Conformal volume and eigenvalue bounds: the Korevaar method revisited

Gerasim Kokarev
The University of Leeds

Bristol, 31 January, 2019

Let (Σ^n, g) be a closed Riemannian manifold. We denote by

$$0 = \lambda_0(g) < \lambda_1(g) \leqslant \lambda_2(g) \leqslant \ldots \leqslant \lambda_k(g) \leqslant \ldots \to +\infty$$

the eigenvalues of the Laplace-Beltrami operator

$$-\Delta_g = -\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|g|} g^{ij} \frac{\partial}{\partial x^j} \right),$$

where (g_{ij}) are the components of the metric g, (g^{ij}) is the inverse tensor, $|g|$ stands for $\det(g_{ij})$, and the summation convention used for repeated indices.

Notation: Laplace-Beltrami operator and its eigenvalues

Let (Σ^n, g) be a closed Riemannian manifold. We denote by

$$0 = \lambda_0(g) < \lambda_1(g) \leqslant \lambda_2(g) \leqslant \ldots \leqslant \lambda_k(g) \leqslant \ldots \to +\infty$$

the eigenvalues of the Laplace-Beltrami operator

$$-\Delta_g = -\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|g|} g^{ij} \frac{\partial}{\partial x^j} \right),$$

where (g_{ij}) are the components of the metric g, (g^{ij}) is the inverse tensor, $|g|$ stands for $\det(g_{ij})$, and the summation convention used for repeated indices. The eigenvalues satisfy the min-max relations:

$$\lambda_k(g) = \inf_{L_{k+1}} \sup \left\{ \left(\int |\nabla u|^2 \, dV_g \right) / \left(\int u^2 dV_g \right) : 0 \neq u \in L^{k+1} \right\},$$

where L_{k+1} ranges over all $(k+1)$-dimensional subspaces of smooth functions.

Notation: Laplace-Beltrami operator and its eigenvalues

Let (Σ^n, g) be a closed Riemannian manifold. We denote by

$$0 = \lambda_0(g) < \lambda_1(g) \leqslant \lambda_2(g) \leqslant \ldots \leqslant \lambda_k(g) \leqslant \ldots \to +\infty$$

the eigenvalues of the Laplace-Beltrami operator

$$-\Delta_g = -\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|g|} g^{ij} \frac{\partial}{\partial x^j} \right),$$

where (g_{ij}) are the components of the metric g, (g^{ij}) is the inverse tensor, $|g|$ stands for $\det(g_{ij})$, and the summation convention used for repeated indices. The eigenvalues satisfy the min-max relations:

$$\lambda_k(g) = \inf_{L_{k+1}} \sup \left\{ \left(\int |\nabla u|^2 \, dV_g \right) / \left(\int u^2 dV_g \right) : 0 \neq u \in L^{k+1} \right\},$$

where L_{k+1} ranges over all $(k+1)$-dimensional subspaces of smooth functions.

<u>Example (the first Laplace eigenvalue):</u>

$$\lambda_1(g) = \inf \left\{ \left(\int |\nabla u|^2 \, dV_g \right) / \left(\int u^2 dV_g \right) : u \neq 0, \int u dVol_g = 0 \right\}$$

Introduction: Classical eigenvalue inequalities

- (Hersch, 1970) Any metric on a 2-dimensional sphere S^2 satisfies the inequality

$$\lambda_1(g) Vol_g(S^2) \leqslant 8\pi.$$

The equality is achieved on the round sphere.

- (Hersch, 1970) Any metric on a 2-dimensional sphere S^2 satisfies the inequality
$$\lambda_1(g) Vol_g(S^2) \leqslant 8\pi.$$
The equality is achieved on the round sphere.
- (Yang and Yau, 1980) Any metric on an orientable closed surface Σ^2 satisfies the inequality
$$\lambda_1(g) Vol_g(\Sigma^2) \leqslant 8\pi(\gamma+1),$$
where γ is the genus of M.

- (Hersch, 1970) Any metric on a 2-dimensional sphere S^2 satisfies the inequality
$$\lambda_1(g) Vol_g(S^2) \leqslant 8\pi.$$
The equality is achieved on the round sphere.
- (Yang and Yau, 1980) Any metric on an orientable closed surface Σ^2 satisfies the inequality
$$\lambda_1(g) Vol_g(\Sigma^2) \leqslant 8\pi(\gamma + 1),$$
where γ is the genus of M.
- (Yau's conjecture, Problem 71 in the 1982 list) Does there exist a constant $C_* > 0$ such that for any closed surface (Σ^2, g) the Laplace eigenvalues satisfy the inequalities
$$\lambda_k(g) Vol_g(\Sigma^2) \leqslant C_*(\gamma + 1)k$$
for any $k \geqslant 1$, where γ is the genus of Σ^2.

Introduction: bounds for higher eigenvalues

- (Korevaar, 1993) Any metric on an orientable closed surface Σ^2 satisfies the inequalities
$$\lambda_k(g) Vol_g(\Sigma^2) \leqslant C_*(\gamma + 1)k$$
for any $k \geqslant 1$, where γ is the genus of Σ^2.

Introduction: bounds for higher eigenvalues

- (Korevaar, 1993) Any metric on an orientable closed surface Σ^2 satisfies the inequalities
$$\lambda_k(g) Vol_g(\Sigma^2) \leqslant C_*(\gamma + 1)k$$
for any $k \geqslant 1$, where γ is the genus of Σ^2.
- (Hassannezhad, 2011) On any closed surface
$$\lambda_k(g) Vol_g(\Sigma^2) \leqslant Ak + B\gamma$$
for any $k \geqslant 1$, where γ is the genus of Σ^2.

Introduction: bounds for higher eigenvalues

- (Korevaar, 1993) Any metric on an orientable closed surface Σ^2 satisfies the inequalities
$$\lambda_k(g) Vol_g(\Sigma^2) \leqslant C_*(\gamma + 1)k$$
for any $k \geqslant 1$, where γ is the genus of Σ^2.
- (Hassannezhad, 2011) On any closed surface
$$\lambda_k(g) Vol_g(\Sigma^2) \leqslant Ak + B\gamma$$
for any $k \geqslant 1$, where γ is the genus of Σ^2.
- (Colin de Verdière, 1987; Lohkamp, 1996) Let Σ^n be a closed manifold of dimension $n \geqslant 3$. Then for any finite sequence of real numbers
$$0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_m$$
there exists a Riemannian metric g on Σ^n such that $Vol_g(\Sigma^n) = 1$ and $\lambda_k(g) = \lambda_k$ for any $0 \leqslant k \leqslant m$.

Introduction: bounds for higher eigenvalues

- (Korevaar, 1993) Any metric on an orientable closed surface Σ^2 satisfies the inequalities
$$\lambda_k(g) Vol_g(\Sigma^2) \leqslant C_*(\gamma + 1)k$$
for any $k \geqslant 1$, where γ is the genus of Σ^2.
- (Hassannezhad, 2011) On any closed surface
$$\lambda_k(g) Vol_g(\Sigma^2) \leqslant Ak + B\gamma$$
for any $k \geqslant 1$, where γ is the genus of Σ^2.
- (Colin de Verdière, 1987; Lohkamp, 1996) Let Σ^n be a closed manifold of dimension $n \geqslant 3$. Then for any finite sequence of real numbers
$$0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_m$$
there exists a Riemannian metric g on Σ^n such that $Vol_g(\Sigma^n) = 1$ and $\lambda_k(g) = \lambda_k$ for any $0 \leqslant k \leqslant m$.
- (Colbois, Dodziuk, 1994) Let Σ^n be a closed manifold of dimension $n \geqslant 3$. Then there exists a sequence of metrics g_n such that $Vol_{g_n}(\Sigma^n) = 1$ and $\lambda_1(g_n) \to +\infty$.

Introduction: bounds for higher eigenvalues

Conclusion: in dimension $n \geqslant 3$ the topology of Σ^n does not control the scale invariant quantities $\lambda_k(g) Vol_g(\Sigma^n)^{2/n}$.

Conclusion: in dimension $n \geqslant 3$ the topology of Σ^n does not control the scale invariant quantities $\lambda_k(g) Vol_g(\Sigma^n)^{2/n}$.

Theorem ("Conformal eigenvalue bounds", Korevaar, 1993)

Let (Σ^n, g_0) be a closed Riemannian manifold. Then there exists a constant $C = C([g_0])$ such that for any conformal metric $g \sim g_0$ the following inequalities hold

$$\lambda_k(g) Vol_g(\Sigma^n)^{2/n} \leqslant C k^{2/n} \qquad \text{for any} \quad k \geqslant 1.$$

Introduction: bounds for higher eigenvalues

Conclusion: in dimension $n \geqslant 3$ the topology of Σ^n does not control the scale invariant quantities $\lambda_k(g) Vol_g(\Sigma^n)^{2/n}$.

Theorem ("Conformal eigenvalue bounds", Korevaar, 1993)

Let (Σ^n, g_0) be a closed Riemannian manifold. Then there exists a constant $C = C([g_0])$ such that for any conformal metric $g \sim g_0$ the following inequalities hold

$$\lambda_k(g) Vol_g(\Sigma^n)^{2/n} \leqslant C k^{2/n} \qquad \textit{for any} \quad k \geqslant 1.$$

- The right hand-side in the theorem depends on the index k in the way that is compatible with Weyl's asymptotic formula;

$$\lambda_k(g) Vol_g(\Sigma^n)^{2/n} \sim C(n) k^{2/n} \qquad \text{as} \quad k \to +\infty.$$

- One of the most influential results in spectral geometry in a few last decades; "Korevaar method" to be discussed later.
- Geometric drawback: the dependance of the constant C on the conformal class is implicit. (We would like it to depend on geometric quantities, conformal invariants.)

Conformal volume: first definition

Li, Yau. Invent. Math., 1982 (197 citations according to MathSciNet)

- Definition 1: the m-dimensional conformal volume of a Riemannian manifold (Σ^n, g) is defined as $V_c(m, \Sigma^n) = \inf V_c(\Sigma^n, \phi)$, where $\phi : \Sigma^n \to S^m$ is a conformal immersion whose conformal volume is defined by the relation

$$V_c(m, \phi) = \sup\{Vol(\Sigma^n, (s \circ \phi)^* g_{can}) : s \in Conf(S^m, g_{can})\}.$$

Conformal volume: first definition

Li, Yau. Invent. Math., 1982 (197 citations according to MathSciNet)

- Definition 1: the m-dimensional conformal volume of a Riemannian manifold (Σ^n, g) is defined as $V_c(m, \Sigma^n) = \inf V_c(\Sigma^n, \phi)$, where $\phi : \Sigma^n \to S^m$ is a conformal immersion whose conformal volume is defined by the relation

$$V_c(m, \phi) = \sup\{Vol(\Sigma^n, (s \circ \phi)^* g_{can}) : s \in Conf(S^m, g_{can})\}.$$

- Link with the Willmore conjecture: for a surface $\Sigma^2 \subset \mathbb{R}^m$ the following inequality holds

$$V_c(m, \Sigma^2) \leqslant \int_{\Sigma^2} |H_\Sigma|^2 \, dVol_\Sigma;$$

  the equality occurs if and only if Σ^2 is the image of a minimal surface in S^m, obtained from stereographic projection.

Conformal volume: first definition

Li, Yau. Invent. Math., 1982 (197 citations according to MathSciNet)

- Definition 1: the *m-dimensional conformal volume* of a Riemannian manifold (Σ^n, g) is defined as $V_c(m, \Sigma^n) = \inf V_c(\Sigma^n, \phi)$, where $\phi : \Sigma^n \to S^m$ is a conformal immersion whose conformal volume is defined by the relation

$$V_c(m, \phi) = \sup\{Vol(\Sigma^n, (s \circ \phi)^* g_{can}) : s \in Conf(S^m, g_{can})\}.$$

- Link with the Willmore conjecture: for a surface $\Sigma^2 \subset \mathbb{R}^m$ the following inequality holds

$$V_c(m, \Sigma^2) \leqslant \int_{\Sigma^2} |H_\Sigma|^2 \, dVol_\Sigma;$$

  the equality occurs if and only if Σ^2 is the image of a minimal surface in S^m, obtained from stereographic projection.

- First eigenvalue inequality: (Li and Yau, 1982; El Soufi and Ilias, 1986) For any Riemannian manifold (Σ^n, g) the following inequality holds

$$\lambda_1(\Sigma^n, g) Vol_g(\Sigma^n)^{2/n} \leqslant n V_c(m, \Sigma^n)^{2/n};$$

  the equality occurs if and only if after rescaling the metric g the manifold (Σ^n, g) admits an isometric minimal immersion into a unit sphere $S^m \subset \mathbb{R}^{m+1}$ by first eigenfunctions.

Applications of conformal volume: minimal surfaces

A few remarks on the relationships with minimal surfaces:

- Suppose that (Σ^n, g) admits a minimal immersion into a sphere (S^m, g_{can}). Then $V_c(m, \Sigma^n) \leqslant Vol_g(\Sigma^n)$. If (Σ^n, g) admits a minimal immersion by first eigenfunctions, then the equality occurs.

A few remarks on the relationships with minimal surfaces:

- Suppose that (Σ^n, g) admits a minimal immersion into a sphere (S^m, g_{can}). Then $V_c(m, \Sigma^n) \leqslant Vol_g(\Sigma^n)$. If (Σ^n, g) admits a minimal immersion by first eigenfunctions, then the equality occurs.
- Corollary. For any conformal metric $g \sim g_\Sigma$ on a minimal submanifold we have
$$\lambda_1(\Sigma^n, g) Vol_g(\Sigma^n)^{2/n} \leqslant n Vol_{g_\Sigma}(\Sigma^n)^{2/n},$$
and if $\Sigma^n \subset S^m$ is immersed by first eigenfunctions, then g_Σ maximises $\lambda_1(\Sigma^n, g) Vol_g(\Sigma^n)^{2/n}$ in its conformal class.

Applications of conformal volume: minimal surfaces

A few remarks on the relationships with minimal surfaces:

- Suppose that (Σ^n, g) admits a minimal immersion into a sphere (S^m, g_{can}). Then $V_c(m, \Sigma^n) \leqslant Vol_g(\Sigma^n)$. If (Σ^n, g) admits a minimal immersion by first eigenfunctions, then the equality occurs.
- Corollary. For any conformal metric $g \sim g_\Sigma$ on a minimal submanifold we have

$$\lambda_1(\Sigma^n, g) Vol_g(\Sigma^n)^{2/n} \leqslant n Vol_{g_\Sigma}(\Sigma^n)^{2/n},$$

  and if $\Sigma^n \subset S^m$ is immersed by first eigenfunctions, then g_Σ maximises $\lambda_1(\Sigma^n, g) Vol_g(\Sigma^n)^{2/n}$ in its conformal class.
- (Montiel and Ros, 1986; El Soufi and Ilias, 1986) Suppose that (Σ^n, g_0) is not conformally equivalent to (S^n, g_{can}). Then there exists at most one conformal metric $g \sim g_0$ that admits a minimal immersion into S^m by first eigenfunctions.

Applications of conformal volume: minimal surfaces

A few remarks on the relationships with minimal surfaces:

- Suppose that (Σ^n, g) admits a minimal immersion into a sphere (S^m, g_{can}). Then $V_c(m, \Sigma^n) \leqslant Vol_g(\Sigma^n)$. If (Σ^n, g) admits a minimal immersion by first eigenfunctions, then the equality occurs.
- Corollary. For any conformal metric $g \sim g_\Sigma$ on a minimal submanifold we have
$$\lambda_1(\Sigma^n, g) Vol_g(\Sigma^n)^{2/n} \leqslant n Vol_{g_\Sigma}(\Sigma^n)^{2/n},$$
and if $\Sigma^n \subset S^m$ is immersed by first eigenfunctions, then g_Σ maximises $\lambda_1(\Sigma^n, g) Vol_g(\Sigma^n)^{2/n}$ in its conformal class.
- (Montiel and Ros, 1986; El Soufi and Ilias, 1986) Suppose that (Σ^n, g_0) is not conformally equivalent to (S^n, g_{can}). Then there exists at most one conformal metric $g \sim g_0$ that admits a minimal immersion into S^m by first eigenfunctions.
- (Generalised Hersch) For any conformal metric $g \sim g_{can}$ on S^m the first Laplace eigenvalue satisfies the inequality
$$\lambda_1(S^m, g) Vol_g(S^m)^{2/m} \leqslant m Vol_{g_{can}}(S^m)^{2/m},$$
and the equality is achieved only on metrics $g = s^* g_{can}$, where s is a conformal diffeomorphism.

Results for higher Laplace eigenvalues I

Theorem (GK, 2017)

Let (Σ^n, g) be a closed Riemannian manifold of dimension $n \geqslant 2$. Then for any integer $m > 0$ such that the m-dimensional conformal volume of Σ^n is defined the following inequality holds

$$\lambda_k(\Sigma^n, g) Vol_g(\Sigma^n)^{2/n} \leqslant C(n,m) V_c(m, \Sigma^n)^{2/n} k^{2/n} \qquad \text{for any} \quad k \geqslant 1,$$

where $C(n,m)$ is a constant that depends on dimensions n and m only.

Results for higher Laplace eigenvalues I

Theorem (GK, 2017)

Let (Σ^n, g) be a closed Riemannian manifold of dimension $n \geqslant 2$. Then for any integer $m > 0$ such that the m-dimensional conformal volume of Σ^n is defined the following inequality holds

$$\lambda_k(\Sigma^n, g) Vol_g(\Sigma^n)^{2/n} \leqslant C(n,m) V_c(m, \Sigma^n)^{2/n} k^{2/n} \qquad \text{for any} \quad k \geqslant 1,$$

where $C(n,m)$ is a constant that depends on dimensions n and m only.

• Clarifies the meaning of the constant in the Korevaar theorem, and relates it to a well-known conformal invariant.

Theorem (GK, 2017)

Let (Σ^n, g) be a closed Riemannian manifold of dimension $n \geqslant 2$. Then for any integer $m > 0$ such that the m-dimensional conformal volume of Σ^n is defined the following inequality holds

$$\lambda_k(\Sigma^n, g) Vol_g(\Sigma^n)^{2/n} \leqslant C(n,m) V_c(m, \Sigma^n)^{2/n} k^{2/n} \qquad \text{for any} \quad k \geqslant 1,$$

where $C(n,m)$ is a constant that depends on dimensions n and m only.

• Clarifies the meaning of the constant in the Korevaar theorem, and relates it to a well-known conformal invariant.

• Gives conformal eigenvalue bounds on minimal surfaces $\Sigma^n \subset S^m$:

$$\lambda_k(\Sigma^n, g) Vol_g(\Sigma^n)^{2/n} \leqslant C(n,m) Vol_{g_\Sigma}(\Sigma^n)^{2/n} k^{2/n} \qquad \text{for any} \quad k \geqslant 1, \quad (*)$$

where $g \sim g_\Sigma$ is a conformal metric. In particular, setting $g = g_\Sigma$, we obtain $\lambda_k(g_\Sigma) \leqslant C(n,m) k^{2/n}$.

Results for higher Laplace eigenvalues I

Theorem (GK, 2017)

Let (Σ^n, g) be a closed Riemannian manifold of dimension $n \geqslant 2$. Then for any integer $m > 0$ such that the m-dimensional conformal volume of Σ^n is defined the following inequality holds

$$\lambda_k(\Sigma^n, g) Vol_g(\Sigma^n)^{2/n} \leqslant C(n,m) V_c(m, \Sigma^n)^{2/n} k^{2/n} \qquad \textit{for any} \quad k \geqslant 1,$$

where $C(n,m)$ is a constant that depends on dimensions n and m only.

• Clarifies the meaning of the constant in the Korevaar theorem, and relates it to a well-known conformal invariant.

• Gives conformal eigenvalue bounds on minimal surfaces $\Sigma^n \subset S^m$:

$$\lambda_k(\Sigma^n, g) Vol_g(\Sigma^n)^{2/n} \leqslant C(n,m) Vol_{g_\Sigma}(\Sigma^n)^{2/n} k^{2/n} \qquad \text{for any} \quad k \geqslant 1, \quad (*)$$

where $g \sim g_\Sigma$ is a conformal metric. In particular, setting $g = g_\Sigma$, we obtain $\lambda_k(g_\Sigma) \leqslant C(n,m) k^{2/n}$.

• There is a generalisation of inequality $(*)$ for minimal submanifolds in rather arbitrary Riemannian manifolds, where the constant on the right hand-side now depends on the ambient geometry only (preprint under revision).

Another application to minimal submanifolds

Recall that the minimal surface equations for a submanifold $\Sigma^n \subset S^m$, where $S^m \subset \mathbb{R}^{m+1}$ is a unit sphere, we have

$$\Delta_\Sigma \, x^i\Big|_\Sigma + n \, x^i\Big|_\Sigma = 0,$$

where $x^1, \ldots, x^{m+1}$ are coordinate functions on $\mathbb{R}^{m+1}$; that is, $n = \lambda_k$ is an eigenvalue of Σ^n. Then, by the theorem above we obtain

$$N_\Sigma(n) = k \geqslant \bar{C}(n,m)\left(Vol(\Sigma^n)/V_c(m,\Sigma^n)\right).$$

Another application to minimal submanifolds

Recall that the minimal surface equations for a submanifold $\Sigma^n \subset S^m$, where $S^m \subset \mathbb{R}^{m+1}$ is a unit sphere, we have

$$\Delta_\Sigma\, x^i\Big|_\Sigma + n\, x^i\Big|_\Sigma = 0,$$

where $x^1, \ldots, x^{m+1}$ are coordinate functions on $\mathbb{R}^{m+1}$; that is, $n = \lambda_k$ is an eigenvalue of Σ^n. Then, by the theorem above we obtain

$$N_\Sigma(n) = k \geqslant \bar{C}(n, m)\left(Vol(\Sigma^n)/V_c(m, \Sigma^n)\right).$$

Worked example. Let $\phi : S^2 \to S^m$ be a branched minimal immersion. Then, we have $Vol(S^2, \phi^* g_{can}) = 4\pi d$, where $d > 0$ is an integer, called the *harmonic degree* of ϕ (Calabi, Barbosa, 197?), and $V_c(m, S^2) = 4\pi$ for any $m \geqslant 2$ (Li and Yau, 1982). Thus, we obtain

$$N_\Sigma(n) \geqslant \bar{C}(2, m) d$$

Another application to minimal submanifolds

Recall that the minimal surface equations for a submanifold $\Sigma^n \subset S^m$, where $S^m \subset \mathbb{R}^{m+1}$ is a unit sphere, we have

$$\Delta_\Sigma \, x^i\Big|_\Sigma + n\, x^i\Big|_\Sigma = 0,$$

where $x^1, \ldots, x^{m+1}$ are coordinate functions on $\mathbb{R}^{m+1}$; that is, $n = \lambda_k$ is an eigenvalue of Σ^n. Then, by the theorem above we obtain

$$N_\Sigma(n) = k \geqslant \bar{C}(n,m)\left(Vol(\Sigma^n)/V_c(m,\Sigma^n)\right).$$

Worked example. Let $\phi : S^2 \to S^m$ be a branched minimal immersion. Then, we have $Vol(S^2, \phi^* g_{can}) = 4\pi d$, where $d > 0$ is an integer, called the *harmonic degree* of ϕ (Calabi, Barbosa, 197?), and $V_c(m, S^2) = 4\pi$ for any $m \geqslant 2$ (Li and Yau, 1982). Thus, we obtain

$$N_\Sigma(n) \geqslant \bar{C}(2,m)d$$

(Ejiri, 1994) Under the hypotheses of the worked example, $N_\Sigma(n) \geqslant d + 1$. (The proof uses very specific properties of harmonic maps of S^2 to S^m.)

About the proofs: Korevaar method "original"

- "Decomposion theorems": constructions that yield collections of disjoint sets with sufficient amount of mass: Korevaar (1993), Grigoryan, Yau (1999), Grigoryan, Netrusov, Yau (2004), etc.

About the proofs: Korevaar method "original"

- "Decomposion theorems": constructions that yield collections of disjoint sets with sufficient amount of mass: Korevaar (1993), Grigoryan, Yau (1999), Grigoryan, Netrusov, Yau (2004), etc.
- Model statement. Let (X, d) be a metric space satisfying certain "metric covering properties". Then for any finite non-atomic measure μ on X and any integer $k > 0$ there exists a collection of pairs (F_i, G_i), where $i = 1, \ldots, k$, such that:

About the proofs: Korevaar method "original"

- "Decomposion theorems": constructions that yield collections of disjoint sets with sufficient amount of mass: Korevaar (1993), Grigoryan, Yau (1999), Grigoryan, Netrusov, Yau (2004), etc.
- Model statement. Let (X, d) be a metric space satisfying certain "metric covering properties". Then for any finite non-atomic measure μ on X and any integer $k > 0$ there exists a collection of pairs (F_i, G_i), where $i = 1, \ldots, k$, such that:
  - $\bar{F}_i \subset G_i$ for any i;

About the proofs: Korevaar method "original"

- "Decomposion theorems": constructions that yield collections of disjoint sets with sufficient amount of mass: Korevaar (1993), Grigoryan, Yau (1999), Grigoryan, Netrusov, Yau (2004), etc.
- Model statement. Let (X, d) be a metric space satisfying certain "metric covering properties". Then for any finite non-atomic measure μ on X and any integer $k > 0$ there exists a collection of pairs (F_i, G_i), where $i = 1, \ldots, k$, such that:
  - $\bar{F}_i \subset G_i$ for any i;
  - all the sets G_i are disjoint;

About the proofs: Korevaar method "original"

- "Decomposion theorems": constructions that yield collections of disjoint sets with sufficient amount of mass: Korevaar (1993), Grigoryan, Yau (1999), Grigoryan, Netrusov, Yau (2004), etc.
- Model statement. Let (X, d) be a metric space satisfying certain "metric covering properties". Then for any finite non-atomic measure μ on X and any integer $k > 0$ there exists a collection of pairs (F_i, G_i), where $i = 1, \ldots, k$, such that:
  - $\bar{F}_i \subset G_i$ for any i;
  - all the sets G_i are disjoint;
  - $\mu(F_i) \geqslant c\mu(X)/k$ for any i, where the constant c depends on the geometry of (X, d) only.

About the proofs: Korevaar method "original"

- "Decomposion theorems": constructions that yield collections of disjoint sets with sufficient amount of mass: Korevaar (1993), Grigoryan, Yau (1999), Grigoryan, Netrusov, Yau (2004), etc.
- Model statement. Let (X, d) be a metric space satisfying certain "metric covering properties". Then for any finite non-atomic measure μ on X and any integer $k > 0$ there exists a collection of pairs (F_i, G_i), where $i = 1, \ldots, k$, such that:
  - $\bar{F}_i \subset G_i$ for any i;
  - all the sets G_i are disjoint;
  - $\mu(F_i) \geqslant c\mu(X)/k$ for any i, where the constant c depends on the geometry of (X, d) only.
- Geometry of the sets F_i and G_i is important for applications (F_i can not be chosen as metric balls, the margin sets $G_i \setminus F_i$ are important for the Dirichlet integral estimates, etc); the power of k in the estimate can not be improved.

Korevaar method: test-functions

- Test-functions are always (to my knowledge) chosen as Federer-Fleming functions (or Plateau functions)!

Korevaar method: test-functions

- Test-functions are always (to my knowledge) chosen as Federer-Fleming functions (or Plateau functions)!
- "Decomposition theorems" above allow to bound the numerator of the Rayleigh quotient from below.

Korevaar method: test-functions

- Test-functions are always (to my knowledge) chosen as Federer-Fleming functions (or Plateau functions)!
- "Decomposition theorems" above allow to bound the numerator of the Rayleigh quotient from below.
- The estimate for the numerator (that is $\int |\nabla u|^2 \, dV_g$) normally uses two main ingredients:

Korevaar method: test-functions

- Test-functions are always (to my knowledge) chosen as Federer-Fleming functions (or Plateau functions)!
- "Decomposition theorems" above allow to bound the numerator of the Rayleigh quotient from below.
- The estimate for the numerator (that is $\int |\nabla u|^2 \, dV_g$) normally uses two main ingredients:
  - Hölder inequality $\int |\nabla u|^2 \, dV_g \leqslant Vol_g(G_i \backslash F_i)^{1-2/n} \left(\int |\nabla u|^n \, dV_g\right)^{2/n}$;

Korevaar method: test-functions

- Test-functions are always (to my knowledge) chosen as Federer-Fleming functions (or Plateau functions)!
- "Decomposition theorems" above allow to bound the numerator of the Rayleigh quotient from below.
- The estimate for the numerator (that is $\int |\nabla u|^2 \, dV_g$) normally uses two main ingredients:
  - Hölder inequality $\int |\nabla u|^2 \, dV_g \leqslant Vol_g(G_i \backslash F_i)^{1-2/n} \left(\int |\nabla u|^n \, dV_g \right)^{2/n}$;
  - volume comparison theorems.

Korevaar method: test-functions

- Test-functions are always (to my knowledge) chosen as Federer-Fleming functions (or Plateau functions)!
- "Decomposition theorems" above allow to bound the numerator of the Rayleigh quotient from below.
- The estimate for the numerator (that is $\int |\nabla u|^2 \, dV_g$) normally uses two main ingredients:
  - Hölder inequality $\int |\nabla u|^2 \, dV_g \leqslant Vol_g(G_i \backslash F_i)^{1-2/n} \left(\int |\nabla u|^n \, dV_g\right)^{2/n}$;
  - volume comparison theorems.
- We choose test-functions differently: they are modelled on the first eigenfunctions of a sphere S^m (or other appropriate space), and use properties of conformal diffeomorphisms. This allows to extend a number of classical eigenvalue inequalities for the first eigenvalue to higher eigenvalues. The applications include:

Korevaar method: test-functions

- Test-functions are always (to my knowledge) chosen as Federer-Fleming functions (or Plateau functions)!
- "Decomposition theorems" above allow to bound the numerator of the Rayleigh quotient from below.
- The estimate for the numerator (that is $\int |\nabla u|^2 \, dV_g$) normally uses two main ingredients:
  - Hölder inequality $\int |\nabla u|^2 \, dV_g \leqslant Vol_g(G_i \backslash F_i)^{1-2/n} \left(\int |\nabla u|^n \, dV_g\right)^{2/n}$;
  - volume comparison theorems.
- We choose test-functions differently: they are modelled on the first eigenfunctions of a sphere S^m (or other appropriate space), and use properties of conformal diffeomorphisms. This allows to extend a number of classical eigenvalue inequalities for the first eigenvalue to higher eigenvalues. The applications include:
  - the version for higher eigenvalues of the Reilly inequality (1977) for the first Laplace eigenvalue,

Korevaar method: test-functions

- Test-functions are always (to my knowledge) chosen as Federer-Fleming functions (or Plateau functions)!
- "Decomposition theorems" above allow to bound the numerator of the Rayleigh quotient from below.
- The estimate for the numerator (that is $\int |\nabla u|^2 \, dV_g$) normally uses two main ingredients:
  - Hölder inequality $\int |\nabla u|^2 \, dV_g \leqslant Vol_g(G_i \backslash F_i)^{1-2/n} \left(\int |\nabla u|^n \, dV_g \right)^{2/n}$;
  - volume comparison theorems.
- We choose test-functions differently: they are modelled on the first eigenfunctions of a sphere S^m (or other appropriate space), and use properties of conformal diffeomorphisms. This allows to extend a number of classical eigenvalue inequalities for the first eigenvalue to higher eigenvalues. The applications include:
  - the version for higher eigenvalues of the Reilly inequality (1977) for the first Laplace eigenvalue,
  - sharpened estimates for Schrödinger eigenvalues, and index bounds for minimal hypersurfaces,

Korevaar method: test-functions

- Test-functions are always (to my knowledge) chosen as Federer-Fleming functions (or Plateau functions)!
- "Decomposition theorems" above allow to bound the numerator of the Rayleigh quotient from below.
- The estimate for the numerator (that is $\int |\nabla u|^2 \, dV_g$) normally uses two main ingredients:
  - Hölder inequality $\int |\nabla u|^2 \, dV_g \leqslant Vol_g(G_i \backslash F_i)^{1-2/n} \left(\int |\nabla u|^n \, dV_g\right)^{2/n}$;
  - volume comparison theorems.
- We choose test-functions differently: they are modelled on the first eigenfunctions of a sphere S^m (or other appropriate space), and use properties of conformal diffeomorphisms. This allows to extend a number of classical eigenvalue inequalities for the first eigenvalue to higher eigenvalues. The applications include:
  - the version for higher eigenvalues of the Reilly inequality (1977) for the first Laplace eigenvalue,
  - sharpened estimates for Schrödinger eigenvalues, and index bounds for minimal hypersurfaces,
  - bounds for Laplace eigenvalues on Kähler manifolds, generalising to higher eigenvalues the classical result by Bourguignon, Li, and Yau (1994).

Note the the first eigenvalue bound via conformal volume implies the first eigenvalue bound via the L^2-norm of mean curvature vector for surfaces $\Sigma^2 \subset \mathbb{R}^m$. There is a more general classical result:

Note the the first eigenvalue bound via conformal volume implies the first eigenvalue bound via the L^2-norm of mean curvature vector for surfaces $\Sigma^2 \subset \mathbb{R}^m$. There is a more general classical result:

Theorem (Reilly, 1977)

Let (Σ^n, g) be a closed Riemannian manifold of dimension $n \geqslant 2$. Then for any isometric immersion $\phi : \Sigma^n \to \mathbb{R}^m$ the following inequality holds

$$\lambda_1(\Sigma^n, g) \leqslant \frac{n}{Vol_g(\Sigma^n)} \int_{\Sigma^n} |H_\phi|^2 \, dVol_g,$$

where H_ϕ is the mean curvature vector of an immersion ϕ. When $n = m - 1$ the equality above occurs if and only if Σ^n is a sphere isometrically embedded into $\mathbb{R}^m$ as a hypersphere. When $n < m - 1$ the equality occurs if and only if after scaling the metric g and making a translation and dilation in $\mathbb{R}^m$, the immersion ϕ is an isometric minimal immersion into a unit sphere $S^{m-1} \subset \mathbb{R}^m$ by first eigenfunctions.

Results for higher eigenvalues II

- (El Soufi, Harrell, Ilias, 2009) For a submanifold $\Sigma^n \subset \mathbb{R}^m$ the Laplace eigenvalues satisfy the inequalities $\lambda_k(\Sigma^n) \leqslant C(n) \max |H| \, k^{2/n}$ for any $k \geqslant 1$.

Results for higher eigenvalues II

- (El Soufi, Harrell, Ilias, 2009) For a submanifold $\Sigma^n \subset \mathbb{R}^m$ the Laplace eigenvalues satisfy the inequalities $\lambda_k(\Sigma^n) \leqslant C(n) \max |H| \, k^{2/n}$ for any $k \geqslant 1$.
- Using $V_c(m, \Sigma^2) \leqslant \int |H|^2 \, dVol$ for $\Sigma^2 \subset \mathbb{R}^m$, by the theorem above we obtain

$$\lambda_k(\Sigma^2, g) Vol_g(\Sigma^2) \leqslant \left(\int_{\Sigma^2} |H|^2 \, dVol_{g_\Sigma} \right) k$$

for any $k \geqslant 1$, and any conformal metric $g \sim g_\Sigma$.

Results for higher eigenvalues II

- (El Soufi, Harrell, Ilias, 2009) For a submanifold $\Sigma^n \subset \mathbb{R}^m$ the Laplace eigenvalues satisfy the inequalities $\lambda_k(\Sigma^n) \leqslant C(n) \max |H| \, k^{2/n}$ for any $k \geqslant 1$.
- Using $V_c(m, \Sigma^2) \leqslant \int |H|^2 \, dVol$ for $\Sigma^2 \subset \mathbb{R}^m$, by the theorem above we obtain

$$\lambda_k(\Sigma^2, g) Vol_g(\Sigma^2) \leqslant \left(\int_{\Sigma^2} |H|^2 \, dVol_{g_\Sigma} \right) k$$

for any $k \geqslant 1$, and any conformal metric $g \sim g_\Sigma$.

Theorem (GK, 2017)

Let $\Sigma^n \subset \mathbb{R}^m$ be an immersed closed submanifold. Then

$$\lambda_k(\Sigma^n) \leqslant C(n, m) \left(\frac{1}{Vol(\Sigma^n)} \int_{\Sigma^n} |H|^2 \, dVol_{g_\Sigma} \right) k$$

for any $k \geqslant 1$, where H is the mean curvature vector of Σ^n.

Results for higher eigenvalues II

- (El Soufi, Harrell, Ilias, 2009) For a submanifold $\Sigma^n \subset \mathbb{R}^m$ the Laplace eigenvalues satisfy the inequalities $\lambda_k(\Sigma^n) \leqslant C(n) \max |H| \, k^{2/n}$ for any $k \geqslant 1$.
- Using $V_c(m, \Sigma^2) \leqslant \int |H|^2 \, dVol$ for $\Sigma^2 \subset \mathbb{R}^m$, by the theorem above we obtain

$$\lambda_k(\Sigma^2, g) Vol_g(\Sigma^2) \leqslant \left(\int_{\Sigma^2} |H|^2 \, dVol_{g_\Sigma} \right) k$$

for any $k \geqslant 1$, and any conformal metric $g \sim g_\Sigma$.

Theorem (GK, 2017)

Let $\Sigma^n \subset \mathbb{R}^m$ be an immersed closed submanifold. Then

$$\lambda_k(\Sigma^n) \leqslant C(n, m) \left(\frac{1}{Vol(\Sigma^n)} \int_{\Sigma^n} |H|^2 \, dVol_{g_\Sigma} \right) k$$

for any $k \geqslant 1$, where H is the mean curvature vector of Σ^n.

- There is a version of the statement above for submanifolds in constant curvature spaces, and even more general Riemannian manifolds.

- Now we consider the eigenvalue problem for the Schrodinger operator on a Riemannian manifold (Σ^n, g):

$$(\Delta_g + V)u + \lambda u = 0,$$

where V is a potential ($V \in L^\infty$ or $V \in L^p$, where $p > n/2$.) Denote by $Neg(V)$ the number of negative eigenvalues.

Number of negative eigenvalues of Schrodinger operators

- Now we consider the eigenvalue problem for the Schrodinger operator on a Riemannian manifold (Σ^n, g):

$$(\Delta_g + V)u + \lambda u = 0,$$

  where V is a potential ($V \in L^\infty$ or $V \in L^p$, where $p > n/2$.) Denote by $Neg(V)$ the number of negative eigenvalues.
- There is a vast literature on various estimates for $Neg(V)$: Rozenblum (1972), Cwikel (1977), Lieb (1980), Li and Yau (1983), Grigoryan, Netrusov, Yau (2004), Grigoryan, Nadirashvili (2011).

Number of negative eigenvalues of Schrodinger operators

- Now we consider the eigenvalue problem for the Schrodinger operator on a Riemannian manifold (Σ^n, g):

$$(\Delta_g + V)u + \lambda u = 0,$$

  where V is a potential ($V \in L^\infty$ or $V \in L^p$, where $p > n/2$.) Denote by $Neg(V)$ the number of negative eigenvalues.

- There is a vast literature on various estimates for $Neg(V)$: Rozenblum (1972), Cwikel (1977), Lieb (1980), Li and Yau (1983), Grigoryan, Netrusov, Yau (2004), Grigoryan, Nadirashvili (2011).

Theorem (Grigoryan, Netrusov, Yau, 2004)

For any closed Riemannian manifold (Σ^n, g) and a non-negative potential $V \geqslant 0$, the following inequality holds

$$Neg(V) \geqslant \frac{C}{Vol_g(\Sigma^n)^{n/2-1}} \left(\int_{\Sigma^n} V dVol_g \right)^{n/2},$$

where the constant C depends on the conformal class of g only.

Number of negative eigenvalues of Schrodinger operators

Theorem (GK, 2017)

Under the hypotheses of the theorem above,

$$Neg(V) \geqslant \frac{C(n,m)}{V_c(m,\Sigma^n)} \frac{1}{Vol_g(\Sigma^n)^{n/2-1}} \left(\int_{\Sigma^n} V \, dVol_g \right)^{n/2}.$$

Theorem (GK, 2017)

Under the hypotheses of the theorem above,

$$Neg(V) \geqslant \frac{C(n,m)}{V_c(m,\Sigma^n)} \frac{1}{Vol_g(\Sigma^n)^{n/2-1}} \left(\int_{\Sigma^n} V \, dVol_g \right)^{n/2}.$$

Corollary

Let $\Sigma^n \subset S^m$ be an immersed minimal submanifold. Then for any non-negative potential $V \geqslant 0$ we have

$$Neg(V) \geqslant C(n,m) \left(\frac{1}{Vol_g(\Sigma^n)} \int_{\Sigma^n} V \, dVol_g \right)^{n/2}.$$

In particular, the index of a two-sided minimal hypersurface $\Sigma^n \subset S^{n+1}$ satisfies the inequality

$$Index(\Sigma^n) \geqslant C(n) \left(n + \frac{1}{Vol_g(\Sigma^n)} \int_{\Sigma^n} |S|^2 \, dVol_g \right)^{n/2}, \qquad (**)$$

where S is the shape operator of Σ^n.

A few remarks:

- For minimal surfaces $\Sigma^2 \subset S^3$ there is a stronger result due to Savo (2010):

$$Index(\Sigma^2) \geqslant C_1 + C_2 \int_{\Sigma^2} |S|^2 \, dVol.$$

A few remarks:

- For minimal surfaces $\Sigma^2 \subset S^3$ there is a stronger result due to Savo (2010):
$$Index(\Sigma^2) \geqslant C_1 + C_2 \int_{\Sigma^2} |S|^2 \, dVol.$$
- Corollary above should be compared with the following result due to Buzano and Sharp (2015) for minimal hypersurfaces $\Sigma^n \subset S^{n+1}$:
$$Vol(\Sigma^n) \leqslant C_1 \quad \text{and} \quad Index(\Sigma^n) \leqslant C_2 \quad \Longrightarrow \quad \int_{\Sigma^n} |S|^n \, dVol_g \leqslant C(C_1, C_2, n)$$
in dimensions $2 \leqslant n \leqslant 6$ only. (There are versions of this result for more general ambient manifolds, but under the same dimension constraints.)

A few remarks:

- For minimal surfaces $\Sigma^2 \subset S^3$ there is a stronger result due to Savo (2010):
$$Index(\Sigma^2) \geqslant C_1 + C_2 \int_{\Sigma^2} |S|^2 \, dVol.$$
- Corollary above should be compared with the following result due to Buzano and Sharp (2015) for minimal hypersurfaces $\Sigma^n \subset S^{n+1}$:
$$Vol(\Sigma^n) \leqslant C_1 \quad \text{and} \quad Index(\Sigma^n) \leqslant C_2 \quad \Longrightarrow \quad \int_{\Sigma^n} |S|^n \, dVol_g \leqslant C(C_1, C_2, n)$$
in dimensions $2 \leqslant n \leqslant 6$ only. (There are versions of this result for more general ambient manifolds, but under the same dimension constraints.)
- There is a version of inequality $(**)$ for minimal hypersurfaces in arbitrary Riemannian manifolds. In this case the constant $C(n, m)$ is replaced by the constant that depends on the geometry of the ambient space. (Preprint under revision.)

Theorem (Bourguignon, Li, Yau, 1994)

Let (Σ^n, J) be a closed complex manifold that admits a holomorphic immersion $\phi : \Sigma^n \to \mathbb{C}P^m$. Suppose that Σ^n is full in the sense that the image $\phi(\Sigma^n)$ is not contained in any hyperplane of $\mathbb{C}P^m$. Then for any Kähler metric g on Σ^n the first Laplace eigenvalue satisfies the inequality

$$\lambda_1(\Sigma^n, g) \leqslant 4n\frac{m+1}{m} d([\phi], [\omega_g]),$$

where $d([\phi], [\omega_g])$ is the holomorphic immersion degree defined by

$$d([\phi], [\omega_g]) = \left(\int_{\Sigma^n} \phi^* \omega_{FS} \wedge \omega_g^{n-1} \right) / \left(\int_{\Sigma^n} \omega_g^n . \right)$$

(Above the Fubini-Study metric is assumed to be normalised so that the diameter of $\mathbb{C}P^m$ equals $\pi/2$.)

Theorem (Bourguignon, Li, Yau, 1994)

Let (Σ^n, J) be a closed complex manifold that admits a holomorphic immersion $\phi : \Sigma^n \to \mathbb{C}P^m$. Suppose that Σ^n is full in the sense that the image $\phi(\Sigma^n)$ is not contained in any hyperplane of $\mathbb{C}P^m$. Then for any Kähler metric g on Σ^n the first Laplace eigenvalue satisfies the inequality

$$\lambda_1(\Sigma^n, g) \leqslant 4n\frac{m+1}{m} d([\phi], [\omega_g]),$$

where $d([\phi], [\omega_g])$ is the holomorphic immersion degree defined by

$$d([\phi], [\omega_g]) = \left(\int_{\Sigma^n} \phi^* \omega_{FS} \wedge \omega_g^{n-1}\right) / \left(\int_{\Sigma^n} \omega_g^n.\right)$$

(Above the Fubini-Study metric is assumed to be normalised so that the diameter of $\mathbb{C}P^m$ equals $\pi/2$.)

Corollary. For any Kähler metric g on $\mathbb{C}P^m$ the following inequality holds:

$$\lambda_1(\mathbb{C}P^m, g) Vol_g(\mathbb{C}P^m)^{1/m} \leqslant 4(m+1)\left(\frac{Vol(S^{2m+1})}{2\pi}\right)^{1/m},$$

and the equality is achieved on the Fubini-Study metric.

Eigenvalue bounds for Kähler metrics

Theorem (Bourguignon, Li, Yau, 1994)

Let (Σ^n, J) be a closed complex manifold that admits a holomorphic immersion $\phi : \Sigma^n \to \mathbb{C}P^m$. Suppose that Σ^n is full in the sense that the image $\phi(\Sigma^n)$ is not contained in any hyperplane of $\mathbb{C}P^m$. Then for any Kähler metric g on Σ^n the first Laplace eigenvalue satisfies the inequality

$$\lambda_1(\Sigma^n, g) \leqslant 4n\frac{m+1}{m} d([\phi], [\omega_g]),$$

where $d([\phi], [\omega_g])$ is the holomorphic immersion degree defined by

$$d([\phi], [\omega_g]) = \left(\int_{\Sigma^n} \phi^* \omega_{FS} \wedge \omega_g^{n-1}\right) / \left(\int_{\Sigma^n} \omega_g^n.\right)$$

(Above the Fubini-Study metric is assumed to be normalised so that the diameter of $\mathbb{C}P^m$ equals $\pi/2$.)

Similar sharp estimates have been obtained on complex Grassmanians (Arezzo, Ghigi, Loi, 2007) and compact Hermitian symmetric spaces (Biliotti, Ghigi, 2013).

Theorem (GK, 2017)

Let (Σ^n, J) be a closed complex manifold that admits a holomorphic immersion $\phi : \Sigma^n \to \mathbb{C}P^m$. Then for any Kähler metric g on Σ^n the Laplace eigenvalues satisfy the inequalities

$$\lambda_k(\Sigma^n, g) \leqslant C(n, m) d([\phi], [\omega_g]) k \qquad \text{for any} \quad k \geqslant 1$$

where $d([\phi], [\omega_g])$ is the holomorphic immersion degree defined above.

- The result answers the questions previously asked by Gromov (???) and Yau (1996).

Eigenvalue bounds for Kähler metrics

Theorem (GK, 2017)

Let (Σ^n, J) be a closed complex manifold that admits a holomorphic immersion $\phi : \Sigma^n \to \mathbb{C}P^m$. Then for any Kähler metric g on Σ^n the Laplace eigenvalues satisfy the inequalities

$$\lambda_k(\Sigma^n, g) \leqslant C(n,m)d([\phi],[\omega_g])k \qquad \textit{for any} \quad k \geqslant 1$$

where $d([\phi],[\omega_g])$ is the holomorphic immersion degree defined above.

- The result answers the questions previously asked by Gromov (???) and Yau (1996).
- The eigenvalue bound above is not asymptotically sharp in the index k.

THANK YOU