On the topology of the evaluation map and rational curves

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Based on a number of talks given during 2006-2007

Introduction: a classical result of Gottlieb

Let M be a closed connected manifold and $Diff(M)$ be its group of diffeomorphisms endowed with the compact open topology. For a given base point $u \in M$ we consider the *evaluation map*

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• As an example with the torus shows the result is sharp, in the sense that the statement no longer holds if $\chi(M)=0$.
• The proof uses Nielsen-Wecken fixed point classes or a version of Lefschetz fixed point theorem.

Main result: symplectic version of Gottlieb's theorem

Let M be a closed symplectic manifold and $Symp(M)$ be its group of symplectomorphisms endowed with the compact open topology. We are interested in the topological properties of the evaluation map on the 2-skeleton of $Symp(M)$.

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GK, arXiv:math/0603255

Let (M, ω) be a closed symplectic manifold. Suppose that there exists an almost complex structure J compatible with ω such that there are no non-trivial J-holomorphic spheres in M. Then:

(i) the evaluation map $\mathrm{ev}_u : Symp(M) \to M$ induces the trivial homomorphism on π_2;

(ii) if $\chi(M) \neq 0$, the evaluation map $\mathrm{ev}_u : Symp(M) \to M$ is homotopically trivial on the 2-skeleton of M.

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Originally, the statement (ii) has been proved for toroidal cycles only. As was pointed out to us by D. Kotschick, this is sufficient to conclude that ev_u is homotopically trivial on the whole 2-skeleton.

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- as in Gottlieb's theorem the hypothesis $\chi(M) \neq 0$ is essential;
- more importantly, so is the hypothesis on the absence of J-holomorphic spheres (so called rational curves); they can be viewed as obstructions to the vanishing of ev_u on 2-cycles.

Example. Let S^2 be a unit sphere in $\mathbb{R}^3$ equipped with the standard symplectic structure ω, and $SO(3) \subset Symp(\omega)$ be its group of orientation preserving isometries. It is straightforward to conclude that the homomorphism on the cohomology

$$\mathrm{ev}_u^* : H^2(S^2, \mathbb{Z}_2) \to H^2(SO(3), \mathbb{Z}_2)$$

is non-trivial, and in particular, the evaluation map is not null-homotopic on the 2-skeleton.
In fact, one can explicitly construct 2-tori that evaluate into homotopically non-trivial cycles.

Basic notation

Let (Σ, i_Σ) be a Riemann surface. For a smooth map $u : \Sigma \to M$ its energy is defined as

$$E(u) = \frac{1}{2} \int_\Sigma |du|^2 \, dVol,$$

where the norm and the volume are taken w.r.t. the metric $\omega(\cdot, J\cdot)$ on M and any metric on Σ in the conformal class of i_Σ.

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As is known, the quantity

$$S_\omega(J) = \inf\{E(u) : u \text{ is a non-constant } J\text{-sphere in } M\}$$

is positive. Here the infimum over the empty set is supposed to be equal to infinity.

Main result: examples

Symplectically aspherical manifolds. These are symplectic manifolds such that $\omega|_{\pi_2} = 0$. As follows from the energy identity, they do not have non-trivial J-spheres for *any* compatible almost complex structure J. There is a large number of examples of such manifolds (Kollar, Gompf) with arbitrary large Euler-Poincaré numbers.

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Monotone symplectic manifolds in dimension 4*.* Recall that a manifold (M, ω) is called non-positive monotone, if

$$[c_1]|_{\pi_2} = \kappa \cdot [\omega]|_{\pi_2}$$

where $\kappa \leq 0$ and c_1 stands for the Chern class. In dimension 4 for a generic compatible almost complex structure J such manifolds do not have non-trivial J-spheres. (The proof uses Fredholm theory setup for Cauchy-Riemann equations.) The examples include a well-known K3-surface, $\chi(K3) = 24$, $\pi_2(K3) \neq 0$.

Remarks: other related results of Gottlieb

Sample results of Gottlieb's theory:

Gottlieb, 1978

Let M be a closed oriented manifold and $Diff(M)$ be its diffeomorphism group. Then the homomorphisms $\chi(M)\,\mathrm{ev}_u^*$ and $c \cdot \sigma(M)\,\mathrm{ev}_u^*$ of the cohomology groups

$$H^k(M, R) \longrightarrow H^k(Diff(M), R)$$

vanish for any $k > 0$ and any unitary ring R. Here c is an appropriate non-zero integer that depends on the dimension of M only, and $\chi(M)$ and $\sigma(M)$ stand for the Euler-Poincaré number and signature respectively.

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As above, the homomorphism ev_u^* itself is not trivial, in general.

More precise vanishing results: preliminaries

Given a surface Σ and a continuous map $\phi : \Sigma \to Symp(M)$ we are interested in the cases when the map $\mathrm{ev}_u \circ \phi : \Sigma \to M$ is null-homotopic or null-homologous.

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The following lemma shows that we can always suppose that the orbits of ϕ are smooth.

Lemma

Continuous maps $\phi : \Sigma \to Symp(M)$ such that $\mathrm{ev}_u \circ \phi$ is smooth for any $u \in M$ form a dense subset in the space of all continuous maps with respect to the compact open topology.

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The proof is based on the Moser isotopy argument.

More precise vanishing results: evaluation energy

For a map $\phi : \Sigma \to Symp(M)$ we define the *basic evaluation energy* as

$$E(\phi, \omega, J) = \frac{1}{2} \int_{\Sigma} \max_{u \in M} |d(\mathrm{ev}_u \circ \phi)|^2 \, dVol_{\Sigma};$$

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the *correction term*

$$\Delta(\phi, \omega) = \int_\Sigma (\max_u \Lambda_u - \min_u \Lambda_u) dVol_\Sigma \geq 0,$$

where Λ_u is defined by the identity $(\mathrm{ev}_u \circ \phi)^* \omega = \Lambda_u dVol_\Sigma$;

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the *full evaluation energy*

$$\mathcal{E}(\phi, \omega, J) = E(\phi, \omega, J) + \Delta(\phi, \omega).$$

More precise vanishing results: homotopy version

For a given homotopy class of maps $\phi : \Sigma \to Symp(M)$ we define

$$\mathcal{E}_{\Pi}([\phi], \omega, J)$$

as the infimum of full evaluation energy over all pairs (ϕ, i_{Σ}), where ϕ represents $[\phi]$ and $i_{\Sigma} \in \mathcal{M}_g$, $g = genus(\Sigma)$.

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"Measuring homotopy classes whose evaluation is trivial", GK

Let $[\phi]$ be a homotopy class of maps $\phi : \Sigma \to Symp(M)$ such that

$$\sup_J (S_\omega(J) - \mathcal{E}_\Pi([\phi], \omega, J)) > 0$$

Then:
(i) if Σ is a sphere, the homotopy class $[\mathrm{ev}_u \circ \phi]$ is trivial;
(ii) if Σ is a torus, the homotopy class $[\mathrm{ev}_u \circ \phi]$ is trivial or $\chi(M) = 0$.

More precise vanishing results: discussion

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  - under the absence of non-trivial J-holomorphic curves $S_\omega(J) = +\infty$;
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- this theorem implies the main "vanishing theorem":
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  - the topological vanishing of ev_u on toroidal cycles implies the topological vanishing on the 2-skeleton.
- there is a version of "measuring homotopy classes" theorem also for higher genus surfaces; however, it is not very natural in view of our methods.
- there is also a completely analogous statement concerned with "measuring homology classes whose image under the evaluation map is null-homologous".

Preliminaries on perturbed Cauchy-Riemann equations

For mappings $u : \Sigma \to M$ we consider the non-linear Cauchy-Riemann operator

$$\bar{\partial} u = \frac{1}{2}(du + J \circ du \circ i_\Sigma).$$

Here the r.h.s. is the J-complex anti-linear part of du.

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$$Maps(\Sigma, M) \ni u \mapsto \bar{\partial} u \in Sections(\tilde{u}^* \Omega^{0,1}),$$

where $\tilde{u}$ is the graph of u, and $\Omega^{0,1}$ is a vector bundle over $\Sigma \times M$ whose fibre over (z, u) is the space of J-anti-linear operators $T_z\Sigma \to T_u M$. Given a section f of $\Omega^{0,1}$ we also consider the *perturbed Cauchy-Riemann equations*

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$$\bar{\partial} u(z) = \mathfrak{f}(z, u(z)), \qquad z \in \Sigma. \qquad (*)$$

We assume that u and f are $W^{2,2}$- and $W^{p,\ell+1}$-smooth in the Sobolev sense respectively, where $p > 2(n+1)$, $\ell > 3$, and $2n$ is $\dim M$. For a given homotopy class $[v]$ of maps $\Sigma \to M$ by $\mathfrak{M}([v], J)$ we denote the *universal moduli space* formed by pairs (u, f) that solve the perturbed equation $(*)$.

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By elliptic regularity, we have

$$\mathfrak{M}([v], J) \subset C^{\ell+1}(\Sigma, M) \times \{C^{\ell}\text{-smooth sections f}\}.$$

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By π we denote the natural projection

$$\pi : \mathfrak{M}([v], J) \ni (u, \mathrm{f}) \mapsto f \in \{W^{p,\ell+1}\text{-smooth sections}\}.$$

Compactness theorem

Let (M, ω) be a closed symplectic manifold and J be a compatible almost complex structure. Further, let $[v]$ be a homotopy class of maps $\Sigma \to M$ such that

$$V = S_\omega(J) - \langle v^*[\omega], \Sigma \rangle > 0, \qquad \text{and}$$

$$\mathfrak{U}_\ell = \left\{ W^{p,\ell+1}\text{-smooth } \mathrm{f} : \int_\Sigma \max_u |\mathrm{f}(\cdot, u)|^2 < V \right\}.$$

Then the projection π restricted to the domain $\pi^{-1}(\mathfrak{U}_\ell)$ is proper.

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The proof follows by the standard rescaling technique together with the energy estimate

$$E(u) \leqslant \int_\Sigma \max_u |\mathrm{f}(\cdot, u)|^2 + \langle v^*[\omega], \Sigma \rangle$$

for solutions of the perturbed Cauchy-Riemann equations.

Preliminaries on perturbed Cauchy-Riemann equations

Picking a linear connection ∇^{Ω} on $\Omega^{0,1}$ we define a linearised operator

$$\mathcal{D}(u, \mathrm{f})v = \nabla^{\Omega}_{\partial/\partial t}\Big|_{t=0} \left(\bar{\partial} u_t - f \circ \tilde{u}_t\right),$$

where u_t is a family of maps $\Sigma \to M$ such that

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It is a linear elliptic differential operator of first order; by the Riemann-Roch theorem its index is given by the formula

$$\operatorname{ind}\mathcal{D}(u,\mathrm{f}) = n\chi(\Sigma) + 2\langle u^*[c_1],\Sigma\rangle,$$

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and depends on the homotopy class of u and topological data only. A section f is called *regular*, if for any solution u of the perturbed Cauchy-Riemann equations $\mathcal{D}(u,\mathrm{f})$ has a trivial cokernel. Any section for which there are no solutions is regular.

Preliminaries on perturbed Cauchy-Riemann equations

Theorem (Folklore)

Let (M, ω) be a closed symplectic manifold and J be a compatible almost complex structure. Further, let $[v]$ be a homotopy class of maps $\Sigma \to M$ such that

$$V = S_\omega(J) - \langle v^*[\omega], \Sigma \rangle > 0,$$

and $\ell > n\chi(\Sigma) + 2\langle u^*[c_1], \Sigma \rangle + 3$. Then for any regular $\mathrm{f} \in \mathfrak{U}_\ell$ the space of solutions $\pi^{-1}(\mathrm{f})$ in $[v]$ is either empty or a closed $C^{\ell-2}$-smooth manifold of dimension $\operatorname{ind} \mathcal{D}$; besides, it carries a natural orientation. Further, any two regular sections f_0 and $\mathrm{f}_1 \in \mathfrak{U}_\ell$ can be joined by a path $\mathrm{f}_t \in \mathfrak{U}_\ell$ such that the set $\cup_t \pi^{-1}(\mathrm{f}_t)$ is a $C^{\ell-2}$-smooth oriented manifold with boundary $\pi^{-1}(\mathrm{f}_0) \cup \pi^{-1}(\mathrm{f}_1)$; the boundary orientation agrees with $\pi^{-1}(\mathrm{f}_1)$ and is opposite to $\pi^{-1}(\mathrm{f}_0)$.

Suppose that the genus of Σ is one, and a given homotopy class $[v]$ of maps $\Sigma \to M$ is such that $\langle v^*[c_1], \Sigma \rangle = 0$. Then for a regular $\mathrm{f} \in \mathfrak{U}_\ell$ the space of solutions is finite, and its oriented cobordism class defines an integer $\deg \pi$ – an algebraic number of solutions.

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For a given smooth section g of the bundle $\Omega^{0,1}$ let $\mathfrak{S}$ be a subspace of $W^{2,2}(\Sigma, M)$ formed by solutions of

$$\bar{\partial} u(z) = \mathrm{g}(z, u(z)), \qquad z \in \Sigma.$$

The space $\mathfrak{S}$ is called:

• *non-degenerate in the sense of Morse-Bott*, if its each connected component $\mathfrak{S}^\alpha$ is a smooth submanifold of $W^{2,2}(\Sigma, M)$ whose dimension is equal to $\dim \ker \mathcal{D}(u, \mathrm{g})$, where $u \in \mathfrak{S}^\alpha$.

• *strongly non-degenerate*, if it is non-degenerate and for any $u \in \mathfrak{S}$ the linearised operator $\mathcal{D}(u, g)$ does not have adjoint vectors corresponding to the zero eigenvalue.

Theorem (GK and Kuksin)

Let (M, ω) be a closed symplectic manifold equipped with a compatible almost complex structure J and $[v]$ be a homotopy class of maps $\Sigma = \mathbb{T}^2 \to M$ such that

$$\langle v^*[c_1], \Sigma\rangle = 0 \quad \text{and} \quad \langle v^*[\omega], M\rangle < S_\omega(J).$$

Suppose there exists a smooth section $\mathrm{g} \in \mathfrak{U}_\ell$, $\ell > 3$, such that the space of solutions $\mathfrak{S}$ in $[v]$ is strongly non-degenerate in the sense of Morse-Bott and the map

$$\Sigma \times \mathfrak{S} \ni (z, u) \mapsto (z, u(z)) \in \Sigma \times M$$

is an embedding. Then $\deg \pi = \sum_\alpha \pm\chi(\mathfrak{S}^\alpha)$, where $\mathfrak{S}^\alpha$ is a connected component of $\mathfrak{S}$ and $\chi(\mathfrak{S}^\alpha)$ is its Euler-Poincare number.

Outline of the method: preliminaries

Let $\phi : \Sigma \to \mathit{Diff}(M)$ be a map such that $\phi_z(u) := (\mathrm{ev}_u \circ \phi)(z)$ is smooth in $z \in \Sigma$ for any $u \in M$. Define a section g of the bundle $\Omega^{0,1}$ by the formula:

$$\mathrm{g}(z,u) = \bar{\partial}(\phi_z(\bar{u}))\big|_{\bar{u}=\phi_z^{-1}(u)} \in \Omega^{0,1}_{z,u} \qquad z \in \Sigma, u \in M.$$

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Clearly, for any $u \in M$ the map $\mathrm{ev}_u \circ \phi$ is a solution of the equation

$$\bar{\partial} u(z) = \mathrm{g}(z, u(z)), \qquad z \in \Sigma. \qquad (**)$$

Thus, we have a family $\{\mathrm{ev}_u \circ \phi\}$ of homotopic solutions of the perturbed equation. The main idea is to apply Morse-Bott theory to equation $(**)$.

Outline of the method: preliminaries

Lemma 1

For any map $\phi : \Sigma \to \mathit{Diff}(M)$ such that $\mathrm{ev}_u \circ \phi$ is smooth for any $u \in M$ the following inequality holds:

$$\int_\Sigma \max_u \left|\bar{\partial}(\mathrm{ev}_u \circ \phi)\right|^2 dVol \leqslant \mathcal{E}(\phi, \omega, J) - \langle (\mathrm{ev}_u \circ \phi)^*[\omega], \Sigma \rangle.$$

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Lemma 2

Suppose that ϕ takes values in $Symp(M)$. Then:

- the maps $\mathrm{ev}_u \circ \phi$, $u \in M$, are the only solutions of equation $(**)$ in their homology class; in particular, there are no other homotopic solutions.
- this space of solutions is non-degenerate in the sense of Morse-Bott, and is strongly non-degenerate if the genus of Σ equals one.

Outline of the method: the proof

• Changing $\omega \mapsto -\omega$ and $J \mapsto -J$ simultaneously, if necessary, we may assume

$$\langle (\mathrm{ev}_u \circ \phi)^* \omega, \Sigma \rangle \leqslant 0. \qquad (!)$$

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• Under the conditions of the "measuring homotopy classes" theorem, there exists a complex structure i_Σ on Σ, a compatible almost complex structure J on M, and a map $\phi : \Sigma \to Symp(M)$, representing a given homotopy class, such that

$$\mathcal{E}(\phi, \omega, J) < S_\omega(J).$$

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$$\mathcal{E}(\phi, \omega, J) < S_\omega(J).$$

• By Lemma 1, it is straightforward to conclude that

$$\int_\Sigma \max_u |\mathrm{g}(\cdot, u)|^2 \, dVol < S_\omega(J) - \langle (\mathrm{ev}_u \circ \phi)^* \omega, \Sigma \rangle;$$

thus, $\mathrm{g} \in \mathfrak{U}_\ell$, where $\mathfrak{U}_\ell$ is the domain from the Compactness theorem.

Outline of the method: the proof

- The vector bundle $(\mathrm{ev}_u \circ \phi)^* TM$ is isomorphic to $T_u M \times \Sigma$ via a morphism that equals $d\phi_z^{-1}(u)$ on the fiber over a $z \in \Sigma$.

Thus, $(\mathrm{ev}_u \circ \phi)^*[c_1]$ vanishes, and the index $\mathrm{ind}\, \mathcal{D}(\mathrm{ev}_u \circ \phi, \mathrm{g})$ equals $n\chi(\Sigma)$, where $2n = \dim M$.

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• Case $\Sigma = \mathbb{T}^2$: the index $\mathrm{ind}\, \mathcal{D}$ vanishes, and the degree $\deg \pi$ is well-defined. By Lemma 2 the space $\pi^{-1}(\mathrm{g})$ is strongly non-degenerate and Morse-Bott theory applies; we obtain $\deg \pi = \pm\chi(M)$.

On the other hand, if the homotopy class of $\mathrm{ev}_u \circ \phi$ is non-trivial, then the energy identity together with relation (!) imply that there are no J-tori homotopic to $\mathrm{ev}_u \circ \phi$. Thus, $\pi^{-1}(0)$ is empty and $\deg \pi = 0$; from this we conclude that the Euler-Poincaré number $\chi(M)$ has to vanish.

Outline of the method: the proof

- *Case $\Sigma = \mathbb{S}^2$*: we have $\operatorname{ind} \mathcal{D} = 2n = \dim M$.

On the other hand, $\pi^{-1}(g) \simeq M$ is non-degenerate in the sense of Morse-Bott, and hence, $\dim \ker \mathcal{D}(\mathrm{ev}_u \circ \phi, g) = \dim M = \operatorname{ind} \mathcal{D}$. Thus, the section g is regular.

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Suppose that the homotopy class of $\mathrm{ev}_u \circ \phi$ is non-trivial. Then by relation (!), it does not contain J-spheres, and $\pi^{-1}(0) = \varnothing$. In particular, 0 is a regular section and $\pi^{-1}(\mathrm{g}) \simeq M$ is null-cobordant.

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Let $N \subset \mathfrak{M}([\mathrm{ev}_u \circ \phi], J)$ be a cobordism – an oriented manifold whose boundary is $\pi^{-1}(g)$. Choose a point $z_* \in \Sigma$ and consider the map

$$N \ni (u, \mathrm{f}) \longmapsto u(z_*) \in M.$$

Its restriction to the boundary $\partial N \simeq M$ is a diffeomorphism $u \mapsto \phi_{z_*}(u)$. Thus, we obtain a continuous map $N \to \partial N$ which is a diffeomorphism on the boundary – a contradiction.

THANK YOU