Multiplicity bounds for Schrodinger eigenvalues on Riemannian surfaces

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Introduction: eigenvalue problems

Let (M, g) be a compact Riemannian surface, possibly with boundary, and Δ_g be the Laplace-Beltrami operator:

$$\Delta_g = -\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}g^{ij}\frac{\partial}{\partial x^j}\right),$$

where (g_{ij}) are the components of the metric g, (g^{ij}) is the inverse tensor, $|g|$ stands for $\det(g_{ij})$, and the summation convention used for repeated indices.

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We consider the eigenvalue problem

$$(\Delta_g + V)u = \lambda u, \qquad Bu = 0, \tag{$*$}$$

where B is a boundary operator ($Bu = u|_{\partial\Omega}$, $Bu = (\partial u/\partial\nu)|_{\partial\Omega}$).

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$$\lambda_0(g,V) < \lambda_1(g,V) \leqslant \ldots \leqslant \lambda_k(g,V) \leqslant \ldots$$

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Cheng'76, Besson'80, Nadirashvili'87

Let (M, g) be a closed compact surface. Then for any C^∞-smooth function V on M the multiplicity $m_k(g, V)$ satisfies the inequality

$$m_k(g, V) \leqslant 2(2 - \chi) + 2k + 1, \qquad k = 1, 2, \dots,$$

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• No such bounds hold in higher dimensions (Colin de Verdiere, Lohkamp).

Proof of Cheng's multiplicity bounds: main ingredients

1. Local properties of eigenfunctions

Bers' 55

Let u be a solution to the equation $(\Delta_g + V)u = 0$ in a neighbourhood of the origin in $\mathbb{R}^2$, where g and V are C^∞-smooth. Then there exists a homogeneous harmonic polynomial $P_N(x)$ of degree $N \geqslant 0$ such that

$$u(x) = P_N(x) + O(|x|^{N+\varepsilon})$$

for some $\varepsilon > 0$.

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• the above statement holds in any dimension for solutions to rather arbitrary elliptic differential equations;

• due to the results by M. and T. Hoffman-Ostenhof (Comm PDE, 1992) together with the unique continuation results by E. Sawyer (1984), S. Canillo and E. Saweyr (1990) a similar statement holds when the potential is L^p, $p > 1$ (or belongs to a Kato class $K^{2,\delta}$).

Proof of Cheng's multiplicity bounds: main ingredients

2. Cheng's structure theorem: "the nodal set $u^{-1}(0)$ is locally diffeomorphic to its tangent cone".

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Let u be a solution to the equation $(\Delta_g + V)u = 0$ on $\Omega \subset \mathbb{R}^2$, where g and V are C^∞-smooth. Then for any $p \in u^{-1}(0)$ there exists its neighbourhood U and a diffeomorphism $\Phi : U \to B(0) \subset \mathbb{R}^2$ such that $u(x) = P_N(\Phi(x))$, where P_N is an approximating homogeneous harmonic polynomial at p.

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• the proof heavily relies on the smoothness of u (can be relaxed to $u \in C^{1,1}$ at the price that Φ becomes a Lipschitz homeomorphism);
• the statement of the theorem fails in higher dimensions;

Proof of Cheng's multiplicity bounds: main ingredients

$\operatorname{ord}_x(u)$ = the degree of an approximating homogeneous harmonic polynomial $P_N(x)$; $\mathcal{N}^2(u) = \{x \in M : \operatorname{ord}_x(u) \geqslant 2\}$.

Corollary of Cheng's structure theorem

Under the hypothesis of the theorem, the nodal set $u^{-1}(0)$ is stratified into a finite collection of smooth arcs and a finite set $\mathcal{N}^2(u)$. In particular, the nodal set is a graph such that the degree of each vertex $x \in \mathcal{N}^2(u)$ equals $2\operatorname{ord}_x(u)$.

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- In higher dimensions one can ask whether $\dim \mathcal{N}^2(u) \leqslant n-2$.
- This was affirmatively answered by R. Hardt and L. Simon (JDG, 1989), with subsequent works by F.-H. Lin (CPAM, 1991), Q. Han (Indiana Univ. Math. J., 1994), Q. Han, R. Hardt, and F.-H. Lin (CPAM, 1998) R. Hardt, M. and T. Hoffman-Ostenhof, N. Nadirashvili (JDG, 1999), C. Bär (Invent. Math. 1999), J. Cheeger, A. Naber, D. Valtorta (arXiv: 1207.4236)

Proof of Cheng's multiplicity bounds: main ingredients

3. Topological argument: global properties of nodal sets.
The number of nodal domains of any solution u is at least

$$\sum_{x \in \mathcal{N}^2(u)} (\mathrm{ord}_x(u) - 1) + \chi + \ell.$$

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Lemma (consequence of Bers' theorem)

Let $u_1, \ldots, u_{2n}$ be a collection of linearly independent solutions to the Schrodiner equation. Then for a given interior point $x \in M$ there exists a non-trivial linear combination $u = \sum \alpha_i u_i$ such that $\mathrm{ord}_x(u) \geqslant n$.

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Proof of the multiplicity bounds: Suppose the contrary. Then for some $x \in M$ we have $\mathrm{ord}_x(u) \geqslant 2 - \chi - \ell + k + 1$. By the estimate above the number of nodal domains is at least $k + 2$. Contradiction. □

Main results

Open question

Do similar multiplicity bounds hold for potentials in L^p, where $p > 1$, or in $K^{2,\delta}$?

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Theorem 1 (GK, arXiv: 1310.2207)

The multiplicity bounds

$$m_k(g, V) \leqslant 2(2 - \chi - \ell) + 2k + 1, \qquad k = 1, 2, \ldots,$$

hold for the Schrodinger eigenvalue problem when $V \in K^{2,\delta}$, and in particular when $V \in L^p$, $p > 1$, where χ is the Euler-Poincare number and ℓ is the number of boundary components of M.

Main results

Theorem 2 (GK, arXiv: 1310.2207)

Let u be a non-trivial solution to the Schrodinger equation

$$(\Delta_g + V)u = 0 \qquad \text{on } M,$$

where $V \in K^{2,\delta}$, that has a finite number of nodal domains. Then the nodal set consists of a finite number of C^1-smooth arcs (the components of $\mathcal{N}(u) \backslash \mathcal{N}^2(u)$) and a finite number of critical points (the set $\mathcal{N}^2(u)$). Besides, for any $x \in \mathcal{N}^2(u)$ the degree $\deg(x)$ is at least $2\,\mathrm{ord}_x(u)$.

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• Without the finiteness of nodal domains hypothesis it is unknown whether the Hausdorff dimension of $\mathcal{N}^2(u)$ equals zero or not.
• C^1-smoothness of "nodal arcs" follows from the regularity theory of nodal sets due to M. and T. Hoffman-Ostenhof, N. Nadirashvili (Comm. PDE, 1995).

New ingredients

Lemma 1

Let $x \in \mathcal{N}^2(u)$ be an isolated point in $\mathcal{N}^2(u)$ such that the degree $\deg(x)$ (= the number of nodal edges incident to x) is finite. Then $\deg(x)$ is an even integer and $\deg(x) \geqslant 2\,\mathrm{ord}_x(u)$.

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The proof is based on the qualitative behaviour of nodal lines around an isolated point $x \in \mathcal{N}^2(u)$, derived from the results by M. and T. Hoffman-Ostenhof, N. Nadirashvili (Comm. PDE, 1995) together with a rescaling argument.

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Lemma2

Suppose that $\mathcal{N}^2(u)$ consists of isolated points. Then $\mathcal{N}^2(u)$ is finite and the complement $\mathcal{N}^1 \backslash \mathcal{N}^2(u)$ has finitely many connected components.

Based on Lemma 1 and a topological argument similar to the one by M. Karpukhin, GK, I. Polterovich (AIF, to appear).

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Lemma 3

Let $x_i \in \mathcal{N}^2(u)$ be a sequence converging to an interior point $x \in \mathcal{N}(u)$. Then $\limsup \operatorname{ord}_{x_i}(u) < \operatorname{ord}_x(u)$.

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Lemma 4

For any point $x \in \mathcal{N}(u)$ and any sufficiently small ball $B_\varepsilon(x)$ centred at x there exists a subdomain $U_\varepsilon(x)$ such that $x \in U_\varepsilon(x) \subset B_\varepsilon(x)$, and the sets $\partial U_\varepsilon(x)$ and $U_\varepsilon(x) \setminus \mathcal{N}(u)$ have finitely many connected components.

The proof is based on the study of Caratheodory prime ends of nodal domains.

Caratheodory prime ends of nodal domains

Let $\Omega \subset M$ be a an open subset. A *prime end* in Ω is an equivalence class of nested sequences of subdomains $D_i \subset \Omega$ s. t.

$$\bar{D}_{i+1} \cap \Omega \subset D_i, \quad \partial D_i \text{ is connected and non-empty,}$$

$$\mathrm{diam}(p \cup \partial D_i) \to 0, \quad \mathrm{dist}(p, \partial D_i) > 0 \text{ for some } p \in \partial\Omega,$$

where by ∂D_i we mean the boundary of D_i in Ω.

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Each prime end (equivalence class) $[D_i]$ has two characteristics:

- the set $P[D_i]$ of *principal points* ($p \in \partial\Omega$ above), and
- the *impression* $I[D_i] = \cap \bar{D}_i$; $P[D_i] \subset I[D_i]$.

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Topological non-regularity of the boundary of a domain Ω can be characterised by the size and relationships between $P[D_i]$ and $I[D_i]$.

Lemma 5 (Topological regularity of nodal domains)

Under the hypotheses of Theorem 2, every prime end of the nodal domain of u is of the *first kind*, that is its impression $I[D_i]$ consists of a single principal point p.

THANK YOU