<b} \phi_{ab}(r)\omega_{ab}, \text{ where } \omega_{ab} = \frac{dz_a - dz_b}{z_a - z_b}$$

where we put

$$\phi_{ab}(r)(v_1 \otimes \ldots \otimes v_a \otimes \ldots \otimes v_b \otimes \ldots \otimes v_n) = \sum_i v_1 \otimes \ldots \otimes s_i \triangleright v_a \otimes \ldots \otimes t_i \triangleright v_b \otimes \ldots \otimes v_n,$$

is flat and S_n-invariant.

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Infinitesimal 2-Yang-Baxter operators

Let $\mathfrak{G} = (\partial\colon \mathfrak{h} \to \mathfrak{g}, \triangleright)$ be a differential crossed module. Define $\bar{\mathfrak{U}}^{(n)}$ as being

$$\frac{(\mathfrak{h} \otimes \mathfrak{g} \otimes \ldots \otimes \mathfrak{g}) \oplus (\mathfrak{g} \otimes \mathfrak{h} \otimes \ldots \otimes \mathfrak{g}) \oplus \cdots \oplus (\mathfrak{g} \otimes \mathfrak{g} \otimes \ldots \otimes \mathfrak{h})}{\ldots \otimes \partial(v) \otimes \ldots \otimes w \otimes \cdots = \ldots \otimes v \otimes \ldots \otimes \partial(w) \otimes \ldots}$$

Example:

$$\mathfrak{U}^{(3)} = \frac{(\mathfrak{h} \otimes \mathfrak{g} \otimes \mathfrak{g}) \oplus (\mathfrak{g} \otimes \mathfrak{h} \otimes \mathfrak{g}) \oplus (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{h})}{\left\{\begin{matrix} \partial(v)\otimes w\otimes X = v\otimes\partial(w)\otimes X \\ \partial(v)\otimes X\otimes w = v\otimes X\otimes\partial(w) \\ X\otimes\partial(v)\otimes w = X\otimes v\otimes\partial(w) \end{matrix}\right\}}$$

We define $\hat{\partial}\colon \mathfrak{U}^{(3)} \to \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ as:

$$\hat{\partial} = \partial \otimes \mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes \partial \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{id} \otimes \partial.$$

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Definition (Totally symmetric infinitesimal 2-Yang-Baxter operator)

Let $\mathfrak{G} = (\partial\colon \mathfrak{h} \to \mathfrak{g}, \triangleright)$ be a differential crossed module. An infinitesimal 2-Yang-Baxter operator in $\mathfrak{G}$ is given by a symmetric tensor $r \in \mathfrak{g} \otimes \mathfrak{g}$, and an element $P \in \bar{\mathfrak{U}}^{(3)}$ such that, in $\bar{\mathfrak{U}}^{(4)}$:

$$\hat{\partial}(P) = [r_{12} + r_{13}, r_{23}],$$

$$r_{14} \triangleright (P_{213} + P_{234}) + (r_{12} + r_{23} + r_{24}) \triangleright P_{314} - (r_{13} + r_{34}) \triangleright P_{214} = 0,$$

$$r_{23} \triangleright (P_{214} + P_{314}) - r_{14} \triangleright (P_{423} + P_{123}) = 0,$$

$$P_{123} + P_{231} + P_{312} = 0,$$

$$P_{123} = P_{132}.$$

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Infinitesimal 2-Yang-Baxter operators

Example Consider a Lie algebra $\mathfrak{g}$, and the crossed module given by the the identity map $\mathfrak{g} \xrightarrow{\mathrm{id}} \mathfrak{g}$ and the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}$. Then $\bar{\mathfrak{U}}^{(n)} = \mathfrak{g}^{\otimes n}$.
Given any tensor $r \in \mathfrak{g} \otimes \mathfrak{g}$, the pair $(r, [r_{12} + r_{13}, r_{23}])$ is an infinitesimal 2-Yang-Baxter operator.

Infinitesimal 2-Yang-Baxter operators

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Theorem

Let (r, P) be an infinitesimal 2-Yang-Baxter operator in the differential crossed module $\mathfrak{G} = (\partial : \mathfrak{h} \to \mathfrak{g}, \triangleright)$. Consider a categorical representation of $\mathfrak{G}$ on a complex of vector spaces $\mathcal{V}$. Consider also the $\mathfrak{gl}(\mathcal{V}^{\overline{\otimes}\, n})$-valued 2-connection (A, B) on the configuration space $\mathbb{C}(n)$, defined as:

$$A = \sum_{a<b} \omega_{ab}\, \bar{\phi}_{ab}(r)\,.$$

$$B = \sum_{a<b<c} \omega_{ab} \wedge \omega_{ac}\, \bar{\phi}_{bac}(P) + \omega_{ab} \wedge \omega_{bc}\, \bar{\phi}_{abc}(P)$$

Then (A, B) is a flat 2-connection, invariant the action of the symmetric group S_n. Therefore its $\mathrm{Aut}(\mathcal{V})$-valued holonomy descends to a two dimensional holonomy in $\mathbb{C}(n)/S_n$.

Lie algebra cohomology

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- Differential crossed modules $\mathfrak{G} = (\partial\colon \mathfrak{h} \to \mathfrak{g}, \triangleright)$ are classified, up to weak equivalence, by a Lie algebra cohomology class $k \in H^3(\mathfrak{k}, M)$, where the differential crossed module $\mathfrak{G}$ sits inside the exact sequence of Lie algebras:

$$\{0\} \to M \to \mathfrak{h} \xrightarrow{\partial} \mathfrak{g} \xrightarrow{\mathrm{proj}} \mathfrak{k} \to \{0\},$$

  with M abelian. $\mathfrak{k}$ has an induced action on M.
- Given a differential crossed module $\mathfrak{G}$, the associated cohomology class (the k-invariant) is denoted by $k(\mathfrak{G})$, and we say that $\mathfrak{G}$ geometrically realises k.

The string Lie 2-algebra

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- The string Lie-2-algebra is a differential crossed module $\mathfrak{String}$ geometrically realizing the Lie algebra 3-cocycle $\omega\colon \mathfrak{sl}_2(\mathbb{C}) \wedge \mathfrak{sl}_2(\mathbb{C}) \wedge \mathfrak{sl}_2(\mathbb{C}) \to \mathbb{C}$ with:

$$\omega(X, Y, Z) = \langle [X, Y], Z \rangle.$$

- Note that $\mathfrak{String}$ is well defined up to weak equivalence (but not up to isomorphism).
- A very explicit realization of $\mathfrak{String}$ is due to Wagemann.

Wagemann's realization of $\mathfrak{String}$

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- Let W_1 be the Lie algebra of vector fields in one variable x:

$$\Big[f(x)\,\frac{d}{dx}\,,\,g(x)\,\frac{d}{dx}\Big]=\Big(f\,\frac{dg}{dx}-\frac{df}{dx}\,g\Big)(x)\,\frac{d}{dx}.$$

.

- Identify $\mathfrak{sl}_2(\mathbb{C})\subset W_1$ as the sub-Lie algebra generated by

$$e_{-1}=\frac{d}{dx}\,,\qquad e_0=x\,\frac{d}{dx}\,,\qquad e_1=x^2\,\frac{d}{dx}.$$

- So that the commutation relations are:

$$[e_0,e_{-1}]=-\,e_{-1}\,,\qquad [e_{-1},e_1]=2e_0\,,\qquad [e_0,e_1]=e_1\,.$$

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Wagemann's realization of $\mathfrak{String}$

- Let $\mathbb{F}_0$ be the space of polynomials in the variable x,
- Let $\mathbb{F}_1$ the space of formal one-forms $f(x)dx$, where $f(x) \in \mathbb{F}_0$.
- We consider $\mathbb{F}_0$ and $\mathbb{F}_1$ to be abelian Lie algebras. They are both W_1-modules via the Lie derivative:

$$\Big(f(x)\frac{d}{dx}\Big) \triangleright g(x) = (fg')(x)$$
$$\Big(f(x)\frac{d}{dx}\Big) \triangleright \big(g(x)dx\big) = (fg' + f'g)(x)\, dx.$$

- Hence they are $\mathfrak{sl}_2(\mathbb{C})$-modules as well
- Consider the 2-cocycle $\alpha \colon \mathfrak{sl}_2(\mathbb{C}) \wedge \mathfrak{sl}_2(\mathbb{C}) \to \mathbb{F}_1$, defined as, in the basis $\{e_{-1}, e_0, e_1\}$ of $\mathfrak{sl}_2(\mathbb{C})$:

$$\alpha(e_0, e_1) = -\alpha(e_1, e_0) = 2dx\,, \quad \text{and zero otherwise}\,.$$

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Wagemann's realization of $\mathfrak{String}$

The differential crossed module $\mathfrak{String}$ has the form

$$(\partial\colon \mathbb{F}_0 \to \mathbb{F}_1 \rtimes_\alpha \mathfrak{sl}_2(\mathbb{C}), \triangleright),$$

where:

- if $(a, y), (b, z) \in \mathbb{F}_1 \rtimes_\alpha \mathfrak{sl}_2(\mathbb{C})$ we have:

$$\big[(a, y), (b, z)\big] := \big(y \triangleright b - z \triangleright a + \alpha(y, z)\, , [y, z]\,\big).$$

- $\partial = (d, 0)$, where d denotes the formal de Rham differential.
- The Lie algebra $\mathbb{F}_1 \rtimes_\alpha \mathfrak{sl}_2(\mathbb{C})$ acts on $\mathbb{F}_0$ via the action of $\mathfrak{sl}_2(\mathbb{C})$ in F_0 and the projection $\pi\colon \mathbb{F}_1 \rtimes_\alpha \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{sl}_2(\mathbb{C})$.
- The string differential crossed module can be embedded into the exact sequence:

$$\{0\} \to \mathbb{C} \xrightarrow{i} \mathbb{F}_0 \xrightarrow{\partial} \mathbb{F}_1 \rtimes_\alpha \mathfrak{sl}_2(\mathbb{C}) \xrightarrow{\pi} \mathfrak{sl}_2(\mathbb{C}) \to \{0\}.$$

An infinitesimal 2-Yang-Baxter operator in $\mathfrak{String}$

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Let find an infinitesimal 2-Yang-Baxter operator $(\bar{r}, P)$ in $\mathfrak{String}$, with $(\pi \otimes \pi)(\bar{r}) = r$, where $r \in \mathfrak{sl}_2(\mathbb{C}) \otimes \mathfrak{sl}_2(\mathbb{C})$ is the infinitesimal Yang-Baxter operator in $\mathfrak{sl}_2(\mathbb{C})$:

$$r = e_{-1} \otimes e_1 + e_1 \otimes e_{-1} - 2\, e_0 \otimes e_0 = \sum_i s_i \otimes t_i\,.$$

It holds that $[r_{12} + r_{13}, r_{23}] = 0$. Put:

$$\bar{r} = (0, e_1) \otimes (0, e_{-1}) + (0, e_{-1}) \otimes (0, e_1) - 2\,(0, e_0) \otimes (0, e_0)$$
$$= \sum_i \overline{s_i} \otimes \overline{t_i}\,.$$

Therefore:

$$[\bar{r}_{12} + \bar{r}_{13}, \bar{r}_{23}] =$$
$$= (0, e_{-1}) \otimes (0, e_0) \otimes (dx, 0) + (0, e_{-1}) \otimes (dx, 0) \otimes (0, e_0) +$$
$$- (0, e_0) \otimes (dx, 0) \otimes (0, e_{-1}) - (0, e_0) \otimes (0, e_{-1}) \otimes (dx, 0)\,.$$

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Let P be the obvious lift of $[\overline{r}_{12} + \overline{r}_{13}, \overline{r}_{23}]$ to $\mathfrak{U}^{(3)}$:

$$P = (0, e_{-1}) \otimes (0, e_0) \otimes x + (0, e_{-1}) \otimes x \otimes (0, e_0) \\ - (0, e_0) \otimes x \otimes (0, e_{-1}) - (0, e_0) \otimes (0, e_{-1}) \otimes x \,.$$

let also:

$$C = \sum_i 2_{\mathbb{F}_0} \otimes \overline{s_i} \otimes \overline{t_i} - \sum_i \overline{s_i} \otimes \overline{t_i} \otimes 1_{\mathbb{F}_0} - \sum_i \overline{s_i} \otimes 1_{\mathbb{F}_0} \otimes \overline{t_i} \in \overline{\mathfrak{U}}^{(3)}.$$

Theorem (Cirio, JFM)

The pair $(\overline{r}, P - 2C)$ is an infinitesimal 2-Yang-Baxter operator.

Therefore:

$$\beta(P) = [\overline{r}_{12} + \overline{r}_{13}, \overline{r}_{23}]$$

$$P_{123} + P_{231} + P_{312} = 0$$

$$\overline{r}_{14} \triangleright (P_{213} + P_{234}) + (\overline{r}_{12} + \overline{r}_{23} + \overline{r}_{24}) \triangleright P_{314} - (\overline{r}_{13} + \overline{r}_{34}) \triangleright P_{214} = 0$$

$$\overline{r}_{23} \triangleright (P_{214} + P_{314}) - \overline{r}_{14} \triangleright (P_{423} + P_{123}) = 0$$

Invariants of braid cobordisms?

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Consider a surface braid $b_1 \xrightarrow{\mathcal{S}} b_2$ without branch points, connecting the braids b_1 and b_2.

This has an associated map $\mathcal{S}' \colon D^2 \to \mathbb{C}(n)/S_n$. By using Chen integrals we can therefore define a surface holonomy

$$H(b_1) \xrightarrow{H(\mathcal{S}')} H(b_2),$$

where $H(b_1)$ and $H(b_2)$ are valued in the algebra of formal power series in the universal enveloping algebra $\mathcal{U}(\mathrm{ch}_n^+)$, and $H(\mathcal{S}')$ is valued in the algebra of formal power series in $\mathcal{U}(2\mathrm{ch}_n)$.

Problem: Extend this surface holonomy to the case when $\mathcal{S}$ has branch points. This requires some new input since, in general, the map $\mathcal{S}' \colon D^2 \setminus \{\text{branch points}\} \to \mathbb{C}(n)/S_n$ is not defined in all of D^2, however having very particular singularities.

Problem Spaces of (general) 2-chord diagrams for any 2-manifold? (This is very related with the previous problem.)

Problem: Categorification of the STU- and IHX- relations?

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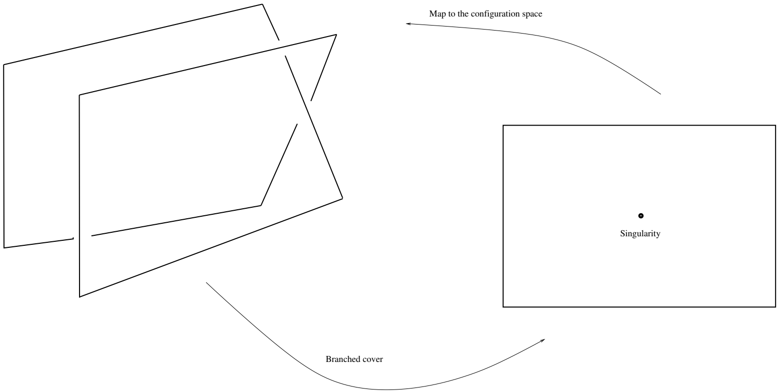
Invariants of braid cobordisms?



Drawing by Rui Carpentier.

Invariants of braid cobordisms from the String Lie-2-algebra?

- Is the holonomy of the Knizhnik-Zamolodchikov 2-connection derived from the infinitesimal 2-Yang-Baxter operator on the string Lie-2-algebra convergent (or can it be regularised) for braid-cobordisms with branch points.
- Does it yield an interesting invariant of braid cobordisms?
- The non-trivial part of the homology would essentially live in

$$H_1(\mathcal{HOM}(\mathfrak{String}, \mathfrak{String})) = \mathrm{Hom}(\mathfrak{sl}_2(\mathbb{C}), \mathbb{C})$$

The k-invariant of the differential crossed module of 2-chord diagrams?

Problem: Describe the kernel M_n of the boundary map $\partial\colon 2\mathrm{ch}_n \to \mathrm{ch}_n^+$ in the differential crossed module $2\mathfrak{ch}_n = (\partial\colon 2\mathrm{ch}_n \to \mathrm{ch}_n^+)$ of totally symmetric horizontal 2-chord diagrams.

By construction the cokernel is the Lie algebra ch_n of horizontal chord diagrams, generated by r_{ab}, where $1 \leq a < b \leq n$, subject to the infinitesimal braid relations

$$r_{ab} = r_{ba}; \quad [r_{ab} + r_{ac}, r_{bc}] = 0; \quad [r_{ab}, r_{a'b'}] = 0 \text{ if } \{a, b\} \cap \{a', b'\} = \emptyset.$$

Address whether the associated cohomology class

$$k(2\mathfrak{ch}_n) \in H^3(\mathrm{ch}_n, M_n)$$

is trivial or not.

Here $2\mathfrak{ch}_n = (\partial\colon 2\mathrm{ch}_n \to \mathrm{ch}_n^+)$ is embedded in the exact sequence:

$$\{0\} \to M_n \xrightarrow{i} 2\mathrm{ch}_n \xrightarrow{\partial} \mathrm{ch}_n^+ \xrightarrow{\mathrm{proj}} \mathrm{ch}_n \to \{0\}.$$

Background
The Knizhnik-Zamolodchikov connection
Crossed modules and differential crossed modules
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Geometric framework for infinitesimal 2-Yang-Baxter operators

Problem: As infinitesimal Yang-Baxter operators in a Lie algebra come naturally from invariant non-degenerate symmetric bilinear forms, it would be important to find a simple geometric way to construct infinitesimal 2R-matrices.

Problem: The most interesting case is when the 1-dimensional holonomy for braids and the 2-dimensional holonomy for braided surfaces derived from the Knizhnik-Zamolodchikov 2-connection are 1- and 2-intertwiners for weak representations differential crossed modules. For this to hold we must impose a refinement of the notion of an infinitesimal 2-Yang-Baxter operator. This categorifies the relation

$$[r, \Delta(a)] = 0, \forall a \in \mathfrak{g}$$

much stronger than the 4-term relation

$$[r_{12} + r_{13}, r_{23}] = 0.$$

Braided monoidal 2-categories

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- **Problem:** Can we define a braided monoidal 2-category from the holonomy of the 2- Knizhnik-Zamolodchikov connection, considering the differential crossed module of horizontal 2-chord diagrams.
- **Problem:** Drinfeld 2-Associators? (Florian Schätz)