On the compactness property of the quasilinearly perturbed harmonic maps equation

Gerasim Kokarev*

Abstract

For maps $u : M \to M'$ between closed Riemannian manifolds we consider the quasilinearly perturbed harmonic maps equation

$$\tau(u)(x) = \mathsf{G}(x, u(x)) \cdot du(x) + \mathsf{g}(x, u(x)),\ x \in M.$$

In the case of a nonpositively curved target manifold M' for a sufficiently small linear part G the space of classical solutions within a homotopy class is proved to be compact. The proof is based on the uniform gradient estimate for the solutions of the perturbed equation in terms of total energies and C^1-norms of G and g. An important ingredient for this analysis is an analogue of the monotonocity inequality.

Introduction

In the recent paper [5] there were studied semilinear and some special quasilinear perturbations of the harmonic maps equation for the case if the domain and the target manifolds M and M' are closed and M' is nonpositively curved. In particular, for semilinear and, if $\dim M \leqslant 3$, quasilinear perturbations the space of classical solutions of fixed homotopy class was proved to be compact. This compactness property is particularly important for the qualitative theory of quasilinear elliptic differential equations for the mappings of manifolds (see the survey [8]). The point is that sometimes the space of harmonic mappings is degenerate. Using such perturbations and the compactness property one can show that for a typical perturbation the set of the solutions of the perturbed equation is finite and the algebraic number of them is an invariant of the manifold M'.

The present note is devoted to a proof of a general compactness theorem for the quasilinear perturbations of the harmonic maps equation. Since this statement is known for small dimensions of M (see [5], [8]) we consider only the case $\dim M \geqslant 3$.

M and M' denote C^∞-smooth connected compact manifolds without boundary. Moreover, the manifold M is supposed to be orientable. We also assume that the manifolds M and M' are endowed with C^∞-smooth Riemannian metrics g and g'. The convention about the sign of the Laplace-Beltrami operator Δ_M is different from [3]. At the center of the normal Riemannian coordinates it is given by

$$\Delta_M = \frac{\partial^2}{(\partial x^1)^2} + \cdots + \frac{\partial^2}{(\partial x^n)^2}.$$

We also use the Einstein convention about the summation for the repeated indices.

The author is grateful to V. Buchstaber, S. Kuksin and A. Shirikyan for the help during the work under this note.

*Heriot-Watt University, Edinburgh, EH14 4AS, UK E-mail: G.Y.Kokarev@ma.hw.ac.uk

1. Basic notation and results

Let (M, g) and (M', g') be closed Riemannian manifolds. The Riemannian metrics on M and M' naturally determine the metric on the bundle $J^1(M, M')$ over the manifold $M \times M'$. For maps $u : M \to M'$ we consider the energy functional

$$E(u) = \int\limits_M e(u)(x)\, d\,Vol_M(x),$$

where $e(u)$ denotes the energy density of the map u

$$e(u)(x) = \frac{1}{2} \|du(x)\|^2, \quad x \in M.$$

Let us denote by $-\tau(u)$ the Euler-Lagrange operator for the functional E. In local coordinates on M and M' it is given by

$$\tau^i(u) = \Delta_M u^i + g^{\alpha\beta} \Gamma'^{\,i}_{jk} \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta}, \tag{1.1}$$

where $\Gamma'^{\,i}_{jk}$ denote the Christiffel symbols of the Levi-Chivita connection on M' and Δ_M is the Laplace-Beltrami operator on M. C^2-smooth solutions of the equation $\tau(u) = 0$ are called harmonic mappings of M to M' and the operator $\tau(u)$ is called the harmonic maps operator.

For a map $u : M \to M'$ in a given homotopy class $[v]$ let us consider the quasilinear elliptic differential equation, see [8],

$$\tau(u)(x) = \mathsf{G}(x, u(x)) \cdot du(x) + \mathsf{g}(x, u(x)), \quad x \in M, \tag{1.2}$$

where G is a morphism of the vector bundles $J^1(M, M')$ and $(\Pi')^* TM'$ ($\Pi' : M \times M' \to M'$ is a natural projection), and g is a nonautonomous vector field on M'. Below we suppose that $\mathsf{G}(x, u)$ and $\mathsf{g}(x, u)$ are C^s-smooth ($s \geqslant 2$) with respect to $x \in M$ and $u \in M'$. Let $\mathcal{F}_\mathsf{G}$ and $\mathcal{F}_\mathsf{g}$ be vector spaces formed by morphisms $\mathsf{G}(x, u)$ and vector fields $\mathsf{g}(x, u)$ correspondingly. By regularity theory of elliptic operators any C^2-smooth solution of (1.2) with $\mathsf{G} \in \mathcal{F}_\mathsf{G}$ and $\mathsf{g} \in \mathcal{F}_\mathsf{g}$ is C^{s+1}-smooth. Let us consider the space

$$\mathfrak{M}_{[v]} = \{(u, \mathsf{G}, \mathsf{g}) : u \text{ is a } C^2\text{-smooth solution of (1.2) }, u \in [v] \text{ and } \mathsf{G} \in \mathcal{F}_\mathsf{G},\ \mathsf{g} \in \mathcal{F}_\mathsf{g}\},$$

which is a subspace of $C^{s+1}(M, M') \times \mathcal{F}_\mathsf{G} \times \mathcal{F}_\mathsf{g}$. Let π denote the natural projection

$$\pi : \mathfrak{M}_{[v]} \to \mathcal{F}_\mathsf{G} \times \mathcal{F}_\mathsf{g}, \quad (u, \mathsf{G}, \mathsf{g}) \to (\mathsf{G}, \mathsf{g}).$$

Recall that a continuous map between topological spaces is called proper if the preimage of any compact set is compact. Below the symbol $\|\mathsf{G}(x, u)\|$ denotes the norm of the linear operator $\mathsf{G}(x, u) : \mathcal{L}(T_x M, T_u M') \to T_u M'$ with respect to the Riemannian metrics on M and M'.

The principal result of this note is the following

Theorem 1. *Let* $\dim M \geqslant 3$ *and suppose that* M' *is a Riemannian manifold of nonpositive sectional curvature. Then for any homotopy class* $[v]$ *of the mappings* M *to* M' *there exists a constant* $C_{[v]} > 0$ *such that the map* $\pi|_{\pi^{-1}(\mathcal{U})}$, *where*

$$\mathcal{U} = \{(\mathsf{G}, \mathsf{g}) \in \mathcal{F}_\mathsf{G} \times \mathcal{F}_\mathsf{g} : \|\mathsf{G}(x, u)\| < C_{[v]} \textit{ for any } x \in M,\ u \in M'\},$$

is proper.

Remark 1. Let M' be a closed Riemannian manifold of negative sectional curvature. Then the constant $C_{[v]}$ in Theorem 1 can be chosen independly of the homotopy class $[v]$. See Remark 4 below.

In particular, Theorem 1 implies that the set of solutions of (1.2) in a fixed homotopy class is compact with respect to the C^{s+1}-topogy. For the case of the harmonic maps equation this follows from the results of [13].

Simple examples given in [8] show that the statement of the theorem no longer holds if one of the following suppositions is dropped: quasilinearity of the r.h.s. of (1.2), smallness of G or nonpositivity of the sectional curvature of M'.

The proof of Theorem 1 is based on the a priori estimate for the solutions of (1.2) (Theorem 2 below) in terms of their total energies and C^1-norms of G and g. An important ingredient for this analysis is an analogue of the monotonocity inequality (Lemma 5). To prove that the energies of the solutions are bounded we use the energy estimate and the version of the Poincaré inequality due to [5].

By $\|\mathsf{g}\|_{C^0}$ and $\|\mathsf{G}\|_{C^0}$ we denote the norms

$$\|\mathsf{g}\|_{C^0} = \max_{\substack{x\in M\\ u\in M'}} \|\mathsf{g}(x,u)\|_{g'}, \quad \|\mathsf{G}\|_{C^0} = \max_{\substack{x\in M\\ u\in M'}} \|\mathsf{G}(x,u)\|$$

of a nonautonomous vector field g on M' and of a morphism G of the vector bundles $J^1(M,M')$ and $(\Pi')^*TM'$. Similarly using the Levi-Chivita connections on M and M' one can define C^1-norms

$$\|\mathsf{g}\|_{C^1} = \max_{\substack{x\in M\\ u\in M'}} \Big(\|\mathsf{g}(x,u)\|_{g'} + \|\nabla\, \mathsf{g}(x,u)\|\Big),$$

$$\|\mathsf{G}\|_{C^1} = \max_{\substack{x\in M\\ u\in M'}} \left(\|\mathsf{G}(x,u)\| + \|\nabla\, \mathsf{G}(x,u)\|\right),$$

where $\|\nabla\, \mathsf{g}(x,u)\|$ and $\|\nabla\, \mathsf{G}(x,u)\|$ denote the natural norms of the linear operators

$$\nabla\, \mathsf{g}(x,u) : T_xM \times T_uM' \to T_uM',$$

$$\nabla\, \mathsf{G}(x,u) : T_xM \times T_uM' \to \mathcal{L}\left(\mathcal{L}(T_xM, T_uM'), T_uM'\right)$$

with respect to the Riemannian metrics on M and M'.

Let us choose positive C_{g} and C_{G} in such way that the following inequalities hold

$$\|\mathsf{g}\|_{C^1} \leqslant C_{\mathsf{g}} \quad \text{and} \quad \|\mathsf{G}\|_{C^1} \leqslant C_{\mathsf{G}}. \tag{1.3}$$

Definition (following [1]). A harmonic map $v : \mathbb{R}^n \to M'$ ($n \geqslant 3$) is called an n-obstruction for the Riemannian manifold M' if $\|dv(x)\| \leqslant \|dv(0)\| \neq 0$ for any $x \in \mathbb{R}^n$ and there exists the constant $E > 0$ such that

$$R^{2-n} \int_{B_R} \|dv(x)\|^2\, dx \leqslant E \quad \text{for any} \quad R > 0,$$

where $B_R = \{x \in \mathbb{R}^n : |x| \leqslant R\}$.

Theorem 2. *Let $n = \dim M \geqslant 3$ and suppose that the Riemannian manifold M' does not admit any n-obstructions. Then for any $C_E > 0$ there exists $C > 0$ depending on the constants C_E, C_{g}, C_{G} and the geometry of M and M' such that for any solution u of problem (1.2)–(1.3) with the energy $E(u) \leqslant C_E$ the following inequality holds*

$$\max_{x\in M} \|du(x)\| \leqslant C. \tag{1.4}$$

Remark 2. Formally the notion of an obstruction makes sense also for $n = 2$. In this case by the extension theorem for harmonic mappings [11] the condition of the absence of 2-obstructions is equivalent to the absence of nontrivial harmonic 2-spheres. For the case $\dim M = 2$ the estimate (1.4) is known (see, for example, the proof of Theorem 1 for the case $\dim M = 2$ given in [8]) and follows from the conformal invariance property of the energy functional [2].

Example. Let M' be a Riemannian manifold of nonpositive sectional curvature. Then for any $n \geqslant 3$ the manifold M' does not admit any n-obstructions.

Indeed, to prove this we recall that for any harmonic map $v : M \to M'$ (for any Riemannian manifolds M and M') in the normal Riemannian coordinates on M with the center at the point x_0 the Bochner formula holds (see [3, p. 22])

$$\Delta_M e(v)(x_0) = \|\nabla dv\|^2 + \sum_{\alpha} \langle dv \cdot Ricci^M e_\alpha, dv \cdot e_\alpha \rangle - \sum_{\alpha,\, \beta} \langle R'(dv \cdot e_\alpha, dv \cdot e_\beta) dv \cdot e_\alpha, dv \cdot e_\beta \rangle,$$

where $Ricci^M$ denotes the Ricci tensor on M, R' is the Riemannian curvature tensor of M' and vectors $e_\alpha = \partial / \partial x^\alpha |_{x = x_0}$.

Assume the contrary. Let the map v be an n-obstruction for M'. Then, since v is a harmonic map from Euclidean space to the space of nonpositive sectional curvature, the Bochner formula implies that the function $e(v)(x)$ is subharmonic, i. e. $\Delta e(v)(x) \geqslant 0$. By the mean value inequality we have the following estimate with the constant E from the definition of an n-obstruction

$$e(v)(0) \leqslant \frac{1}{\omega_n R^n} \int\limits_{B_R} e(v)(x) dx \leqslant \frac{1}{2\omega_n R^2} E,$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. Proceeding to limit as $R \to \infty$ in this inequality we get $\|dv(0)\| = 0$. This contradicts the definition of an n-obstruction and ends the proof.

At the end of this section we note some corollaries of Theorem 2.

Corollary 3. *Let $n = \dim M \geqslant 3$ and suppose that the Riemannian manifold M' does not admit any n-obstructions. Then for any positive C_E the family of mappings*

$$\{u : M \to M' \text{ is a } C^2\text{-smooth solution of (1.2) with } E(u) \leqslant C_E\}$$

is compact with respect to the C^{s+1}-topology.

The proof uses the Arzela-Ascoli theorem, which localizes the problem, and the Shauder estimate from elliptic regularity theory (see the proof of Lemma 7 below).

Setting the r.h.s. of (1.2) to be equal to zero we get

Corollary 4. *Let $n = \dim M \geqslant 3$ and suppose that the Riemannian manifold M' does not admit any n-obstructions. Then for any $C_E > 0$ there exists a positive $C = C(C_E, M, M')$ such that for any harmonic map u with the energy $E(u) \leqslant C_E$ the following estimate holds*

$$\max_{x \in M} \|du(x)\| \leqslant C.$$

Remark 3. The problem of an a priori estimate of $\max_{x \in M} \|du(x)\|$ for the class of stationary harmonic mappings was considered in [9]. One of the results of this paper is a statement which is similar to Corollary 4 with the assumption that M' does not admit nontrivial harmonic spheres S^ℓ for $\ell = 2, \ldots, n-1$, where $n = \dim M \geqslant 3$. We do not know any relationships between this constraint on M' and the absence of n-obstructions.

2. Monotonocity inequality

The monotonocity inequality for harmonic maps (and for the larger class of stationary harmonic maps) is well-known (see [10], [12]). In this section we prove a similar result for the solutions of equation (1.2). It is important that this inequality will be uniform with respect to the C^0-norms of G and g in the r.h.s. of (1.2). Therefore we will assume that

$$\|\mathsf{G}\|_{C^0} \leqslant C^0_{\mathsf{G}}, \quad \|\mathsf{g}\|_{C^0} \leqslant C^0_{\mathsf{g}} \tag{2.1}$$

for some positive constants C^0_{G} and C^0_{g}. Below we set $n = \dim M$.

By the Nash theorem one can consider M' as a Riemannian submanifold of $\mathbb{R}^N$ for some positive integer N. Let λ_M denote the injectivity radius of M. By $\rho < \lambda_M$ let us denote a positive constant such that for any point $x_0 \in M$ the Riemannian metric g_{ij} in the normal Riemannian coordinates with the center at the point x_0 satisfies the following inequalities in the ball $B_\rho(x_0)$

$$|g_{ij}(x) - \delta_{ij}| \leqslant K\,|x|^2, \quad |\partial_k g_{ij}(x)| \leqslant K\,|x|, \tag{2.2}$$

where $|x|$ denotes Euclidean distance from 0 to x, and

$$\Lambda^{-1}\,|\xi| \leqslant |\xi|_{g(x)} \leqslant \Lambda\,|\xi| \tag{2.3}$$

for any $\xi \in \mathbb{R}^n$. Here the positive constants K and Λ depend only on the geometry of M and $|\xi|_g = \left(g_{ij}\xi^i\xi^j\right)^{1/2}$, $|\xi| = \left(\sum_{i=1}^n (\xi^i)^2\right)^{1/2}$. For the map $u : M \to M'$ we denote by $E^{x_0}_r(u)$ the integral

$$E^{x_0}_r(u) = \int\limits_{B_r(x_0)} e(u)(x)\,d\,Vol_g(x)$$

over the ball of the radius $0 < r \leqslant \rho$ with the center at the point $x_0 \in M$.

Following the ideas of [1, Proposition 2.1], [10] we prove the following inequality:

Lemma 5. *There exists a positive constant $\bar{r} = \bar{r}(M) \leqslant \rho$ such that for any solution u of equation (1.2) with the r.h.s. satisfying (2.1) and for any $0 < r_1 < r_2 \leqslant \bar{r}$ the inequality*

$$r_1^{2-n} E^{x_0}_{r_1}(u) \leqslant C_1 r_2^{2-n} E^{x_0}_{r_2}(u) + C_2 \tag{2.4}$$

holds, where C_1, $C_2 > 0$ depend only on the constants C^0_{g}, C^0_{G} and the geometry of M.

Proof. Let us consider a Lipshitz vector field ξ on M and the corresponding one-parameter family of diffeomorphisms φ_s. Let $u_s = u \circ \varphi_s$, then $\partial u_s/\partial s|_{s=0} = du(\xi)$. By the first variation formula we have

$$\left.\frac{d}{ds}\right|_{s=0} E(u_s) = -\int\limits_M \langle \tau(u), du(\xi)\rangle\,d\,Vol_g. \tag{2.5}$$

On the other hand, changing variables

$$E(u_s) = \frac{1}{2}\int\limits_M \sum_{\ell=1}^N g^{\alpha\beta}\frac{\partial u_s^\ell}{\partial x^\alpha}\frac{\partial u_s^\ell}{\partial x^\beta}d\,Vol_g = \frac{1}{2}\int\limits_M \sum_{\ell=1}^N g_s^{\alpha\beta}\frac{\partial u^\ell}{\partial x^\alpha}\frac{\partial u^\ell}{\partial x^\beta}d\,Vol_{g_s},$$

where $g_s = \varphi^*_{-s}g$ is a metric induced by the diffeomorphism φ_{-s}. Differentiating the above expression with respect to s at $s = 0$ we get (see [10, p. 146])

$$\left.\frac{d}{ds}\right|_{s=0} E(u_s) = -\frac{1}{2}\int\limits_M \|du\|^2 \operatorname{div}\xi\,d\,Vol_g + \int\limits_M \langle du \circ \nabla\xi, du\rangle\,d\,Vol_g. \tag{2.6}$$

Indeed, note that $dVol_{g_s} = \det(\partial\varphi_{-s}/\partial x)dVol_g$. Then the first term in the r.h.s. of expression (2.6) follows from

$$\frac{d}{ds}\bigg|_{s=0} \det\left(\frac{\partial\varphi_{-s}}{\partial x}\right) = \operatorname{div}\left(\frac{d\varphi_{-s}}{ds}\right)\bigg|_{s=0} = -\operatorname{div}\xi.$$

The formula

$$\frac{d}{ds}\bigg|_{s=0} g_s^{\alpha\beta} = \frac{d}{ds}\bigg|_{s=0}\left(g^{\alpha'\beta'}\frac{\partial\varphi_s^\alpha}{\partial x^{\alpha'}}\frac{\partial\varphi_s^\beta}{\partial x^{\beta'}}\right) = 2g^{\alpha'\beta'}\frac{d}{ds}\bigg|_{s=0}\left(\frac{\partial\varphi_s^\alpha}{\partial x^{\alpha'}}\right)\delta_{\beta'}^{\beta} = 2g^{\alpha'\beta}\nabla_{\frac{\partial}{\partial x^{\alpha'}}}\xi^\alpha$$

gives us the second one.

For $0 < t' < t \leqslant \rho$ let us consider the vector field $\xi = \eta(r)r\partial/\partial r = \eta(|x|)x^\alpha\partial/\partial x^\alpha$, where

$$\eta(r) = \begin{cases} 1, & \text{if} \quad r \leqslant t' \\ \dfrac{t-r}{t-t'}, & \text{if} \quad t' < r < t \\ 0, & \text{if} \quad t \leqslant r. \end{cases}$$

Below by $O(r^l)$ we denote quantities such that $\left|O(r^l)\right| \leqslant Ar^l$, where A is a positive constant depending only on the geometry of M. Taking the covariant derivative of the vector field ξ we have

$$\xi_\alpha^\gamma = \frac{\partial}{\partial x^\alpha}\left(\eta(|x|)x^\gamma\right) + \Gamma_{\alpha\beta}^\gamma\eta(|x|)x^\beta = \eta(|x|)\delta_\alpha^\gamma + \eta'(|x|)\frac{x_\alpha x^\gamma}{|x|} + \Gamma_{\alpha\beta}^\gamma\eta(|x|)x^\beta.$$

Therefore, by (2.2)

$$\operatorname{div}\xi = \xi_\alpha^\alpha = n\eta(r) + r\eta'(r) + O(r^2), \tag{2.7}$$

where $r = |x|$. Now we calculate the second summand in formula (2.6):

$$\begin{aligned}\langle du\circ\nabla\xi, du\rangle &= \sum_{\ell=1}^N g^{\alpha\beta}\xi_\alpha^\gamma\frac{\partial u^\ell}{\partial x^\beta}\frac{\partial u^\ell}{\partial x^\gamma} = \eta(r)\sum_{\ell=1}^N g^{\alpha\beta}\frac{\partial u^\ell}{\partial x^\alpha}\frac{\partial u^\ell}{\partial x^\beta} \\ &+ \frac{\eta'(r)}{r}\sum_{\ell=1}^N g^{\alpha\beta}\left(\frac{\partial u^\ell}{\partial x^\beta}x_\alpha\right)\left(\frac{\partial u^\ell}{\partial x^\gamma}x^\gamma\right) + \eta(r)\sum_{\ell=1}^N g^{\alpha\beta}\frac{\partial u^\ell}{\partial x^\beta}\frac{\partial u^\ell}{\partial x^\gamma}\Gamma_{\alpha\delta}^\gamma x^\delta \\ &= \eta(r)\left\|du\right\|^2 + I + II.\end{aligned}$$

By (2.2) and (2.3) we have $|II| \leqslant B\left\|du\right\|^2|x|^2$, where the constant $B > 0$ depends only on M, i. e. $II = \left\|du\right\|^2 O(r^2)$. The summand I can be expressed in the following way

$$\begin{aligned}I = \frac{\eta'(r)}{r}\sum_{\ell=1}^N \delta^{\alpha\beta}\left(\frac{\partial u^\ell}{\partial x^\beta}x_\alpha\right)\left(\frac{\partial u^\ell}{\partial x^\gamma}x^\gamma\right) + \frac{\eta'(r)}{r}\sum_{\ell=1}^N(g^{\alpha\beta} - \delta^{\alpha\beta})\left(\frac{\partial u^\ell}{\partial x^\beta}x_\alpha\right)\left(\frac{\partial u^\ell}{\partial x^\gamma}x^\gamma\right) \\ = \eta'(r)r\left|\frac{\partial u}{\partial r}\right|^2 + \left\|du\right\|^2 O(r^3),\end{aligned}$$

where $\partial/\partial r = r^{-1}x^\alpha\partial/\partial x^\alpha$. Finally we have

$$\langle du\circ\nabla\xi, du\rangle = \left\|du\right\|^2\left(\eta(r) + O(r^2)\right) + \eta'(r)\left(r\left|\frac{\partial u}{\partial r}\right|^2 + \left\|du\right\|^2 O(r^3)\right). \tag{2.8}$$

Substituting (2.7) and (2.8) into expression (2.6) and combining with the first variation formula (2.5) we get

$$-2\int\limits_{B_t}\langle\tau(u),du(\xi)\rangle\,dVol_g=(2-n)\int\limits_{B_t}\|du\|^2\left(\eta(r)+O(r^2)\right)dVol_g$$
$$+\frac{1}{t-t'}\int\limits_{B_t/B_{t'}}\left(\|du\|^2\left(r+O(r^3)\right)-2r\left|\frac{\partial u}{\partial r}\right|^2\right)dVol_g.$$

Here $B_t=B_t(x_0)$ and $B_{t'}=B_{t'}(x_0)$ denote the balls at the point x_0 whose radii are t and t' correspondingly. Taking the limit as $t'\to t$ we arrive at the equality

$$-2\int\limits_{B_t}\left\langle\tau(u),du\left(r\frac{\partial}{\partial r}\right)\right\rangle dVol_g=(2-n)\left(1+O(t^2)\right)\int\limits_{B_t}\|du\|^2\,dVol_g$$
$$+\left(t+O(t^3)\right)\int\limits_{\partial B_t}\|du\|^2\,dS-2t\int\limits_{\partial B_t}\left|\frac{\partial u}{\partial r}\right|^2 dS.$$

Let us denote by $\mathfrak{I}$ the integral in the l.h.s. Neglecting the last term in the r.h.s. and assuming that $t\in(0,\bar{r})$, where $\bar{r}\leqslant\rho$ is sufficiently small, we have

$$-2\left(1+O(t^2)\right)|\mathfrak{I}|\leqslant t\frac{d}{dt}E_t^{x_0}(u)-\left(n-2+O(t^2)\right)E_t^{x_0}(u),\tag{2.9}$$

where $(d/dt)E_t^{x_0}(u)=\int_{\partial B_t}\|du\|^2\,dS$. Since u is a solution of (1.2) and the r.h.s. of the equation satisfies (2.1), the following estimate holds:

$$|\mathfrak{I}|\leqslant t\int\limits_{B_t}\|\tau(u)\|\,\|du\|\,dVol_g\leqslant 2tC^0_{\mathsf{G}}E_t^{x_0}(u)+t\int\limits_{B_t}\|\mathsf{g}\|\,\|du\|\,dVol_g\leqslant 2tC^0_{\mathsf{G}}E_t^{x_0}(u)$$
$$+\frac{t}{2}\int\limits_{B_t}\left(\|\mathsf{g}\|^2+\|du\|^2\right)dVol_g\leqslant 2tC^0_{\mathsf{G}}E_t^{x_0}(u)+tE_t^{x_0}(u)$$
$$+\frac{t}{2}\int\limits_{B_t}\|\mathsf{g}\|^2\,dVol_g\leqslant\left(2C^0_{\mathsf{G}}+1\right)tE_t^{x_0}(u)+(C^0_{\mathsf{g}})^2C_*t^{n+1}.$$

Combining this with (2.9) we get

$$-C''t^{n+1}\leqslant t\frac{d}{dt}E_t^{x_0}(u)-\left(n-2-C't\right)E_t^{x_0}(u)$$

for some positive constants C' and C''. Multiplying both sides of the above relation by $e^{C't}$ we get an inequality which is equivalent to the following

$$-C''t^2e^{C't}\leqslant\frac{d}{dt}\left(t^{2-n}e^{C't}E_t^{x_0}(u)\right).$$

Integrating this inequality over the interval (r_1,r_2) and making elementary transformations we get (2.4) as desired. $\square$

3. Proof of Theorem 2

Assume the contrary. Then there exist sequences $\mathsf{G}_l(x,u)$ and $\mathsf{g}_l(x,u)$, $l \in \mathbb{N}$ satisfying (1.3) and a sequence $u_l : M \to M'$, $l \in \mathbb{N}$ of the solutions

$$\tau(u_l)(x) = \mathsf{G}_l(x, u_l(x)) \cdot du_l(x) + \mathsf{g}_l(x, u_l(x)), \quad x \in M,$$

whose energies are bounded $E(u_l) \leqslant C_E$ such that

$$\max_{x \in M} \|du_l(x)\| = \|du_l(x_l)\| = D_l \to \infty, \quad \text{as } l \to \infty.$$

Since M is compact one can suppose that $x_l \to x_0 \in M$. Let $\bar{r} < \lambda_M$ denote the positive constant satisfying the assumptions of Lemma 5. Let us also assume that the Riemannian normal coordinates are chosen in the ball $B_{\bar{r}}(x_0)$. For any $R > 0$ there exists an integer $l_0(R)$ such that for any $l \geqslant l_0(R)$ the map

$$\phi_l : B_R = \{x \in \mathbb{R}^n : |x| \leqslant R\} \to B_{\bar{r}}(x_0), \quad x \mapsto x/D_l + x_l$$

is well-defined. Let us endow B_R with the metric $g_{l,\alpha\beta} = g_{\alpha\beta} \circ \phi_l$. Direct calculations show that the map $w_l = u_l \circ \phi_l : B_R \to M'$ satisfies the equation

$$\tau_l(w_l)(x) = \frac{1}{D_l} \mathsf{G}_l(\bar{x}, w_l) \cdot dw_l(x) + \frac{1}{D_l^2} \mathsf{g}_l(\bar{x}, w_l), \tag{3.1}$$

where $\bar{x} = x/D_l + x_l$ and τ_l denotes the operator of the harmonic mappings of the ball B_R with the metric $g_{l,\alpha\beta}$ into M'. Note that the maps w_l satisfy

$$\|dw_l(x)\| \leqslant 1, \quad \|dw_l(0)\| = 1. \tag{3.2}$$

Moreover, since $g_{l,\alpha\beta} \to \delta_{\alpha\beta}$ as $l \to \infty$ on B_R with all derivatives, the coefficients of the operator τ_l converge with all derivatives to the coefficients of the operator of harmonic mappings of Euclidean ball B_R into M'. Let us show that from the sequence w_l one can extract the subsequence C^2-converging on B_R to a harmonic map of Euclidean ball B_R into M'.

At first we note that for a sufficiently large l the maps w_l are defined on some ball $B_{R+\varepsilon}$, where ε is a positive constant. Viewing w_l as maps $B_{R+\varepsilon} \to B_{\bar{r}}(x_0) \subset \mathbb{R}^n$ we see that they satisfy the system of differential equations given by

$$g_l^{\alpha\beta}(x) \frac{\partial^2 w_l}{\partial x^\alpha \partial x^\beta} = h_l(x), \tag{3.3}$$

where $g_l^{\alpha\beta}(x)$ denotes the tensor which is the inverse of the metric $g_{l,\alpha\beta}(x)$, and $h_l(x)$ denotes the sum of all other summands of (3.1) transfered to the r.h.s. (see the local presentation of the harmonic maps operator given by (1.1)). The operators in the l.h.s. of (3.3) are uniformly elliptic (i. e. the sequence of their constants of ellipticity is bounded) and their coefficients are C^2-bounded. By the inequality in (3.2) the vectors $h_l(x)$ are bounded in the space $W^{0,p}(B_{R+\varepsilon}, \mathbb{R}^n)$ for any $1 < p < \infty$. Combining this with the Shauder estimate (see [4, Theorem 9.11]) we get that the sequence w_l is bounded with respect to the norm of the space $W^{2,p}(B_{R+\varepsilon/2}, \mathbb{R}^n)$. Now inequalities (1.3) (C^1-boundedness of the sequences G_l and g_l) provide the boundedness of $h_l(x)$ in the space $W^{1,p}(B_{R+\varepsilon/2}, \mathbb{R}^n)$ for any $1 < p < \infty$. Again using the Shauder estimate we have the boundedness of the sequence w_l in the space $W^{3,p}(B_{R+\varepsilon/4}, \mathbb{R}^n)$. Since if $p > n$ this space can be compactly embedded into $C^2(B_{R+\varepsilon/4}, \mathbb{R}^n)$, there is a converging subsequence w_{l_k} of the sequence w_l. Proceeding to a limit in equation (3.1) shows that w_{l_k} converges to a harmonic map $v_R : B_R \to M'$.

By direct calculations we have

$$R^{2-n}\int\limits_{B_R}\|dw_l(x)\|^2(\det g_{l,\alpha\beta})^{1/2}(x)dx=\left(\frac{R}{D_l}\right)^{2-n}\int\limits_{B_{\frac{R}{D_l}}}\|du_l\|^2(\det g_{\alpha\beta})^{1/2}(x)dx$$

for any $l\geqslant l_0(R)$. Moreover, by Lemma 5 we get

$$\left(\frac{R}{D_l}\right)^{2-n}E^{x_0}_{\frac{R}{D_l}}(u_l)\leqslant C_1\bar{r}^{2-n}E^{x_0}_{\bar{r}}(u_l)+C_2$$

for any l such that $R/D_l<\bar{r}$. Combining this with the above equality we get

$$R^{2-n}\int\limits_{B_R}\|dw_l(x)\|^2(\det g_{l,\alpha\beta})^{1/2}(x)dx\leqslant 2\left(C_1\bar{r}^{2-n}E^{x_0}_{\bar{r}}(u_l)+C_2\right).$$

Proceeding to a limit over the subsequence l_k finally leads to the following inequality

$$R^{2-n}\int\limits_{B_R}\|dv_R(x)\|^2dx\leqslant 2\left(C_1\bar{r}^{2-n}C_E+C_2\right) \tag{3.4}$$

for any $R>0$.

Now let us choose some positive R_0. In the manner described above one can construct the sequence of mappings $w_{l_k}:B_{R_0}\to B_{\bar{r}}(x_0)$ converging to the harmonic map $v_{R_0}:B_{R_0}\to M'$. Doubling the radius of the ball and using the same procedure for the subsequence u_{l_k} we find the harmonic map $v_{2R_0}:B_{2R_0}\to M'$ such that $v_{2R_0}|_{B_{R_0}}=v_{R_0}$. Continuing this process we finally get the harmonic map $v:\mathbb{R}^n\to M'$.

Our aim now is to show that the map v is an n-obsruction. Conditions (3.2) imply that

$$\|dv(x)\|\leqslant\|dv(0)\|=1.$$

Therefore it is sufficient to check that the quantities $R^{2-n}\int_{B_R}\|dv\|^2$ are uniformly bounded with respect to all $R>0$. We shall show that for any positive R the inequality

$$R^{2-n}\int\limits_{B_R}\|dv(x)\|^2dx\leqslant 2\left(C_1\bar{r}^{2-n}C_E+C_2\right) \tag{3.5}$$

holds. Indeed, for $R=2^mR_0$ this follows from inequality (3.4). For any other R we can find an integer m such that $R<2^mR_0$ and in this case inequality (3.5) follows from its validity for 2^mR_0 and the monotonocity inequality

$$R^{2-n}\int\limits_{B_R}\|dv(x)\|^2dx\leqslant(2^mR_0)^{2-n}\int\limits_{B_{2^mR_0}}\|dv(x)\|^2dx$$

for the harmonic mappings of Euclidean domains of $\mathbb{R}^n$ (see, for instance, [9], [10], [12]).

Hence, the constructed map v is an n-obstruction for M'. This contradiction ends the proof. □

4. Energy boundedness

For a proof of Theorem 1 we will need the statement about the energy boundedness of the solutions of (1.2) (Corollary 6). This follows from the following results in the paper [5]: the energy estimate for the solutions of (1.2) and the version of the Poincaré inequality for the mappings of manifolds. For the completeness of the exposition we state these results below.

Let u and v be C^1-smooth mappings M to M'. We call a C^1-smooth homotopy $H(s,x)$, $s \in [0,1]$, $x \in M$, between u and v geodesic, if for any $x \in M$ the curve

$$[0,1] \ni s \mapsto H_s(x) = H(s,x) \in M'$$

is geodesic in M'. If M' has nonpositive sectional curvature, then for any homotopic maps there is a geodesic homotopy. Indeed, the geodesic homotopy can be obtained by replacing the curve $s \mapsto H_s(x)$ by the unique homotopic geodesic.

For any homotopic C^1-maps u and v one can define the distance function

$$N_p(u,v) = \inf\left\{N_p(H) : H \text{ is a } C^1\text{-homotopy between } u \text{ and } v\right\},$$

where $1 \leqslant p < \infty$ and

$$N_p(H) = \left(\int\limits_M \left(\int\limits_0^1 \left\|\frac{d}{ds}H_s(x)\right\|_{g'} ds\right)^p d\,Vol_g(x)\right)^{1/p}.$$

By Hölder's inequality, for any $1 \leqslant p_1 \leqslant p_2 < \infty$ we have

$$N_{p_1}(u,v) \leqslant N_{p_2}(u,v)\,(\,Vol\,M)^{(p_2-p_1)/p_2p_1}.$$

Proposition 1. *Let M' be a Riemannian manifold of nonpositive sectional curvature. Then for any solution u of equation (1.2) contained in the homotopy class $[v]$*

$$E(u) \leqslant \|\mathsf{g}\|_{C^0} N_1(u,v) + \sqrt{2}\,\|\mathsf{G}\|_{C^0} E(u)^{1/2} N_2(u,v) + E(v).$$

Proof. Let $H(s,x)$ be a geodesic homotopy between u and v. Then, since the sectional curvature is nonpositive, the second variation formula for the energy functional (see [3, p. 28]) impies that

$$\frac{\partial^2}{\partial s^2}E(H_s) \geqslant 0 \text{ for any } s \in [0,1].$$

Therefore the function $s \mapsto E(H_s)$ is convex and we have the inequality

$$E(u) \leqslant E(v) + (\partial/\partial s)|_{s=1}\, E(H_s). \tag{4.1}$$

Since u is a solution of equation (1.2) the first variation formula implies that

$$\left.\frac{\partial}{\partial s}\right|_{s=1} E(H_s) = -\int_M \left.\left\langle \mathsf{G}\cdot du + \mathsf{g},\, \frac{\partial H}{\partial s}\right\rangle\right|_{s=1} d\,Vol_g.$$

Making elementary transformations we get

$$\begin{aligned}\left|\int_M \left\langle \mathsf{g} + \mathsf{G}\cdot du,\, \frac{\partial H}{\partial s}\right\rangle d\,Vol_g\right| &\leqslant \|\mathsf{g}\|_{C^0} N_1(H) + \|\mathsf{G}\|_{C^0}\int_M \sqrt{2e(u)}\left\|\frac{\partial H}{\partial s}\right\| d\,Vol_g \\ &\leqslant \|\mathsf{g}\|_{C^0} N_1(H) + \sqrt{2}\,\|\mathsf{G}\|_{C^0} E(u)^{1/2} N_2(H).\end{aligned}$$

This inequality and (4.1) imply the desired estimate. □

For a proof of the following geometric inequality (the version of the Poincaré inequality) see in [5].

Proposition 2. *Let M' be a Riemannian manifold of nonpositive sectional curvature and $[v]$ be an arbitrary homotopy class of C^1-mappings from M to M'. Then there exists a constant $\bar{C} > 0$ (depending on $[v]$) such that for any C^1-smooth map $u \in [v]$*

$$N_2(u, v) \leqslant \bar{C} \left(E(u)^{1/2} + E(v)^{1/2} + 1 \right). \tag{4.2}$$

Remark 4. Due to results in [6] for manifolds of negative sectional curvature the constant $\bar{C}$ in inequality (4.2) can be chosen independly of the homotopy class.

Corollary 6. *Let M' be a manifold of nonpositive sectional curvature and $[v]$ be an arbitrary homotopy class of C^1-mappings from M to M'. Then for any positive C^0_{g} and $C^0_{\mathsf{G}} < C_{[v]} = (\sqrt{2}\bar{C})^{-1}$ there exists a positive $C_E \geqslant 0$ such that for any solution $u \in [v]$ of equation (1.2) with the r.h.s. satisfying*

$$\|\mathsf{G}\|_{C^0} \leqslant C^0_{\mathsf{G}}, \quad \|\mathsf{g}\|_{C^0} \leqslant C^0_{\mathsf{g}},$$

the estimate $E(u) \leqslant C_E$ holds.

5. Proof of Theorem 1

Following the notation of Theorem 1 let

$$\mathcal{U} = \{(\mathsf{G}, \mathsf{g}) \in \mathcal{F}_{\mathsf{G}} \times \mathcal{F}_{\mathsf{g}} : \|\mathsf{G}(x, u)\| < C_{[v]} \text{ for any } x \in M, u \in M'\}$$

be a domain of the space $\mathcal{F}_{\mathsf{G}} \times \mathcal{F}_{\mathsf{g}}$. Using elliptic regularity arguments one can prove the following simple condition for the map $\pi|_{\pi^{-1}(\mathcal{U})}$ to be proper.

Lemma 7. *Suppose that for any sequence $(u_l, \mathsf{G}_l, \mathsf{g}_l) \in \mathfrak{M}_{[v]}$, $(\mathsf{G}_l, \mathsf{g}_l) \in \mathcal{U}$ such that $(\mathsf{G}_l, \mathsf{g}_l) \to (\mathsf{G}, \mathsf{g}) \in \mathcal{U}$ with respect to the C^s-topology there exists a constant $C > 0$ such that*

$$\max_{x \in M} \|du_l(x)\| \leqslant C \text{ for any } l \in \mathbb{N}. \tag{5.1}$$

Then for problem (1.2) the map $\pi|_{\pi^{-1}(\mathcal{U})}$ is proper.

Proof of Lemma 7. Note that to prove the lemma it is sufficient to show that for any sequence $(u_l, \mathsf{G}_l, \mathsf{g}_l)$ satisfying the conditions above there exists a subsequence of maps u_{i_m} which is converging with respect to the C^{s+1}-topology to the map u such that $(u, \mathsf{G}, \mathsf{g}) \in \mathfrak{M}_{A,[v]}$. Indeed, this implies that in any sequence $(u_l, \mathsf{G}_l, \mathsf{g}_l) \in \pi^{-1}(K)$, where K is a compact subset of $\mathcal{U}$, there is a subsequence converging to some point of $\pi^{-1}(K)$ in the space $C^{s+1}(M, M') \times \mathcal{F}_{\mathsf{G}} \times \mathcal{F}_{\mathsf{g}}$. This means that $\pi^{-1}(K)$ is compact.

Suppose that the sequence $(u_l, \mathsf{G}_l, \mathsf{g}_l) \in \mathfrak{M}_{A,[v]}$ satisfies the suppositions of the lemma. Since the maps u_l satisfy (5.1), then by the Arzela-Ascoli theorem one can extract a subsequence (we also denote it by u_l) which is C^0-converging to some map u. Hence, one can find a finite covering by charts $\{Q_i\}$ of the manifold M and a finite system $\{Q'_i\}$ of charts on M' such that for almost all indices l and for any i the inclusion $u_l(Q_i) \subset Q'_i$ holds. (Without loss of generality one can assume that the closures $\overline{Q}_i$ are compact and contained in some charts of M). Let us denote by $\{U_i\}$ an open covering of the manifold M such that $\overline{U}_i \subset Q_i$. The restrictions of the maps u_l on Q_i viewed as maps $Q_i \to Q'_i \subset \mathbb{R}^{n'}$, where $n' = \dim M'$, satisfy the system of elliptic equations

$$g^{\alpha\beta}(x)\frac{\partial^2 u_l}{\partial x^\alpha \partial x^\beta} = h_l(x), \tag{5.2}$$

where vectors $h_l(x) = (h_l^1(x), \dots, h_l^{n'}(x))$ are given by

$$h_l^\ell(x) = (\mathsf{G}_l)_k^{\ell\gamma}(x, u_l(x))\frac{\partial u_l^k}{\partial x^\gamma} + (\mathfrak{g}_l)^\ell(x, u_l(x)) - g^{\alpha\beta}(x)\Gamma'^{\ell}_{kj}(u_l(x))\frac{\partial u_l^k}{\partial x^\alpha}\frac{\partial u_l^j}{\partial x^\beta} + g^{\alpha\beta}(x)\Gamma^\gamma_{\alpha\beta}(x)\frac{\partial u_l^\ell}{\partial x^\gamma}.$$

Inequalities (5.1) and the uniform boundedness of the sequences G_l and $\mathfrak{g}_l$ imply that the sequence $h_l(x)$ on the r.h.s. of (5.2) is bounded with respect to the norm of $W^{0,p}(Q_i, \mathbb{R}^{n'})$. Now let us denote by $\{V_i^1\}$ an open covering of M such that

$$\overline{U}_i \subset V_i^1 \subset \overline{V}_i^1 \subset Q_i \quad \text{for any} \quad i.$$

Applying the Shauder estimate (as in the proof of Theorem 2) we get the boundedness of the sequence u_l in the space $W^{2,p}(V_j^1, \mathbb{R}^{n'})$. Now C^1-boundedness of the sequences G_l and $\mathfrak{g}_l$ implies that the sequence of vectors $h_l(x)$ is bounded in $W^{1,p}(V_j^1, \mathbb{R}^{n'})$. Covering M by open sets $\{V_i^2\}$ such that

$$\overline{U}_i \subset V_i^2 \subset \overline{V}_i^2 \subset V_i^1,$$

and again applying the Shauder estimate on the domains V_i^1, we see that the sequence u_l is bounded in $W^{3,p}(V_i^2, \mathbb{R}^{n'})$. Perfoming this procedure the sufficient number of times we obtain the $W^{s+2,p}(U_i, \mathbb{R}^{n'})$-boundedness of the sequence u_l. Recall that if $p > n$, the space $W^{s+2,p}(U_i, \mathbb{R}^{n'})$ is compactly emedded into $C^{s+1}(U_i, \mathbb{R}^{n'})$ and that the finite system $\{U_i\}$ is an open covering of M. Therefore there exists a subsequence u_{l_m} of the sequence u_l which converges with respect to the C^{s+1}-topology to the map u. Proceeding to a limit in equations (5.2) we get that $(u, \mathsf{G}, \mathfrak{g})$ belongs to the space $\mathfrak{M}_{A,[v]}$. This ends the proof of the lemma. □

Thus for a proof of the theorem it is sufficient to show that for any sequence $(u_l, \mathsf{G}_l, \mathfrak{g}_l)$ satisfying the conditions of Lemma 7 there exists a constant $C > 0$ such that relation (5.1) holds. Let us choose positive $C^0_{\mathfrak{g}}$ and $C^0_{\mathsf{G}} < C_{[v]}$ such that

$$\|\mathsf{G}_l\|_{C^0} \leqslant C^0_{\mathsf{G}}, \quad \|\mathfrak{g}_l\|_{C^0} \leqslant C^0_{\mathfrak{g}} \quad \text{for any } l \in \mathbb{N}.$$

By Corollary 6 there exists $C_E \geqslant 0$ such that for any $l \in \mathbb{N}$ the inequality $E(u_l) \leqslant C_E$ holds. Since the sequence $(\mathsf{G}_l, \mathfrak{g}_l)$ converges to some $(\mathsf{G}, \mathfrak{g})$ with respect to the C^2-topology, then, in particular, it is C^1-bounded. Since manifolds of nonpositive sectional curvature does not admit any n-obstructions for $n \geqslant 3$ (see the example in Section 1), Theorem 2 gives us the desired estimate. □

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