Variational aspects of Laplace eigenvalues on Riemannian surfaces

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Introduction: Laplace eigenvalues

- M is a compact Riemannian surface, possibly with boundary.
- For a Riemannian metric g on M we denote by

$$0 = \lambda_0(g) < \lambda_1(g) \leqslant \lambda_2(g) \leqslant \ldots \leqslant \lambda_k(g) \leqslant \ldots$$

the eigenvalues of the Laplace-Beltrami operator

$$-\Delta_g = -\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}g^{ij}\frac{\partial}{\partial x^j}\right),$$

where (g_{ij}) are the components of the metric g, (g^{ij}) is the inverse tensor, $|g|$ stands for $\det(g_{ij})$, and the summation convention used for repeated indices.

If the boundary ∂M is not empty, then we assume the Neumann conditions on it.

(Hersch, 1970) Any metric on a 2-dimensional sphere S^2 satisfies the inequality

$$\lambda_1(g) Vol_g(S^2) \leqslant 8\pi.$$

The equality is achieved on the round sphere.

Introduction: classical inequalities and bounds

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(Li and Yau, 1983) Any metric on a projective plane $\mathbf{R}P^2$ satisfies the inequality

$$\lambda_1(g) Vol_g(\mathbf{R}P^2) \leqslant 12\pi.$$

There are also bounds for $\lambda_1(g) Vol_g(M)$ in terms of genus on non-orientable surfaces.

Introduction: classical inequalities and bounds

Higher eigenvalues:

Korevaar, 1993

Let M be an orientable surface of genus γ. Then there exists a universal constant C_* such that for any metric g on M the kth Laplace eigenvalue satisfies the inequality

$$\lambda_k(g) Vol_g(M) \leqslant C_* k(\gamma + 1)$$

for any $k \geqslant 0$.

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$$\lambda_k(g) Vol_g(M) \leqslant C_* k(\gamma + 1)$$

for any $k \geqslant 0$.

There is also a similar uniform bound for metrics on non-orientable surfaces with a different, possibly, greater constant.

In sequel, the constant C_* is referred to as the *Korevaar constant*.

Looking for the metrics that maximise the first Laplace eigenvalue (among metrics of unit volume) on a given surface M.

Extremal problems for Laplace eigenvalues

Looking for the metrics that maximise the first Laplace eigenvalue (among metrics of unit volume) on a given surface M.

Hersch, 1970; Li and Yau, 1983

Complete answer for the zero genus case:

- S^2: the round metric (and its pull-backs under conformal diffeomorphisms);
- $\mathbf{R}P^2$: the metrics that admit a (minimal) Veronese embedding into S^4 by first eigenfunctions.

The maximal metrics on surfaces with boundary are limits of the metrics obtained by excising small disks from the maximal surfaces above and tending their radii to zero.

The work on the higher genus cases:

Berger's question, 1973

Does the equilateral flat metric on the torus $\mathbf{T}^2$ maximises $\lambda_1(g)Vol_g(\mathbf{T}^2)$ among all metrics?

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Jakobson, Levitin, Nadirasvili, Nigam, Polterovich, 2005

Let M be a Bolza surface, that is a genus two Riemann surface with the highest order (=48) of the automorphism group. Does the metric on M obtained by pulling back the round metric from the sphere maximises $\lambda_1(g)Vol_g(M)$ among all metrics?

Confirmed by numerical computations only...

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Extremal problems for Laplace eigenvalues

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2. Some of the constructions, as well as numerical evidence, indicate that the maximal metrics are expected to be singular or degenerate. (For example, so is the metric on the Bolza surface.) What sort of singularities can occur, in general? Is it possible to describe or classify them?

3. The above discussion suggests that the class of smooth Riemannian metrics is not natural for such extremal problems. What is the right setup for the extremal problems on degenerate/singular metrics? What is the right notion of extremality for such metrics?

Talk purposes and plan:

We consider the above issues for the extremal problems in *conformal classes* on Riemannian surfaces, that is we look for metrics maximising (and more generally extremal for) $\lambda_1(g)Vol_g(M)$ among metrics in a given conformal class.

Talk purposes and plan:

We consider the above issues for the extremal problems in *conformal classes* on Riemannian surfaces, that is we look for metrics maximising (and more generally extremal for) $\lambda_1(g)Vol_g(M)$ among metrics in a given conformal class.

We use the *direct method of calculus of variations* which allows to answer almost all questions posed above in a uniform fashion. In particular, this includes:

- extending the Laplace eigenvalues to Radon measures, which (together with other data) can be naturally identified with singular metrics;
- extending the notion of λ_k-*extremality* (under conformal variations) to such measures;
- studying general regularity properties of such λ_k-extremal metrics (or measures).

Generalised conformal metrics as Radon measures

Let (M, c) be a compact surface equipped with a conformal class of Riemannian metrics. For a finite Radon measure μ we define its kth eigenvalue by the relation

$$\lambda_k(\mu, c) = \inf_{\Lambda^{k+1}} \sup_{u \in \Lambda^{k+1}} \mathrm{R}_c(u, \mu),$$

where $\Lambda^{k+1} \subset L_2(M, \mu)$ is a $(k+1)$-dimensional subspace formed by C^∞-smooth functions, and

$$\mathrm{R}_c(u, \mu) = \left(\int_M |\nabla u|^2 \, dVol_g \right) / \left(\int_M u^2 d\mu \right),$$

where $g \in c$ is a reference metric. By the conformal invariance of the Dirichlet integral the choice of $g \in c$ does not affect the definition above.

Generalised conformal metrics as Radon measures

Singular metrics. Let h be a metric with conical singularities conformal to a smooth metric $g \in c$ away from the singular points. Then the value $\lambda_k(Vol_h, c)$ coincides with other definitions of $\lambda_k(h)$ used in the literature.

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Alexandrov surfaces of bounded integral curvature. These are metric spaces (M, d), where M is a smooth compact surface, and d is a distance function that is a C^0-limit of distances d_n of smooth Riemannian metrics on M whose total integral curvatures are bounded,

$$d_n = \mathrm{dist}_{g_n}, \quad \int_M |K_{g_n}|\, dVol_{g_n} \leqslant C < \infty.$$

This is a large class of singular surfaces that contains, for example, all polyhedral surfaces as well as surfaces with conical singularities and their limits under the integral curvature bound.

Generalised conformal metrics as Radon measures

The signed curvature measures $d\omega_n = K_{g_n} dVol_{g_n}$ converge weakly to a signed measure $d\omega$ on M, called the *curvature measure* of an Alexandrov surface. By Alexandrov's theorem it is an intrinsic characteristic, that is, it does not depend on a sequence g_n.

Cusp is a point $x \in M$ where $\omega(x) = 2\pi$.

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Reshetnyak, Huber, 1960

Any Alexandrov surface (M, d) of bounded integral curvature and without cusps is "conformally equivalent" to a smooth Riemannian metric on a background compact surface. More precisely, its distance function d has the form

$$d(x, y) = \inf_{\gamma} \left\{ \int_0^1 e^{u(\gamma(t))} \left|\dot{\gamma}(t)\right|_g dt \right\},$$

where $\gamma \in \mathcal{C}^1(x, y)$, and $e^u \in L^p(M, Vol_g)$, $p > 1$.

Generalised conformal metrics as Radon measures

Steklov eigenvalues. Our definition contains as a partial case other popular eigenvalue problems. For example, suppose that M has a non-empty boundary and for $g \in c$ let μ_g be its boundary length measure. Then the eigenvalues $\lambda_k(\mu_g, c)$ coincide with the so-called Steklov eigenvalues of a metric g, representing the spectrum of the Dirichlet-to-Neumann map.

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Possible pathologies. Let μ be a discrete measure supported at ℓ distinct points. Using the fact that the capacity of a point is equal to zero, it is straightforward to show that

$$\lambda_k(\mu, c) = \begin{cases} 0, & \text{if } \ell > k, \\ +\infty, & \text{if } \ell \leqslant k, \end{cases}$$

for an arbitrary conformal class c on M.

Generalised conformal metrics as Radon measures

Grigor'yan and Yau, 1999

Let M be an orientable surface of genus γ, possibly with boundary. Then there exists a universal constant C_* such that for any finite continuous Radon measure μ and any conformal class c on M the kth Laplace eigenvalue satisfies the inequality

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As an application, we obtain bounds (isoperimetric inequalities) for Steklov eigenvalues as well as eigenvalues on Alexandrov surfaces.

Existence of a weak maximiser

Let M be a compact surface, possibly with boundary, equipped with a conformal class c such that

$$\sup\{\lambda_1(\mu, c)\mu(M) : \mu \text{ is a continuous Radon measure on} M\} > 8\pi.$$

Then any λ_1-maximising sequence of Radon probability measures contains a subsequence that converges weakly to a continuous Radon measure μ at which the supremum above is achieved.

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Ingredients:

- upper semi-continuity property: for $\mu_n \rightharpoonup \mu$

$$\limsup \lambda_k(\mu_n, c) \leqslant \lambda_k(\mu, c);$$

- "no atoms" lemma: if $\mu \neq \delta_x$ and has a non-trivial discrete part, then $\lambda_1(\mu, c)$ vanishes.
- if $\mu_n \rightharpoonup \delta_x$, then $\limsup \lambda_1(\mu_n, c) \leqslant 8\pi$.

Extremal measures

For a given Radon probability measure μ on M by its conformal deformation we mean the family of probability measures

$$\mu_t(X) = \left(\int_X e^{\phi t} d\mu\right) / \left(\int_M e^{\phi t} d\mu\right),$$

where $X \subset M$ is Borel set, and $\phi \in L^\infty(M)$.

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Definition

A Radon probability measure μ on (M, c) is called λ_k-*extremal*, if for any $\phi \in L^\infty(M)$ the function $t \mapsto \lambda_k(\mu_t, c)$ satisfies either the inequality

$$\lambda_k(\mu_t, c) \leqslant \lambda_k(\mu, c) + o(t) \quad \text{as } t \to 0,$$

or the inequality

$$\lambda_k(\mu_t, c) \geqslant \lambda_k(\mu, c) + o(t) \quad \text{as } t \to 0.$$

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- when the right and left derivatives at $t = 0$ of $\lambda_k(\mu_t, c)$ exist, the relations above can be re-phrased in the following form

$$\frac{d}{dt}\bigg|_{t=0+} \lambda_k(\mu_t, c) \cdot \frac{d}{dt}\bigg|_{t=0-} \lambda_k(\mu_t, c) \leqslant 0,$$

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  which is a natural notion of extremality for "non-smooth" functionals.
- when μ is a genuine Riemannian metric this definition coincides with the one introduced by Nadirashvili and also studied by Colbois, El Soufi, Ilias, and others;

Regularity issues

Any Radon measure decomposes into the sum of its absolutely continuous and singular parts:

$$\mu = \int \phi dVol_g + \mu \lfloor \Sigma,$$

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2. What are the properties of the singular set Σ and, in particular, when is it empty? (If $\lambda_k(\mu, c) > 0$ and μ is continuous, then Σ is either empty or has positive capacity.)

3. What are the properties of the "collapsing set" $M \setminus \operatorname{supp} \mu$?

Singularities are unavoidable

There are extremal measures with non-trivial singular set. Let M be a unit disk

$$D = \{z \in \mathbb{C} : |z| \leqslant 1\}$$

equipped with a Euclidean metric g normalized so that the boundary circle ∂D has length 1; denote by ℓ the boundary length measure. The "generalized Weinstock inequality" says that ℓ maximizes $\lambda_1(\mu, [g])$ among all probability measures μ supported in ∂D,

$$\lambda_1(\mu, [g]) \leqslant \lambda_1(\ell, [g]).$$

Since the conformal deformations do not change the support of the measure, we conclude that ℓ is λ_1-extremal.

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• It is unknown whether there are λ_k-maximal measures with a non-trivial singular set.

A general regularity result for λ_k-extremal measures

GK, arXiv: 1103.2448

Let (M, c) be a compact surface, possibly with boundary, and μ be a λ_k-extremal measure such that the embedding

$$L_2(M,\mu) \cap L_2^1(M, Vol_g) \subset L_2(M,\mu) \qquad (*)$$

is compact. Then μ is absolutely continuous in the interior of its support where its density is C^∞-smooth and vanishes at isolated points only.

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Example. Let h be a metric with conical singularities conformal to a genuine metric $g \in c$ away from the singular points. If h is λ_k-extremal under conformal deformations, then it is smooth everywhere and the angles at its singularities are integer multiples of 2π.

A general regularity result for λ_k-extremal measures

Version of Nadirashvili's lemma

Let (M, c) be a compact surface, possibly with boundary, and μ be a Radon probability measure on M such that the embedding $(*)$ is compact. Then the following hypotheses are equivalent:

(i) the measure μ is λ_k-extremal;

(ii) the quadratic form

$$u \longmapsto \int_M u^2 \phi d\mu$$

is indefinite on the eigenspace E_k for any zero mean-value function $\phi \in L^\infty(M)$;

(iii) there exists a finite collection of λ_k-eigenfunctions (u_i) such that $\sum u_i^2 = 1$ on $\operatorname{supp} \mu$.

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- The possible zeroes of the density of μ inside its support in the theorem are always conical singularities whose angles are integer multiples of 2π.

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- If $\operatorname{supp} \mu \neq M$, then the singular set Σ is non-empty.

A general regularity result for λ_k-extremal measures

Comments:

- The possible zeroes of the density of μ inside its support in the theorem are always conical singularities whose angles are integer multiples of 2π.
- If $\operatorname{supp}\mu \neq M$, then the singular set Σ is non-empty.
- The hypothesis that the embedding $(*)$ is compact can be characterised via the behaviour of sharp constants in the so-called *isocapacitory inequalities*. This indicates on the non-trivial relationship between the singular set Σ and the set of points where the decay

$$\mu(B(x,r)) \to 0 \quad \text{as } r \to 0,$$

is slow (in comparison with $\ln(1/r)$ and its powers).

Existence of a partially regular maximiser

GK, arXiv: 1103.2448

Let M be a compact surface, possibly with boundary, equipped with the conformal class c such that

$$\sup\{\lambda_1(\mu, c)\mu(M) : \mu \text{ is a continuous Radon measure on } M\} > 8\pi.$$

Then there exists a maximiser for $\lambda_1(\mu, c)\mu(M)$, understood as a continuous Radon measure μ, such that

it defines a C^∞-smooth Riemannian metric, conformal to the ones in c, in the interior $\operatorname{supp}\mu$ except maybe a nowhere dense singular set Σ of zero Lebesgue measure.

If the embedding $(*)$ is compact, then Σ does not have any points in the interior of $\operatorname{supp}\mu$.

Concentration-compactness of extremal metrics

GK, arXiv: 1103.2448, see also arXiv:0904.3264

Let M be a closed surface endowed with a conformal class c, and (g_n) be a sequence of λ_1-extremal smooth metrics in c (possibly with conical singularities) normalized to have a unit volume. Then there exists a subsequence (g_n) such that one of the following holds:

(i) the volume measures Vol_{g_n} converge weakly to a pure Dirac measure δ_x for some $x \in M$, and $\lambda_1(g_n) \to 8\pi$;

(ii) the subsequence (g_n) converges smoothly to a λ_1-extremal metric $g \in c$, which may have a finite number of conical singularities, and $\lambda_1(g_n) \to \lambda_1(g)$.

Thus, the set of conformal λ_1-extremal metrics whose first eigenvalues are bounded away from 8π is always compact. As an example with S^2 shows, the case (i) occurs.

Comments:

- comparisons with Montel-Ros;
- the statement of the theorem continues to hold for variable conformal classes as long as they stay in a bounded domain in the moduli space;
- there are versions for higher eigenvalues that are much more implicit.

THANK YOU