Quasi-linear elliptic differential equations for mappings of manifolds, II

Gerasim Kokarev
School of Mathematics, The University of Edinburgh
King's Bulidings, Mayfield Road, Edinburgh EH9 3JZ, UK
E-mail: G.Kokarev@ed.ac.uk

Sergei Kuksin
School of Mathematical and Computer Sciences,
Heriot-Watt University, Edinburgh EH14 4AS, UK;
Steklov Mathematical Institute, St. Gubkina 8, Moscow 117966, Russia
E-mail: S.B.Kuksin@ma.hw.ac.uk

Abstract

We study questions related to the orientability of the infinite-dimensional moduli spaces formed by solutions of elliptic equations for mappings of manifolds. The principal result states that the first Stiefel-Whitney class of such a moduli space is given by the $\mathbb{Z}_2$-spectral flow of the families of linearised operators. Under an additional compactness hypotheses, we develop elements of Morse-Bott theory and express the algebraic number of solutions of a non-homogeneous equation with a generic right-hand side in terms of the Euler characteristic of the space of solutions corresponding to the homogeneous equation. The applications of this include estimates for the number of homotopic maps with prescribed tension field and for the number of the perturbed pseudoholomorphic tori, sharpening some known results.

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Contents

0. Introduction

0.1. Description and aims of the paper

In this work we continue to study elliptic differential equations for mappings of closed manifolds and the moduli spaces of their solutions. Namely, for mappings $u : M \to M'$ we consider the equation

$$Au(x) + \mathsf{f}(x, u(x)) = 0, \qquad x \in M, \tag{0.1}$$

where $Au(x) \in T_{u(x)}M'$ is an elliptic differential operator and f belongs to a Banach space $\mathcal{F}$ formed by sufficiently smooth non-autonomous vector fields. For a fixed homotopy class $[v]$ of mappings $M \to M'$ the *universal moduli space* $\mathfrak{M}_{A,[v]}$ is defined as a collection of pairs (u, f) such that $\mathsf{f} \in \mathcal{F}$ and $u \in [v]$ is a (classical) solution of equation (0.1). Basic properties of this infinite-dimensional moduli space are discussed in the preceding paper [18]. Now we assume that the linearised operators

$$A_*(u) + \mathsf{f}_*(\cdot, u), \tag{0.2}$$

which are elliptic differential operators on the bundle u^*TM', have zero index. Then $\mathfrak{M}_{A,[v]}$ is a Banach manifold modelled on the space $\mathcal{F}$. Moreover, it admits a *quasi-finite* *dimensional structure* – an atlas whose transition functions have the form $\mathrm{Id} + K$, where K is a finite-dimensional map. Besides, the natural projection π_A

$$\mathfrak{M}_{A,[v]} \ni (u, \mathsf{f}) \longmapsto \mathsf{f} \in \mathcal{F}, \tag{0.3}$$

whose fiber $\pi_A^{-1}(\mathsf{f})$ is a space of homotopic solutions of equation (0.1), is also of this form.

The purpose of this paper is to study the orientability of the moduli space $\mathfrak{M}_{A,[v]}$, calculate the degree of the projection π_A, and describe related examples. Similar questions were raised and exploited in a number of works during the last years. For example, in the literature on symplectic geometry (see [9, 25] and references therein) a related analysis heavily uses almost complex structures on the involved manifolds as well as the specific type of equations. In series of papers by Tromba [32, 33] the orientability of similar moduli spaces is proved for the class of variational equations for vector-valued functions. In this case the situation is simplified due to the linear structure on the space of maps $M \to M'$ and the variational nature of the problem.

Here we present an analysis for rather general elliptic equations for mappings of manifolds. Our first result is a calculation of the obstruction to the orientability of $\mathfrak{M}_{A,[v]}$, i.e. its first Stiefel-Whitney class, for elliptic differential equations of type (0.1). More precisely, under some natural hypotheses, we prove the following result:

Statement A (Theorem 3.1). *The first Stiefel-Whitney class of the universal moduli space $\mathfrak{M}_{A,[v]}$ coincides with the first cohomology class given by the values of the $\mathbb{Z}_2$-spectral flows of linearised operators* (0.2)*.*

The notion of the spectral flow is explained in the preliminary Sections 1 and 2; it is similar to that in [25, 27].

We describe a number of situations when the first Stiefel-Whitney class vanishes and the corresponding moduli space $\mathfrak{M}_{A,[v]}$ is orientable. This, for example, occurs when the target manifold is almost complex and the linearised operators $A_*(u)$ are complex-linear.

An orientation on the moduli space $\mathfrak{M}_{A,[v]}$ allows to define an algebraic number of solutions of equation (0.1) for a generic vector field $\mathsf{f} \in \mathcal{F}$ – each solution enters into the sum with a sign '+' or '−', depending on whether the linearised map $d(\pi_A)_*(u, \mathsf{f})$ preserves the orientation or not. The index of such a solution is a certain integer $\mathrm{ind}(u, \mathsf{f})$ such that the sign of u coincides with $(-1)^{\mathrm{ind}(u,\mathsf{f})}$. We give a description of the index in terms of spectral properties of the linearised operators. In certain cases, for example, when the linearised operators are bounded from below, it coincides with the Morse-type index – the number of real negative eigenvalues.

Assume that equation (0.1) has the compactness property, i.e. the projection π_A is proper. (This is a strong assumption; some examples are described in the last sections.) Then for a generic f the number of solutions is finite and their algebraic number equals the degree of the projection π_A. The calculation of this degree is another problem addressed in the paper. For this we develop elements of Morse-Bott theory; see [20] for its $\mathbb{Z}_2$-version and [16, 23] for the related results in subsequent examples. More precisely, under an appropriate non-degeneracy condition on the space $\mathfrak{S}_{A,[v]}$ formed by solutions of the homogeneous equation

$$Au(x) = 0, \qquad x \in M,$$

we show that there exists its orientable neighbourhood in the universal moduli space. The degree of π_A is, then, calculated using special perturbations of the homogeneous equation.

Statement B (Theorem 6.2). *Suppose that the compactness hypothesis holds and the space $\mathfrak{S}_{A,[v]}$ is strongly non-degenerate in the sense of Morse-Bott. Suppose also that the evaluation map $M \times \mathfrak{S}_{A,[v]} \to M \times M'$, $(x, u) \mapsto (x, u(x))$ is an immersion. Then the degree of π_A – the algebraic number of solutions of equation* (0.1) *for a generic non-autonomous vector field* f *– is given by the algebraic sum of the Euler-Poincaré numbers of the connected components of the space $\mathfrak{S}_{A,[v]}$.*

In Section 6 we prove Statement B for some special case which, however, covers all essential examples and allows to construct explicitly a perturbation which computes the degree of π_A. The details for the general case will appear elsewhere; the general $\mathbb{Z}_2$-version is sketched in [20].

Sections 7–9 are devoted to some examples: first order ordinary differential equations on manifolds, perturbed harmonic map equation, and semi-linear Cauchy-Riemann equations. The results there are related to a number of questions interesting on its own. Here we highlight some of them.

Example C (Section 7). The classical Lefschetz formula implies that the algebraic number of 1-periodic solutions of the equation

$$\frac{d}{dt}u(t) = \mathsf{f}(t, u(t)), \qquad t \in \mathbb{R}/\mathbb{Z},$$

for a generic 1-periodic in t vector field f on M', is equal to the Euler-Poincaré number $\chi(M)$. Our theory implies a finer statement (which can also be proved by Neilsen-Wecken theory of fixed point classes): *the algebraic number of* 1*-periodic solutions within a fixed homotopy class is equal to $\chi(M)$ or zero with respect to the cases the class is trivial or not.* Moreover, the solutions in the algebraic sums are counted with the same index as in the Lefschetz formula if the homotopy class is orientable (i.e. the pull-back bundle u^*TM' is trivial) and with the opposite sign if the class is not orientable. The general theory also implies the existence of a contractible 1-periodic solution when the Euler-Poincaré number does not vanish. As an application of this one can prove a weak version of the classical Gottlieb theorem [10]: the evaluation map

$$\mathrm{ev}_u : \mathit{Diff}(M') \to M', \qquad \phi \mapsto \phi(u),$$

where $u \in M'$, induces the trivial homomorphism on the fundamental groups provided $\chi(M') \neq 0$. In the recent preprint [21] the developed here Morse-Bott theory is used to establish a similar vanishing theorem for certain 2-cycles in the diffeomorphism group.

Example D (Section 8). Let M and M' be Riemannian manifolds and $\tau(u)$ be a harmonic map operator; the equation $-\tau(u) = 0$ is the Euler-Lagrange equation for the energy functional

$$E(u) = \frac{1}{2}\int_M \|du(x)\|^2 \, d\,Vol_M(x), \qquad x \in M.$$

The vector field $\tau(u)(x) \in T_{u(x)}M'$, where $x \in M$, along a map u is called the tension field and coincides with the mean curvature vector when u is a Riemannian immersion. Theorem 8.2 shows that *if the target manifold M' has negative sectional curvature or is a locally symmetric space of non-positive sectional curvature, then for a generic non-autonomous vector field* f *the number of homotopic maps $u(x)$ with the prescribed tension field $\mathsf{f}(x, u(x))$ is finite and their algebraic number is equal to the Euler-Poincaré number of the space formed by harmonic maps within the same homotopy class.* In particular, when the domain manifold is a circle $\mathbb{R}/\mathbb{Z}$ we have the following analogue of the Lefschetz theorem in the previous example: *suppose that M' is as above, then for a generic vector field f the number of 1-periodic curves with the prescribed geodesic curvature $\mathsf{f}(t, u(t))$ within a fixed homotopy class is equal to the Euler-Poincaré number $\chi(M)$ or zero with respect to the cases the homotopy class is trivial or not.*

These results for negatively curved manifolds were previously announced in [15] and independently treated in [16]. Note also that, as is pointed out in [22], the main statement (Theorem 8.2) holds for arbitrary non-positively curved manifolds M'; i.e. the conditions which guarantee the Morse-Bott non-degeneracy of the space of homotopic harmonic mappings are unnecessary. Moreover, if the perturbation f is gradient, then Morse inequalities for the number of the corresponding solutions hold.

Example E (Section 9). Let M' be an almost Kähler manifold, i.e. endowed with an almost complex structure J and a symplectic form ω such that ω is J-invariant and tames J. Denote by S the infimum of the energies of J-holomorphic spheres in M'. For maps $u : \mathbb{T}^2 \to M'$ consider the perturbed Cauchy-Riemann equation

$$\frac{\partial}{\partial \bar{z}} u(z) = \mathsf{f}(z, u(z)), \qquad z \in \mathbb{T}^2. \tag{0.4}$$

In Section 9 we show that *if M' satisfies the generalised Einstein condition, the first Chern class is a non-zero multiple of ω, $[c_1] = k[\omega]$, then for a generic vector field f such that $\int \max_u \|\mathsf{f}(z, u)\|^2 < (S/2)$ the algebraic number of isolated solutions of equation (0.4) within a fixed homotopy (homology) class is equal to $\chi(M')$ or zero with respect to the cases the class is trivial or not.*

Conventions. Throughout the text we follow the notation in the paper [18]. In particular, M and M' always denote connected smooth manifolds of dimensions n and n' respectively and without boundary. Moreover, M is assumed to be compact. By g and g' we denote Riemannian metrics on M and M' respectively. We adopt that smoothness means C^∞-smoothness, unless there is an explicit statement to the contrary.

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0.2. Basic notation

Here we recall the notation and some results of the paper [18].

Let A be a quasi-linear elliptic differential operator of order $k > 0$ defined on mappings of M to M'; for definitions see [18, Section 2] . Let us denote by f the system of coefficients of a differential operator whose order is not greater than $k-1$. For mappings $u : M \to M'$ of a fixed homotopy class $[v]$ consider the differential equation

$$Au(x) + \mathsf{f}(j^{k-1}u(x)) = 0, \qquad x \in M. \tag{0.5}$$

Here we suppose that f belongs to some reflexive Banach space $\mathcal{F}$ which is contained in the space formed by systems of C^s-smooth ($s > n/2 + 2$) coefficients of differential operators whose order is not greater than $k - 1$. For example, one can suppose that $\mathcal{F}$ is formed by the coefficients of such operators which are sufficiently smooth in the Sobolev sense; e.g. $W^{p,s+1}$-smooth, where p is greater than the dimension of $J^{k-1}(M, M')$.[1] The solutions of (0.5) are supposed to be H^m-smooth, where $m = s + k - 3$. The space

$$\mathfrak{M}_{A,[v]} = \{(u, \mathsf{f}) : u \in H^m(M, M'),\ u \in [v],\ \mathsf{f} \in \mathcal{F},\ \text{and } (u, \mathsf{f}) \text{ satisfies (0.5)}\} \tag{0.6}$$

with the topology induced by the inclusion

$$\mathfrak{M}_{A,[v]} \subset C^{m+2}(M, M') \times \mathcal{F}$$

is called the *universal moduli space* of solutions (within the homotopy class $[v]$) corresponding to problem (0.5). The map $\pi_A : \mathfrak{M}_{A,[v]} \to \mathcal{F}$ acting by the rule (0.3) is called the *natural projection*.

The linearised operator $A_*(u) + \mathsf{f}_*(j^{k-1}u)$ is an elliptic differential operator on the bundle u^*TM'. It is defined with respect to a fixed connection on M'; for details see [18, Section 3]. As is known [28] its index is a homotopy invariant and, in particular, does not depend on $u \in [v]$. We denote this index by $\operatorname{ind}_{[v]} A$ and suppose that it vanishes from now on.

Under some suppositions on the space of parameters $\mathcal{F}$ in [18, Theorem 5.1] it is proved that the moduli space $\mathfrak{M}_{A,[v]}$ is a C^2-smooth submanifold of $H^m(M, M') \times \mathcal{F}$ and the projection π_A is a Fredholm map of the same smoothness. Moreover, $\mathfrak{M}_{A,[v]}$ admits a structure of the quasi finite-dimensional Banach manifold [18, Theorem 5.5], i.e. there exists an atlas whose transition functions have the form $\operatorname{Id} + K$ where K is a finite-dimensional map.

A quasi finite-dimensional Banach structure is a particular case of the so-called Fredholm structure, see [7, 8], and [4], i.e. a class of atlases whose transition functions have the form $\operatorname{Id} + K$ with a compact operator K. There is a natural notion of orientability for Fredholm manifolds. More precisely, a Fredholm atlas is called oriented, if for any intersecting charts the differential of the transition map belongs to the connected component of the identity in the space of linear isomorphisms of the form $\operatorname{Id} + K$. In other words, one can say that the Fredholm structure is orientable if its first Stiefel-Whitney class vanishes; see details in [4].

In Section 3 we prove that the universal moduli space $\mathfrak{M}_{A,[v]}$ is orientable under some suppositions on the spectra of linearised operators $A_*(u) + \mathsf{f}_*(j^{k-1}u)$.

1. Preliminaries I

In this section we recall some well-known facts and introduce the notation to be used in sequel.

1.1. The spectrum of a linear elliptic differential operator

Let A_* be a linear elliptic differential operator of order k (with sufficiently smooth coefficients) on a real vector bundle V over a closed manifold M. Let us denote by A_*^c the complexified differential operator on the complexified vector bundle $V \otimes \mathbb{C}$. The operator A_*^c defines a linear Fredholm operator (also denoted by A_*^c) of the Sobolev space $H^k(V \otimes \mathbb{C})$ to $L_2(V \otimes \mathbb{C})$; see, for example, [28, 30]. The spectrum $\sigma(A_*^c)$ is a subset of the complex plane that can be defined in the following way: $\lambda \notin \sigma(A_*^c)$ if and only if the operator $A_*^c - \lambda \operatorname{Id}$ has a bounded inverse operator defined on $L_2(V \otimes \mathbb{C})$. The following alternative holds; for the case of operators on functions see [30, Chapter 1].

[1]the actual order of smoothness is not important – only the fact that $\mathcal{F}$ is a reflexive Banach space is used.

Proposition 1.1. *Let A_* be a linear elliptic differential operator of order $k > 0$. Then one of the following possibilities holds:*

(i) $\sigma(A_*^c) = \mathbb{C}$ *(this is, in particular, so if* $\operatorname{ind} A_*^c \neq 0$*);*

(ii) $\sigma(A_*^c)$ *is a discrete (possibly, empty) subset of the complex plane* $\mathbb{C}$ *and any spectral point is an eigenvalue of finite multiplicity.*

The set $\sigma(A_*^c)$ is called the complex spectrum of an operator A_*. In particular, for any $\lambda \in \mathbb{R}$ the operator $A_* - \lambda \operatorname{Id} : H^k(V) \to L_2(V)$ has a bounded inverse if and only if $\lambda \notin \sigma(A_*^c)$. Note also that since the operator A_*^c commutes with the complex conjugation in $V \otimes \mathbb{C}$, the spectrum $\sigma(A_*^c)$ is symmetric with respect to the real line. Below by the multiplicity of $\lambda \in \sigma(A_*^c)$ we mean its (complex) algebraic multiplicity as an eigenvalue of the operator A_*^c; see [17]. In particular, if the vector bundle V is endowed with an almost complex structure and A_* is a complex linear differential operator (i.e., A_* commutes with the almost complex structure), then by our definition the multiplicity of its any eigenvalue is even.

Suppose that V is endowed with a metric and its base M with a measure whose density is smooth. This introduces a scalar product on the space of sections of V. (In sequel we always assume that a scalar product on the space of sections under consideration is given in such a way). If a linear elliptic operator A_* is formally self-adjoint with respect to this scalar product, then the spectrum $\sigma(A_*^c)$ is real, and consists of eigenvalues $\{\lambda_j\}$, $j = 1, 2, \ldots$ of finite multiplicity, and $|\lambda_j| \to +\infty$ as $j \to +\infty$.

The resolvent set of A_*^c is a complement of the spectrum $\sigma(A_*^c)$. In sequel we use the following lemma.

Lemma 1.2. *Let A_* be an elliptic differential operator of order $k > 0$ with smooth coefficients. Suppose that* $\operatorname{ind} A_*$ *vanishes and zero is an eigenvalue without adjoint vectors, i.e. its algebraic multiplicity equals the dimension of* $\operatorname{Ker} A_*$*. Then the resolvent set of A_* is not empty.*

Sketch of the proof. Denote by $\operatorname{Im} A_*$ the image of the operator A_* acting on smooth sections of a vector bundle V. Due to the Fredholm theorem [28, Chapter XI] the spaces $\operatorname{Ker} A^*$ (the kernel of a formally adjoint operator) and $\operatorname{Im} A_*$ are orthogonal with respect to the L_2-product and span the space of all smooth sections. Since there are no adjoint vectors for the zero eigenvalue, $\operatorname{Ker} A_*$ and $\operatorname{Im} A_*$ intersect trivially and, since $\operatorname{ind} A_* = 0$, are complementary. This implies that spaces $\operatorname{Ker} A_*^c$ and $\operatorname{Im} A_*^c$ are transversal in the space of all smooth sections of $V \otimes \mathbb{C}$.

Now denote by $\mathcal{W}$ and $\mathcal{H}$ the closures of $\operatorname{Im} A_*^c$ in H^k- and L_2-topologies respectively. Clearly, the identity induces a compact embedding $\mathcal{W} \subset \mathcal{H}$ as a dense subset and the operator

$$D = A_*^c|_{\mathcal{W}} : \mathcal{W} \longrightarrow \mathcal{H}$$

is an isomorphism. Moreover, D is a closed operator $\mathcal{H} \to \mathcal{H}$ whose domain of definition is $\mathcal{W}$ and has a non-empty resolvent set. Since

$$A_*^c - \lambda \operatorname{Id} = (D - \lambda \operatorname{Id}) \oplus (-\lambda) \operatorname{Id}_{\operatorname{Ker} A_*^c},$$

the resolvent set of D without 0 coincides with the resolvent set of A_*^c. Since the former is non-empty, so is the latter. □

1.2. The spectral flow

Here we recall properties of the spectral flow for families of certain Fredholm operators. For the self-adjoint case the best for our purposes reference is [27]; see also [25, Appendix A.2] for some relevant facts on the $\mathbb{Z}_2$-spectral flow (and its relation to the determinant line bundles).

Let $\mathcal{W}$ and $\mathcal{H}$ be separable real Hilbert spaces and $\mathcal{W}$ be compactly embedded into $\mathcal{H}$ as a dense subset. Below by $\mathcal{L}(\mathcal{W},\mathcal{H})$ we mean the vector space of linear bounded operators from $\mathcal{W}$ to $\mathcal{H}$. Further, let us denote by $\mathcal{F}^0 = \mathcal{F}^0(\mathcal{W},\mathcal{H})$ the set of linear bounded Fredholm operators $\mathcal{W} \to \mathcal{H}$ of zero index and let $\mathcal{F}^0_* = \mathcal{F}^0_*(\mathcal{W},\mathcal{H})$ be a set formed by all operators $A_* \in \mathcal{F}^0$ such that:

- A_* defines a closed operator $\mathcal{H} \to \mathcal{H}$ whose domain of definition is equal to $\mathcal{W}$;
- the resolvent set of the complexified operator $A^c_* : \mathcal{H} \otimes \mathbb{C} \to \mathcal{H} \otimes \mathbb{C}$ is not empty.

It is known [17, Chapter IV] that $\mathcal{F}^0_*$ is an open subset of $\mathcal{F}^0$. Denote by $\Gamma(\mathcal{W},\mathcal{H})$ the space of continuous paths

$$[0,1] \ni t \mapsto A_*(t) \in \mathcal{F}^0_*(\mathcal{W},\mathcal{H}).$$

By $\Omega(\mathcal{W},\mathcal{H})$ and $\Omega^1(\mathcal{W},\mathcal{H})$ we mean respectively the spaces of continuous and continuously differentiable (in the norm topology) paths from $\Gamma(\mathcal{W},\mathcal{H})$ such that the ends $A_*(0)$ and $A_*(1)$ are invertible.

Proposition 1.3. *Let $\mathcal{F}^0_k$, where $k \in \mathbb{N}$, be a subset of $\mathcal{F}^0$ formed by operators D such that the dimension of* $\operatorname{Ker} D$ *equals k. Then $\mathcal{F}^0_k$ is a submanifold of $\mathcal{F}^0$ of codimension k^2, whose tangent space $T_D\mathcal{F}^0_k$ can be identified with the space*

$$\{P \in \mathcal{L}(\mathcal{W},\mathcal{H}) : Px \in \operatorname{Im} D \ \forall x \in \operatorname{Ker} D\}. \tag{1.1}$$

Sketch of the proof. Let D be a point from $\mathcal{F}^0_k$. The spaces $\mathcal{W}$ and $\mathcal{H}$ can be decomposed into the direct sums

$$\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2 \quad \text{and} \quad \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$$

such that $\mathcal{W}_1 = \operatorname{Ker} D$ and $D_{22} = D|_{\mathcal{W}_2}$ is an isomorphism onto $\mathcal{H}_2$. Let M be a sufficiently small operator and denote by M_{ij}, $i, j \in 1, 2$, its components with respect to the decompositions above. Then the dimension of $\operatorname{Ker}(D + M)$ is equal to k if and only if

$$M_{11} = M_{12} \circ (D_{22} + M_{22})^{-1} \circ M_{21}.$$

This gives a parametrisation of a neighbourhood of D in $\mathcal{F}^0_k$ in terms of M_{12}, M_{21}, and M_{22}. Clearly, it has codimension k^2 in the space $\mathcal{F}^0$. The assertion that the tangent space can be identified with the space (1.1) follows straightforward. □

By $\overline{\mathcal{F}^0_1}$ we denote the closure of $\mathcal{F}^0_1$, which coincides with the disjoint union $\cup_{k \geqslant 1} \mathcal{F}^0_k$. For a path $A_*(t)$ from $\Omega^1(\mathcal{W},\mathcal{H})$ the intersection number modulo 2 of $A_*(t)$ with the cycle $\overline{\mathcal{F}^0_1}$ is well-defined. Indeed, due to the transversality theory in [1, Chapter 4] one can show that the set of paths transverse to all $\mathcal{F}^0_k$ is open and dense in $\Omega^1(\mathcal{W},\mathcal{H})$ and that the set of homotopies transverse to all $\mathcal{F}^0_k$ is dense. By standard arguments (see, e.g., [27, Remark 4.24]) this intersection number function extends to $\Omega(\mathcal{W},\mathcal{H})$.

Definition 1.1. By the *spectral flow* $\mu_2(A_*)$ along a path $A_*(t)$ from $\Omega(\mathcal{W},\mathcal{H})$ we call the intersection number mod 2 of $A_*(t)$ with the cycle $\overline{\mathcal{F}^0_1}$ formed by degenerate Fredholm operators of zero index.

Thus, the spectral flow μ_2 is a function defined on $\Omega(\mathcal{W}, \mathcal{H})$ and taking values in $\mathbb{Z}_2$ such that:

(*i*) μ_2 is constant on connected components of $\Omega(\mathcal{W}, \mathcal{H})$;

(*ii*) μ_2 is equal to zero on constant paths;

(*iii*) μ_2 has the catenation property, i.e. if paths $A^1_*(t)$ and $A^2_*(t)$ are such that $A^1_*(1) = A^2_*(0)$, then for the path

$$A^1_* \# A^2_*(t) = \begin{cases} A^1_*(2t), & 0 \leqslant t \leqslant \frac{1}{2} \\ A^2_*(2t-1), & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

the following relation holds:

$$\mu_2(A^1_* \# A^2_*) = \mu_2(A^1_*) + \mu_2(A^2_*) \mod 2.$$

To work with the spectral flow it is useful to introduce the following notation. Let $A_*(t)$ be a continuously differentiable path from $\Omega(\mathcal{W}, \mathcal{H})$. We say that a point $t_0 \in (0,1)$ is a crossing (of the cycle $\overline{\mathcal{F}^0_1}$) if the operator $A_*(t_0)$ has a non-trivial kernel. A crossing t_0 is called regular if

$$\nu \in \operatorname{Ker} A_*(t_0) \text{ and } \dot{A}_*(t_0)\nu \in \operatorname{Im} A_*(t_0) \text{ implies that } \nu = 0.$$

According to (1.1) this condition means that the derivative $\dot{A}_*(t_0)$ does not belong to the tangent cone to the cycle $\overline{\mathcal{F}^0_1}$ at the point $A_*(t_0)$. In particular, this implies that any regular crossing is isolated. A crossing is called simple if it is regular and $\dim \operatorname{Ker} A_*(t_0) = 1$. Due to the definition of the spectral flow if $A_*(t)$ has only simple crossings, then $\mu_2(A_*)$ is equal to the parity of the number of crossings. More generally, as is shown in [27] and [25, Appendix A.2] for a path with only regular crossings the spectral flow is given by the formula

$$\mu_2(A_*) = \sum_t \dim \operatorname{Ker} A_*(t) \mod 2,$$

where the sum is taken over all crossings t. Below we give a different formula (Corollary 1.6) for the spectral flow.

Let $A^0_*(t)$ and $A^1_*(t)$ be paths from $\Omega(\mathcal{W}, \mathcal{H})$ that are homotopic with free ends as paths in $\Gamma(\mathcal{W}, \mathcal{H})$. Let us denote by $H(t,\tau)$, where t and $\tau \in [0,1]$, a corresponding homotopy such that $H(t,0) = A^0_*(t)$ and $H(t,1) = A^1_*(t)$. By $\bar{A}^0_*(\tau)$ and $\bar{A}^1_*(\tau)$ we denote the paths of the end points under this homotopy, i.e. the paths given by

$$\bar{A}^0_*(\tau) = H(0,\tau), \quad \bar{A}^1_*(\tau) = H(1,\tau), \quad \tau \in [0,1]. \tag{1.2}$$

Since the paths $A^0_* \# \bar{A}^1_*(t)$ and $\bar{A}^0_* \# A^1_*(t)$ are homotopic in $\Omega(\mathcal{W}, \mathcal{H})$, by properties (*i*) and (*iii*) we arrive at the following statement.

Lemma 1.4. *Suppose that paths $A^0_*(t)$ and $A^1_*(t)$ belong to $\Omega(\mathcal{W}, \mathcal{H})$ and are homotopic in the space $\Gamma(\mathcal{W}, \mathcal{H})$. Let H be a corresponding homotopy and denote by $\bar{A}^0_*(\tau)$ and $\bar{A}^1_*(\tau)$ the paths given by (1.2). Then we have the following relation*

$$\mu_2(A^0_*) + \mu_2(A^1_*) = \mu_2(\bar{A}^0_*) + \mu_2(\bar{A}^1_*) \mod 2.$$

In particular, the spectral flows along any closed homotopic paths are equal.

In sequel we often refer to the following auxiliary statements.

Lemma 1.5. *Let $A_*(t)$ be a path from $\Gamma(\mathcal{W},\mathcal{H})$ and λ and μ be real numbers such that the paths $A_*(t)-\lambda\,\mathrm{Id}$ and $A_*(t)-\mu\,\mathrm{Id}$ belong to $\Omega(\mathcal{W},\mathcal{H})$. Then we have*

$$\mu_2(A_*-\lambda\,\mathrm{Id})+\mu_2(A_*-\mu\,\mathrm{Id})=\nu_0(\lambda,\mu)+\nu_1(\lambda,\mu)\mod 2, \tag{1.3}$$

where $\nu_0(\lambda,\mu)$ and $\nu_1(\lambda,\mu)$ denote the number of real eigenvalues (taking into account their multiplicities) between λ and μ of operators $A_*(0)$ and $A_*(1)$ respectively.

Proof. First, suppose that the operators $A_*(0)$ and $A_*(1)$ are such that all their real eigenvalues between λ and μ are simple. In this case relation (1.3) follows from Lemma 1.4 and the definition of the spectral flow. Indeed, let us join the curves $A_*(t)-\lambda\,\mathrm{Id}$ and $A_*(t)-\mu\,\mathrm{Id}$ by the homotopy

$$H(t,\tau)=A_*(t)-((1-\tau)\lambda+\tau\mu)\,\mathrm{Id},\quad t,\tau\in[0,1].$$

It follows straightforward that the corresponding paths $\bar{A}^0_*(\tau)$ and $\bar{A}^1_*(\tau)$, given by (1.2), have only simple crossings which coincide with the eigenvalues between λ and μ of $A_*(0)$ and $A_*(1)$ respectively.

To demonstrate the general case we use the following observation. Let $D\in\mathcal{F}^0_*$ be an operator with real eigenvalues $\lambda_1,\ldots,\lambda_\ell$ situated between λ and μ. Then D can be deformed in $\mathcal{F}^0_*$ to an operator with simple real eigenvalues between λ and μ. The deformation can be chosen through operators whose spectra outside of the interval with the ends λ and μ is the same; in particular, the spectral flow along this deformation is equal to zero. To see this recall that one can decompose D into the sum of a finite-dimensional operator and an operator which does not have $\lambda_1,\ldots,\lambda_\ell$ as its eigenvalues [17, Chapter 3]. Now the deformation is given by the deformation of the finite-dimensional part to an operator with simple eigenvalues. This can be done such that at any time the eigenvalues always stay between λ and μ.

Applying this observation to the operators $A_*(0)$ and $A_*(1)$ we reduce the situation to the case of simple eigenvalues between λ and μ. Combining this with the discussion in the beginning we prove the lemma. □

As a consequence of Lemma 1.5 and the property (*iii*) we are able to give a general formula for the spectral flow of a curve with arbitrary crossings in terms of their algebraic multiplicities.

Corollary 1.6. *Let $A_*(t)$ be a path from $\Gamma(\mathcal{W},\mathcal{H})$. Suppose that a given partition $0=t_0<t_1<\ldots<t_{\ell+1}=1$ of the interval $[0,1]$ and a collection of real numbers $\lambda_0,\ldots,\lambda_\ell$ are such that $\lambda_i\notin\sigma(A^c_*(t))$ for any $t\in[t_i,t_{i+1}]$, where $i=1,\ldots,\ell$. Then for any $\lambda\in\mathbb{R}$ such that $A_*(t)-\lambda\,\mathrm{Id}$ belongs to $\Omega(\mathcal{W},\mathcal{H})$ we have the relation*

$$\mu_2(A_*-\lambda\,\mathrm{Id})=\nu_0(\lambda,\lambda_0)+\nu_1(\lambda,\lambda_\ell)+\sum_i\nu_{t_i}(\lambda_{i-1},\lambda_i)\mod 2,$$

where $\nu_{t_i}(\lambda_{i-1},\lambda_i)$ denotes the number of real eigenvalues (taking into account their multiplicities) of $A_*(t_i)$ between λ_{i-1} and λ_i.

2. Preliminaries II

Here we consider families of Fredholm operators depending on a parameter which belongs to some topological space. We discuss properties of this parameter space determined by the behaviour of the spectra of those Fredholm operators. First, we describe the necessary framework and introduce notions needed to state the main result (Theorem 3.1) in Section 3. The second subsection is devoted to the results which will be used for the construction of an oriented atlas on the moduli space of solutions.

2.1. Characteristics of sets related to spectral properties of Fredholm operators

Let $\mathbf{M}$ be a topological, possibly infinite-dimensional, manifold. Following the notation in Section 1 by A_* *we denote Fredholm operators depending on* $w \in \mathbf{M}$ such that the operator $A_*(w)$ belongs to the space $\mathcal{F}^0_*(\mathcal{W}, \mathcal{H})$ and the spectrum of its complexification $A^c_*(w) : \mathcal{H} \otimes \mathbb{C} \to \mathcal{H} \otimes \mathbb{C}$, whose domain of definition is equal to $\mathcal{W} \otimes \mathbb{C}$, consists only of eigenvalues of finite multiplicity. In particular, the set $\mathcal{R}(w) = \mathbb{R} \backslash \sigma(A^c_*(w))$, called below the real resolvent set, is formed by real numbers λ such that $A_*(w) - \lambda \operatorname{Id}$ is invertible.

First cohomology class given by the spectral flow. For any path-connected subset $\mathcal{M}$ of the parameter space $\mathbf{M}$ the spectral flow of the family A_* defines in a natural way the first cohomology class $w_A(\mathcal{M}) \in H^1(\mathcal{M}, \mathbb{Z}_2)$. Indeed, to any closed continuous path γ_t in $\mathcal{M}$, where $t \in [0, 1]$, and a real number λ from the resolvent set of $A^c_*(\gamma_0)$ we can assign an element of $\mathbb{Z}_2$ by the rule:

$$\gamma \longmapsto \mu_2(A_*(\gamma) - \lambda \operatorname{Id}).$$

By Lemma 1.5 this function does not depend on $\lambda \in \mathbb{R} \backslash \sigma(A^c_*(\gamma_0))$. Further, by the properties (i)–(iii) of the spectral flow this defines a homomorphism of the fundamental group $\pi_1(\mathcal{M}, \gamma_0)$ into $\mathbb{Z}_2$, which can be also interpreted as an element of $H^1(\mathcal{M}, \mathbb{Z}_2)$. It is clear that this cohomology class behaves naturally with respect to the inclusion of path-connected subsets.

Lemma 2.1. *Let* A_* *and* $\widetilde{A}_*$ *be families of Fredholm operators as above with the same parameter space* $\mathbf{M}$ *such that the family* A_* *can be continuously deformed to* $\widetilde{A}_*$ *through operators in* $\mathcal{F}^0_*(\mathcal{W}, \mathcal{H})$*. Then for any path-connected subset* $\mathcal{M}$ *of the space* $\mathbf{M}$ *its* w_A*- and* $w_{\widetilde{A}}$*-classes coincide.*

Proof. For a proof of the lemma it is sufficient to show that for any closed path γ in the space $\mathbf{M}$ and an appropriate real number λ the relation

$$\mu_2(A_*(\gamma) - \lambda \operatorname{Id}) = \mu_2(\widetilde{A}_*(\gamma) - \lambda \operatorname{Id}) \mod 2$$

holds. This is a direct consequence of Lemma 1.4. □

Definition 2.1. The family of Fredholm operators A_* is called *spectrally degenerate* along a curve $\gamma_t \in \mathbf{M}$ (or, more precisely, along the homotopy class given by γ) if:

- all real eigenvalues of the operators $A^c_*(\gamma_0)$ and $A^c_*(\gamma_1)$ at the end-points have even multiplicities;
- $\mu_2(A_*(\gamma) - \lambda \operatorname{Id}) \equiv 1 \mod 2$ for some real number λ which belongs to the both resolvent sets of $A^c_*(\gamma_0)$ and $A^c_*(\gamma_1)$.

If one of these conditions fails the family is called *spectrally non-degenerate* along the curve. A path-connected subset $\mathcal{M}$ is called spectrally non-degenerate with respect to a family A_* if the latter family is spectrally non-degenerate along any curve in $\mathcal{M}$.

Note that, by Lemma 1.5, the value of the spectral flow at $A_*(\gamma) - \lambda\,\mathrm{Id}$, in the above definition, does not depend on λ as long as the first condition is satisfied. Moreover, if points γ_0 and γ_1, as in the first condition, belong to a subset $\mathcal{M}$ whose w_A-class vanishes, then this value also does not depend on a curve $\gamma_t \in \mathcal{M}$ joining them.

Example 2.1. Let $\mathcal{M}$ be a path-connected subset of $\mathbf{M}$. Suppose that for any $w_0 \in \mathcal{M}$ there exists its neighbourhood $\mathcal{U}(w_0) \subset \mathcal{M}$ and a real number C such that for any $w \in \mathcal{U}(w_0)$ the resolvent set of $A^c_*(w)$ contains the interval $(-\infty, C)$. Then the space $\mathcal{M}$ is spectrally non-degenerate and, moreover, its w_A-class vanishes. Indeed, let $\gamma_t \in \mathcal{M}$ be an arbitrary path. Due to the suppositions above the constant C such that there are no real eigenvalues of $A_*(\gamma_t)$ less than C can be chosen uniformly with respect to t. Hence, for any $\lambda < C$ the spectral flow $\mu_2(A_*(\gamma) - \lambda\,\mathrm{Id})$ is equal to zero, etc.

Lemma 2.2. *Let $\mathcal{M}$ be a spectrally non-degenerate subset of $\mathbf{M}$. Then $\mathcal{M}$ has a spectrally non-degenerate neighbourhood.*

Proof. For any point w_0 from the space $\mathcal{M}$ let us choose its neighbourhood $\mathcal{U}(w_0)$ in the space $\mathbf{M}$ in the following way. If the operator $A^c_*(w_0)$ has a real eigenvalue of odd multiplicity, then as $\mathcal{U}(w_0)$ we choose a neighbourhood of w_0 such that for any $w \in \mathcal{U}(w_0)$ so does the operator $A^c_*(w)$. Now suppose that the multiplicities of all real eigenvalues of $A^c_*(w_0)$ are even. Then the point w_0 belongs to a connected component $\mathcal{E}(w_0)$ of the closed set $\mathcal{E}$ formed by points $w \in \mathbf{M}$ such that the operator $A_*(w)$ has real eigenvalues only of even multiplicities. In this case as $\mathcal{U}(w_0)$ we choose a neighbourhood of the point w_0 which does not intersect other connected components of $\mathcal{E}$.

We state that the union of the neighbourhoods $\mathcal{U}(w)$ chosen above over all points $w \in \mathcal{M}$ forms a spectrally non-degenerate neighbourhood of $\mathcal{M}$. Indeed, let γ_t, where $t \in [0,1]$, be a path in this neighbourhood such that the multiplicities of all real eigenvalues of the operators $A^c_*(\gamma_0)$ and $A^c_*(\gamma_1)$ are even. Then by our construction there exist points $\bar\gamma_0$ and $\bar\gamma_1$ from the set $\mathcal{M}$ such that they belong to the connected components $\mathcal{E}(\gamma_0)$ and $\mathcal{E}(\gamma_1)$ respectively. Let $\bar\gamma_t$ be an arbitrary path in $\mathcal{M}$ joining the points $\bar\gamma_0$ and $\bar\gamma_1$ and denote by ω^0_t and ω^1_t paths in $\mathcal{E}(\gamma_0)$ and $\mathcal{E}(\gamma_1)$ joining the points γ_0 and $\bar\gamma_0$ and γ_1 and $\bar\gamma_1$, respectively. Due to the catenation property the value $\mu_2(A_*(\gamma) - \lambda\,\mathrm{Id})$ of the spectral flow along γ is equal to the sum

$$\mu_2(A_*(\bar\gamma) - \lambda\,\mathrm{Id}) + \mu_2(A_*(\omega^0) - \lambda\,\mathrm{Id}) + \mu_2(A_*(\omega^1) - \lambda\,\mathrm{Id}) \mod 2,$$

where λ is an appropriate real number. The first term in the sum above is equal to zero by the spectral non-degeneracy of $\mathcal{M}$. Further, since the paths ω^0 and ω^1 lie in the connected components of the set $\mathcal{E}$, the spectral flow along them also vanish. Thus, the value $\mu_2(A_*(\gamma) - \lambda\,\mathrm{Id})$ is equal to zero and the lemma is demonstrated. □

The following notion will be useful in sequel.

Definition 2.2. A path-connected subset $\mathcal{M}$ of the parameter space $\mathbf{M}$ is called *weakly spectrally non-degenerate* (with respect to a family A_*) if there exists a continuous family of Fredholm operators $\widetilde{A}_*(w) \in \mathcal{F}^0_*(\mathcal{W}, \mathcal{H})$ such that:

- the operator $\widetilde{A}_*(w) - A_*(w)$ is compact for any $w \in \mathcal{M}$
- the subset $\mathcal{M}$ is spectrally non-degenerate with respect to the family $\widetilde{A}_*$.

It seems plausible (possibly, under some weak additional hypotheses) that any path-connected subset $\mathcal{M}$ is weakly spectrally non-degenerate.

2.2. Spectral covering

For a family of Fredholm operators $A_*(w)$ satisfying the hypotheses in the previous subsection consider the real resolvent set $\mathcal{R}(w)$ of the operator $A^c_*(w)$, where $w \in \mathbf{M}$. There is a natural equivalence relation on $\mathcal{R}(w)$. Namely, two real resolvent points of $A^c_*(w)$ are called equivalent if the number of real eigenvalues (counting their multiplicities) of $A^c_*(w)$ between them is even. Alternatively, one can say that resolvent points $\lambda_1, \lambda_2 \in \mathcal{R}(w)$ are equivalent if

$$\text{L-S index of } (A_*(w) - \lambda_1 \operatorname{Id})^{-1} \circ (A_*(w) - \lambda_2 \operatorname{Id}) = 1.$$

Here L-S index denotes the Leray-Schauder degree of a linear operator of the form $\operatorname{Id} + K$, where K is compact; see [26]. By $[\lambda]_w$ we shall denote the equivalence class corresponding to a given point $\lambda \in \mathcal{R}(w)$. Let us consider the set

$$|\mathbf{M}| = \{(w, [\lambda]_w) : \ w \in \mathbf{M}, \ \lambda \in \mathcal{R}(w)\}$$

and endow it with the minimal topology such that the projection

$$p : |\mathbf{M}| \to \mathbf{M}, \qquad (w, [\lambda]_w) \longmapsto w,$$

is continuous. For $w_0 \in \mathbf{M}$ and $\lambda \in \mathcal{R}(w)$ denote by $W_\lambda(w_0)$ a neighbourhood of w_0 such that for any $w \in W_\lambda(w_0)$ the real number λ is a resolvent point of $A_*(w)$. It is clear that the sets $\{(w, [\lambda]_w) : w \in W_\lambda(w_0)\}$ are open in $|\mathbf{M}|$ and the collection of these for all $w_0 \in \mathbf{M}$ and $\lambda \in \mathcal{R}(w_0)$ is a base of the introduced topology.

For a path-connected subspace $\mathcal{M}$ of the parameter space $\mathbf{M}$ we can also consider the space $|\mathcal{M}| = p^{-1}(\mathcal{M})$. The restriction of the projection p on this subspace is called the *spectral covering* over $\mathcal{M}$. If there is no $w \in \mathcal{M}$ such that all real eigenvalues of $A^c_*(w)$ have even multiplicities (and, hence, $\mathcal{M}$ is spectrally non-degenerate), then the space $|\mathcal{M}|$ is a topological double covering over $\mathcal{M}$ which is trivial if and only if the class $w_A(\mathcal{M})$ vanishes (see Lemma 2.3 below). In general, one can view $|\mathcal{M}|$ as a space obtained from this double covering by removing points (over $w \in \mathcal{M}$ such that all real eigenvalues of $A^c_*(w)$ have even multiplicities) corresponding to empty equivalence classes.

A continuous section of the spectral covering is a continuous map $s : \mathcal{M} \to |\mathcal{M}|$ such that $p \circ s = \operatorname{Id}$. In other words, such a section is simply a continuous family of resolvent classes $[\lambda]_w$, $w \in \mathcal{M}$, corresponding to operators $A_*(w)$. For the sake of simplicity we will also call it a *field of resolvent classes* or, shorter, an *A-resolvent field*. Obviously, such a resolvent field always exists locally. The following lemma gives a criterion for its global existence on a given subspace $\mathcal{M}$.

Lemma 2.3. *The spectral covering over a path-connected subspace $\mathcal{M}$ admits a continuous section if and only if the space $\mathcal{M}$ is spectrally non-degenerate and the first cohomology class $w_A(\mathcal{M})$ vanishes.*

The rest of this subsection is essentially devoted to the proof of Lemma 2.3. We start with the following simple statements.

Lemma 2.4. *A continuous path $\gamma : [0,1] \to \mathbf{M}$ admits a continuous lifting $|\gamma|$ – a path in the spectral covering such that $p \circ |\gamma| = \gamma$ – if and only if the curve γ is spectrally non-degenerate (i.e., the family A_* is spectrally non-degenerate along any arc of γ). Moreover, if there is no $t \in [0,1]$ such that all real eigenvalues of $A^c_*(\gamma_t)$ have even multiplicities, then for any $|\gamma_0| \in p^{-1}(\gamma(0))$ there exists a lifting $|\gamma|$ such that $|\gamma|(0) = |\gamma_0|$.*

Proof. First, note that the existence of a continuous lifting of a path γ is equivalent to the existence of a partition $0 = t_0 < t_1 \ldots < t_{\ell+1} = 1$ of the interval $[0,1]$ and a collection of real numbers $\lambda_0, \ldots, \lambda_\ell$ such that:

(i) $\lambda_i \notin \sigma(A_*^c(\gamma_t))$ for any $t \in [t_i, t_{i+1}]$, where $i = 1, \ldots, \ell$;

(ii) the number of real eigenvalues (counting their multiplicities) of $A_*(\gamma_{t_i})$ between λ_{i-1} and λ_i is even for any $i = 1, \ldots, \ell$.

Then the necessity of the spectral non-degeneracy of γ directly follows from Corollary 1.6.

We show that this condition is sufficient. First, suppose that there exists a point $t_0 \in [0,1]$ such that every real eigenvalue of $A_*^c(\gamma_{t_0})$ has even multiplicity. Then, since γ is spectrally non-degenerate, it follows that for any $t \in [0,1]$ there exists $\lambda(t) \in \mathcal{R}(\gamma_t)$ such that for an appropriate real μ the relation

$$\mu_2(A_*(\gamma_{t_0,t}) - \mu\,\mathrm{Id}) = \nu_t(\mu, \lambda(t)) \mod 2 \tag{2.1}$$

holds. Here $\nu_t(\mu, \lambda(t))$ stands for the parity of the number of real eigenvalues (counting their multiplicities) of $A_*^c(\gamma_t)$ between μ and $\lambda(t)$ and $\gamma_{t_0,t}$ denotes the arc

$$\gamma_{t_0,t}(\tau) = \gamma(t_1 + (t_2 - t_1)\tau), \quad \text{where } \tau \in [0,1].$$

It is clear that for any $t \in [0,1]$ there is a vicinity $U(t)$ of t such that $\lambda(t) \in \mathcal{R}(\gamma_\tau)$ for any $\tau \in U(t)$. Hence, one can cover the interval $[0,1]$ by a finite number of such the $U(\bar{t}_i)$'s, $\bar{t}_i \in [0,1]$, and suppose that a partition $t_0 < t_1 \ldots < t_{\ell+1}$, where t_0 and $t_{\ell+1}$ equal 0 and 1, is chosen such that each interval $[t_i, t_{i+1}]$ is contained in $U(\bar{t}_{i+1})$, $i = 0, \ldots, \ell$. It is easy to see that a collection of $\lambda_i = \lambda(\bar{t}_i)$, $i = 1, \ldots, \ell$, chosen in accordance with relation (2.1), satisfy the condition (ii) for this partition. Thus, in this case the existence of a continuous lifting of γ is demonstrated.

Now suppose that for any $t \in [0,1]$ there exists a real eigenvalue of $A_*^c(\gamma_t)$ whose multiplicity is odd. Let $\lambda(0)$ be a real resolvent point of $A_*^c(\gamma(0))$. Then for any $t \in [0,1]$ we choose $\lambda(t)$ in accordance with the formula

$$\mu_2(A_*(\gamma_{0,t}) - \mu\,\mathrm{Id}) = \nu_0(\mu, \lambda(0)) + \nu_t(\mu, \lambda(t)) \mod 2,$$

where μ is an appropriate real. Clearly, such $\lambda(t)$ always exists. Now the desired partition of the interval $[0,1]$ and the collection of reals can be constructed in the manner similar to that above. Thus, the lemma is proved. $\square$

Lemma 2.5. *Let γ_1 and γ_2 be paths starting at a point w_0 and ending at w_1. Suppose that $|\gamma_1|$ and $|\gamma_2|$ are their liftings starting at the same point, $|\gamma_1|(0) = |\gamma_2|(0)$. Then these liftings have the same end point if and only if there exists a real number λ from the resolvent sets of $A_*^c(w_0)$ and $A_*^c(w_1)$ such that*

$$\mu_2(A_*(\gamma_1) - \lambda\,\mathrm{Id}) = \mu_2(A_*(\gamma_2) - \lambda\,\mathrm{Id}).$$

The proof directly follows from Corollary 1.6. Combining the last two lemmas we obtain

Corollary 2.6. *A closed continuous path γ in the parameter space* $\mathbf{M}$ *admits a closed lifting if and only if γ is spectrally non-degenerate and $w_A(\gamma)$ vanishes.*

Proof of Lemma 2.3. Let us suppose that a path-connected subspace $\mathcal{M}$ is spectrally non-degenerate and its first w_A-class vanishes. Then we can construct a section s of the spectral covering in the following way. First, suppose that there exists a point $w_0 \in \mathcal{M}$ such that any real eigenvalue of $A_*^c(w_0)$ has even multiplicity. Then the preimage $p^{-1}(w_0)$ consists of a single point $|w_0| \in |\mathcal{M}|$ and we set $s(w_0) = |w_0|$. For another point $w \in \mathcal{M}$

let us choose a path γ joining w_0 and w such that $\gamma(0) = w_0$. Since $\mathcal{M}$ is spectrally non-degenerate, it follows from Lemma 2.4 that there exists a lifting $|\gamma|$ of the path γ and we set $s(w)$ to be equal to $|\gamma|(1)$. The vanishing of $w_A(\mathcal{M})$ and Lemma 2.5 imply that the value $s(w)$, defined so, does not depend on a choice of a path γ. Thus, we obtain the section of the spectral covering which is clearly continuous.

In the case when for any $w \in \mathcal{M}$ there exists a real eigenvalue of $A_*^c(w)$ whose multiplicity is odd, we can start with any point w_0 and any class $[\lambda]_{w_0}$ in order to spread the chosen equivalence class over $\mathcal{M}$ in the similar manner.

Now we demonstrate the necessity. Let s be a continuous section of the spectral covering. Then for any path γ in $\mathcal{M}$ this defines its lifting $|\gamma| = s \circ \gamma$. Hence, by Lemma 2.4 the path-connected subspace $\mathcal{M}$ is spectrally non-degenerate. Moreover, if a path γ is closed, then the lifting $|\gamma|$ is a closed path in $|\mathcal{M}|$. By Lemma 2.5 this implies that the element $w_A(\mathcal{M}) \in H^1(\mathcal{M}, \mathbb{Z}_2)$ is trivial. □

2.3. Basic framework for the moduli spaces

Consider elliptic differential equation (0.5) for mappings $u : M \to M'$ in a fixed homotopy class $[v]$. For any point (u, f) from the universal moduli space $\mathfrak{M}_{A,[v]}$, given by relation (0.6), we have the operator $A_*(u, \mathsf{f})$ obtained by the linearisation of the left-hand side in equation (0.5),

$$A_*(u, \mathsf{f}) = A_*(u) + \mathsf{f}_*(j^{k-1}u), \qquad (u, \mathsf{f}) \in \mathfrak{M}_{A,[v]}.$$

The operator acts on sections of the vector bundle u^*TM' and is defined with respect to a fixed connection on M', see [18, 20]. In particular, $A_*(u, \mathsf{f})$ defines a Fredholm operator on Sobolev spaces $H^m(u^*TM') \to H^{m-k}(u^*TM')$. Thus, one can consider the universal moduli space (and its any subdomain) as a parameter space for the family of linearised operators. Suppose that $A_*^c(u, \mathsf{f})$ has a non-empty resolvent set. Now we specify the notion of the spectral flow along paths in $\mathfrak{M}_{A,[v]}$.

Let $\gamma_t = (u_t, \mathsf{f}_t)$, where $t \in [0, 1]$, be a smooth path in the moduli space and λ be a real number from the resolvent sets of $A_*^c(\gamma_0)$ and $A_*^c(\gamma_1)$. By the spectral flow $\mu_2(A_*(\gamma) - \lambda\,\mathrm{Id})$ we understand the spectral flow of the family

$$A_*(t) - \lambda\,\mathrm{Id} : H^m(u_0^*TM') \to H^{m-k}(u_0^*TM'), \qquad t \in [0, 1]$$

where the operator $A_*(t)$ is given by the formula

$$A_*(t) = \Phi(u_t, u_0) \circ A_*(u_t, \mathsf{f}_t) \circ \Phi^{-1}(u_t, u_0), \qquad t \in [0, 1].$$

Here $\Phi(u_t, u_0)$ denotes the parallel transport (with respect to a fixed connection on M') along the curves $u_t(x)$, $x \in M$, which defines an isomorphism of the spaces $H^\ell({u_t}^*TM')$ and $H^\ell(u_0^*TM')$ for any $0 \leqslant \ell \leqslant m$. For other (non-smooth) paths γ the values $\mu_2(A_*(\gamma) - \lambda\,\mathrm{Id})$ are defined in accordance with the homotopy property.

In fact, such a definition of the spectral flow also carries over paths in the product $H^m(M, M') \times \mathcal{F}$. As a result, for any its path-connected subspace – for example, a component of $\mathfrak{M}_{A,[v]}$ – the first cohomology class w_A is well-defined. Thus, we see that the non-triviality of the w_A-class for $\mathfrak{M}_{A,[v]}$ is 'bounded' by the topology of the space $H^m(M, M')$. In particular, if the first $\mathbb{Z}_2$-homology group of the latter space is trivial, then the w_A-class of the moduli space vanishes.

More conventions. In sequel Sections 3 and 4 we always suppose that elliptic differential equations are such that the resolvent set of $A_*^c(u,\mathsf{f})$ is not empty for any (u,f). We also adopt the following terminology: a domain of the universal moduli space is called *(weakly) spectrally non-degenerate* if it is (weakly) spectrally non-degenerate with respect to the family of linearised operators $A_*(u,\mathsf{f})$.

3. The first Stiefel-Whitney class of the universal moduli space

3.1. Statement of the main result

Now we describe one of our principal results. As is known [4] the orientability of a Fredholm manifold is equivalent to the vanishing of its first Stiefel-Whitney class. Below we show that for the moduli space of solutions under the weak spectral non-degeneracy condition this cohomology class coincides with the w_A-class, defined above in terms of the spectral flow. Thus, the orientability of the moduli space is essentially determined only by the spectral properties of the family of equations (0.5). The main result applies equally to the whole moduli space of solutions and any its connected subdomain.

As in [18, Theorem 5.1] we assume that the space of parameters $\mathcal{F}$ is sufficiently large in the sense that the natural projection $J^{k-1}(M,M') \to M\times M'$ induces an embedding of the space of sufficiently smooth non-autonomous vector fields on M' into the space $\mathcal{F}$. Since $\operatorname{ind}_{[v]} A = 0$, under this supposition [18, Theorem 5.5] implies that the moduli space $\mathfrak{M}_{A,[v]}$ is a finite-dimensional Banach manifold modeled on the space $\mathcal{F}$ and the projection π_A in corresponding local coordinates has the quasi finite-dimensional form. Further, the following result holds.

Theorem 3.1. *Suppose that elliptic differential equations* (0.5) *are such that the resolvent set of $A_*^c(u,\mathsf{f})$ is not empty for any $(u,\mathsf{f})\in\mathfrak{M}_{A,[v]}$ and the universal moduli space of solutions is weakly spectrally non-degenerate. Then the first Stiefel-Whitney class of the quasi finite-dimensional Banach manifold $\mathfrak{M}_{A,[v]}$ coincides with its w_A-class, given by the $\mathbb{Z}_2$-spectral flow.*

It seems plausible that for any family of elliptic equations (0.5), parameterised by f, the moduli space of solutions is weakly spectrally non-degenerate; see Definition 2.2. In other words, it seems plausible that the corresponding 'weak spectral non-degeneracy' assumption in Theorem 3.1 can be dropped. However, in applications this condition verifies relatively easy.

A proof of the theorem appears in the next subsection. We now give examples in which, due to our result, the universal moduli space is orientable.

Example 3.1 (Perturbations of a complex operator). Let M' be an almost complex manifold. Suppose an operator A in equations (0.5) is such that its linearisation $A_*(u) = A_*(u,0)$ for any H^m-smooth $u\in[v]$ is a complex-linear differential operator on the bundle u^*TM' (i.e. it commutes with an almost complex structure). Consider the family of operators $\widetilde{A}_*(u,\mathsf{f}) = A_*(u)$ on the moduli space. Since $\widetilde{A}_*(u,\mathsf{f})$ is complex linear, the (real) multiplicity of any its eigenvalue is even (see Section 1) and Corollary 1.6 implies that the spectral flow $\mu_2(\widetilde{A}_*(\gamma) - \lambda\,\mathrm{Id})$ for any path γ in the moduli space and an appropriate $\lambda\in\mathbb{R}$ is equal to zero. Therefore, the moduli space (and its any subdomain) is spectrally non-degenerate with respect to $\widetilde{A}_*$ and its $w_{\widetilde{A}}$-class vanishes. We also obtain that the moduli space is weakly spectrally non-degenerate and due to Lemma 2.1

its w_A-class vanishes. Thus, Theorem 3.1 applies and the moduli space of solutions is orientable.

In the examples below we use Theorem 3.1 in the following simplified form: *if the moduli space is spectrally non-degenerate, its quasi-finite dimensional structure is orientable if and only if the w_A-class vanishes.*

Example 3.2 (Local uniform semi-boundedness). Suppose that a differential operator A in equations (0.5) is such that for any $(u_0, \mathsf{f}_0) \in \mathfrak{M}_{A,[v]}$ there exists its neighbourhood $\mathcal{U}(u_0, \mathsf{f}_0)$ and a real constant C such that for any $(u, \mathsf{f}) \in \mathcal{U}(u_0, \mathsf{f}_0)$ the linearised operator $A_*(u, \mathsf{f})$ satisfies the estimate

$$(A_*(u, \mathsf{f})\nu, \nu) \geqslant C(\nu, \nu) \quad \text{for any smooth vector field } \nu \text{ along } u,$$

where the brackets denote an L_2-scalar product on the space of vector fields along u. This implies that the spectrum $\sigma(A_*^c(u, \mathsf{f}))$ does not have points in the interval $\{x \in \mathbb{R} : x < C\}$. Thus, the statement in Example 2.1 applies and we see that any path-connected subset of the moduli space is spectrally non-degenerate and has the trivial w_A-class.

The following example describes conditions in terms of the symbol of equations (0.5) which guarantee the orientability of the moduli space and are useful in practice.

Example 3.3 (Symbol condition). Denote by σ_A the symbol of the operator of order k on the left-hand side in equations (0.5). Suppose that for any H^m-smooth $u \in [v]$ there exists a closed angle $\Lambda(u)$ in the complex plane with the vertex at the origin such that:

- the interval $(-\infty, 0]$ belongs to $\Lambda(u)$;
- the eigenvalues of the linear map $\sigma_A(j^{k-1}u(x), w) : T_{u(x)}M' \to T_{u(x)}M'$ (for the notation see [18]) do not belong to the angle $i^k\Lambda(u)$ for $x \in M$ and $0 \neq w \in T_x^*M$.

For example, if A has a self-adjoint symbol, i.e. the operator $\sigma_A(j^{k-1}u(x), w)$ is self-adjoint for any $u \in [v]$, $x \in M$, and $0 \neq w \in T_x^*M$, then this supposition is equivalent to the following: either $k \equiv 1 \mod 2$ or the eigenvalues of $\sigma_A(j^{k-1}u(x), w)$ are negative for the case $k/2 \equiv 1 \mod 2$ and positive otherwise.

Due to the continuity argument, the hypothesis above implies that for any H^m-smooth $u_0 \in [v]$ there exists its neighbourhood $\mathcal{U}(u_0)$ such that the second condition above is satisfied uniformly for $u \in \mathcal{U}(u_0)$ with the same angle $\Lambda = \Lambda(u_0)$. Now combining the result in Example 2.1 with [30, Theorem 9.1] (see also [30, Theorem 9.3]), we obtain that the family of equations (0.5) is spectrally non-degenerate along any curve in $H^m_{[v]}(M, M') \times \mathcal{F}$. In particular, its any path-connected subset (including the universal moduli space) is spectrally non-degenerate and has the vanishing w_A-class.

Due to results in [7, 8] we have the following statement.

Proposition 3.2. *Suppose that the moduli space of solutions is orientable and its projection π_A to the space of parameters $\mathcal{F}$ is proper. Then the degree* $\deg \pi_A$ *– the algebraic number of solutions $\pi_A^{-1}(\mathsf{f})$ for a generic $\mathsf{f} \in \mathcal{F}$ – is well-defined (i.e. it does not depend on a choice of* f*) and is an invariant of the family of equations* (0.5)*. Moreover, if* $\deg \pi_A \neq 0$*, then for any* f *there exists a solution of equation* (0.5)*.*

In examples it happens that the projection π_A is proper only on a preimage $\pi_A^{-1}(\mathcal{U})$ of a domain $\mathcal{U}$ in the space of parameters $\mathcal{F}$. If $\pi_A^{-1}(\mathcal{U})$ is weakly spectrally non-degenerate, then it is orientable if and only if its w_A-class vanishes. In the latter case the degree $\deg \pi_A|_{\pi^{-1}(\mathcal{U})}$ is also well-defined. In Section 6 we calculate this degree for certain elliptic operators appropriately perturbed by a parameter $\mathsf{f} \in \mathcal{F}$.

3.2. Proof of Theorem 3.1

Quasi finite-dimensional structure on the moduli space. We start with recalling the construction of a quasi finite-dimensional atlas on $\mathfrak{M}_{A,[v]}$, following [18, 20].

Let (u_0, f_0) be a point from $\mathfrak{M}_{A,[v]}$. In the notation [18, Section 5.4], suppose that the decompositions

$$H^m(u_0^* TM') = H_1 \oplus H_2, \qquad \mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2 \tag{3.1}$$

are given such that:

(1) $\dim H_1 = \dim \mathcal{F}_1 < +\infty$;

(2) $d\Psi(u_0, \mathsf{f}_0)|_{H_2 \times \mathcal{F}_1} : H_2 \times \mathcal{F}_1 \to H^{m-k}(u_0^* TM')$ is an isomorphism.

Here Ψ denotes the map $W(u_0) \times \mathcal{F} \to H^{m-k}(u_0^* TM')$ defined by the rule

$$(u, \mathsf{f}) \overset{\Psi}{\longmapsto} \Phi(u, u_0) \circ \big(A(u) + \mathsf{f}(j^{k-1}u)\big)\,;$$

where $\Phi(u, u_0)$ is an isomorphism of $H^{m-k}(u^* TM')$ to $H^{m-k}(u_0^* TM')$ given by the parallel transport along shortest geodesics and $W(u_0)$ is a small neighbourhood of u_0 in the space $H^m(M, M')$. Near the point (u_0, f_0) the space $\mathfrak{M}_{A,[v]}$ is a set of solutions of the equation $\Psi(u, \mathsf{f}) = 0$. Hence, by the property (2) in some vicinity of (u_0, f_0) the implicit function theorem applies and we can express $(u_2, \mathsf{f}_1) \in H_2 \times \mathcal{F}_1$ as a sufficiently smooth function of $(u_1, \mathsf{f}_2) \in H_1 \times \mathcal{F}_2$. The graph of this function is a chart $\mathcal{C}$ on $\mathfrak{M}_{A,[v]}$, which is homeomorphic to an open subset in $H_1 \times \mathcal{F}_2$ under the natural projection onto the domain. Choosing an arbitrary isomorphism $M : H_1 \simeq \mathcal{F}_1$ we can regard $\mathcal{C}$ as an open subset in $\mathcal{F}$.

Thus, a chart $\mathcal{C}$ constructed in this way, called a *chart centred at a point* (u_0, f_0), is determined by decompositions (3.1) satisfying the properties (1)–(2) and by an isomorphism $M : H_1 \simeq \mathcal{F}_1$. The collection of such charts corresponding to all points $(u_0, \mathsf{f}_0) \in \mathfrak{M}_{A,[v]}$, all decompositions (3.1), and isomorphisms M as above forms an atlas on $\mathfrak{M}_{A,[v]}$. It was shown in [18, Section 5.4] that this atlas is quasi finite-dimensional and in corresponding local coordinates the projection π_A has the quasi finite-dimensional form.

Good coverings and its Čech cohomology. We say that an open covering $\mathfrak{V} = \{V_\alpha\}$ of a topological space is *good*, if all non-empty intersections $V_{\alpha_0} \cap \ldots \cap V_{\alpha_\ell}$ are contractible.

The quasi finite-dimensional atlas on the moduli space, described above, can be chosen good.

Indeed, to demonstrate this note that any point of the manifold $H^m(M, M')$ has a basis formed by neighbourhoods $\{W_\alpha\}$ which are the images of sufficiently small balls under the exponential map

$$H^m(u^* TM') \ni \nu(x) \longmapsto \exp_{u(x)} \nu(x) \in H^m(M, M'), \qquad x \in M.$$

The intersection of a finite number of such neighbourhoods is convex with respect to geodesic homotopies and, hence, contractible. Further, any open covering $\{B_\beta\}$ of the Banach space $\mathcal{F}$ by balls is also good. Now, since the universal moduli space is a submanifold of the product $H^m(M, M') \times \mathcal{F}$, one can suppose that the charts of a quasi finite-dimensional atlas above have the form of the intersection of $\mathfrak{M}_{A,[v]}$ and $W_\alpha \times B_\beta$. If the neighbourhoods W_α and the balls B_β are sufficiently small, then the corresponding atlas is good. The reason for this is essentially the combination of the implicit function theorem and the fact that a local C^2-diffeomorphism of reflexive Banach spaces maps sufficiently small balls onto the convex sets. Below we also call such charts, which form a good quasi finite-dimensional atlas, *good* charts.

Good coverings are useful when we deal with the Čech cohomology of spaces. In particular, the Čech cohomology of such coverings are naturally isomorphic to the simplicial cohomology of the corresponding topological space. We recall this isomorphism now – we need it only for the first cohomology groups.

Let $\mathfrak{V} = \{V_\alpha\}$ be a good covering of a topological space X. First, let us construct a *nerve graph* $G(\mathfrak{V})$ in the space X in the following way. Choose a point $p_\alpha \in V_\alpha$ for any α, a point $p_{\alpha\beta}$ in any non-empty intersection $V_\alpha \cap V_\beta$, and fix contractions of the sets V_α to the vertices p_α; the latter exist, since the covering $\mathfrak{V}$ is good. The points p_α are the vertices of $G(\mathfrak{V})$. Further, the contractions of the sets V_α define the paths from $p_{\alpha\beta}$ to p_α. Their compositions give paths from p_α to p_β for any α and β such that $V_\alpha \cap V_\beta$ is non-empty. These are the edges of the graph $G(\mathfrak{V})$. One can show [2, Theorem 13.4] that the fundamental group of the space X coincides with the fundamental group $\pi_1(G(\mathfrak{V}))$ of this graph.

Now we describe the isomorphism between $\check{H}^1(\mathfrak{V}, \mathbb{Z}_2)$ and the group $H^1(X, \mathbb{Z}_2)$; we view the latter as the group formed by homomorphisms from $\pi_1(G(\mathfrak{V}))$ to $\mathbb{Z}_2$. Given a Čech 1-cocycle $c_{\alpha\beta}$ and a closed path γ formed by edges $\overline{p_{\alpha_0}p_{\alpha_1}}, \ldots, \overline{p_{\alpha_\ell}p_{\alpha_0}}$ we define its value on γ as the sum $c_{\alpha_0\alpha_1} + \ldots + c_{\alpha_\ell\alpha_0} \mod 2$. This does not depend on a representative of the cohomology class of $c_{\alpha\beta}$ and defines an isomorphism of the first cohomology groups; for the details we refer to [2].

The first Stiefel-Whitney class of the moduli space. Consider a good quasi finite-dimensional atlas on the moduli space. Since the transition functions between any two charts of this atlas have the quasi finite-dimensional form, we can say that two its charts *agree* if the L-S index of the differential of the transition functions equals 1. Now we define the first Stiefel-Whitney class w_1 of the quasi finite-dimensional structure. First, consider the $\mathbb{Z}_2$-Čech cochain whose value on the intersection of charts $\mathcal{C}_\alpha$ and $\mathcal{C}_\beta$ is given by the formula

$$(w_1)_{\alpha\beta} = \begin{cases} 0, & \text{if the charts } \mathcal{C}_\alpha \text{ and } \mathcal{C}_\beta \text{ agree} \\ 1, & \text{if the charts } \mathcal{C}_\alpha \text{ and } \mathcal{C}_\beta \text{ do not agree.} \end{cases} \tag{3.2}$$

Due to the multiplicity property of the L-S index, see [26], $(w_1)_{\alpha\beta}$ is a cocycle, i.e.

$$(w_1)_{\alpha\beta} + (w_1)_{\beta\gamma} + (w_1)_{\gamma\alpha} = 0 \mod 2,$$

and its Čech cohomology class is called the first Stiefel-Whitney class. It vanishes if and only if the corresponding cocycle is a coboundary – there exists a 0-cochain χ_α such that $(w_1)_{\alpha\beta} = \chi_\alpha - \chi_\beta$.

Now we make last preparations for the proof of Theorem 3.1. Two auxiliary statements (Lemmas 3.3 and 3.4), discussed below, are proved in Appendix A.1. The corresponding constructions are improved versions of those due to [23].

Charts centred at the same point. For a chart $\mathcal{C}$ centred at a point $(u_0, \mathsf{f}_0) \in \mathfrak{M}_{A,[v]}$, which is given by some decompositions (3.1) and an isomorphism $M : H_1 \simeq \mathfrak{F}_1$, define the linear operator

$$\mathscr{A}_{\mathcal{C}}(u_0, \mathsf{f}_0) = (-\Psi'_{\mathsf{f}}(u_0, \mathsf{f}_0) \circ M) \oplus \Psi'_u(u_0, \mathsf{f}_0) : H_1 \oplus H_2 \to H^{m-k}(u_0^* TM'). \tag{3.3}$$

Due to the condition (2), p. 18, this is an isomorphism. Moreover, recall that by [18, Lemma 5.9] the operator $\Psi'_u(u_0, \mathsf{f}_0)$ is equal to $A_*(u_0, \mathsf{f}_0)$. In particular, if a chart $\mathcal{C}$ is given by decompositions (3.1) such that the spaces H_1 and $\mathfrak{F}_1$ are trivial, then the operator $\mathscr{A}_{\mathcal{C}}(u_0, \mathsf{f}_0)$ coincides with $A_*(u_0, \mathsf{f}_0)$.

Lemma 3.3. *Let $\mathcal{C}$ and $\mathcal{C}'$ be good charts of the described above atlas that are centred at the same point (u_0, f_0). Then these charts agree if and only if*

$$\text{L-S index } \textit{of } (\mathscr{A}_{\mathcal{C}'}(u_0, \mathsf{f}_0))^{-1} \circ \mathscr{A}_{\mathcal{C}}(u_0, \mathsf{f}_0) = 1.$$

Homotopic charts. Given charts $\mathcal{C}_0$ and $\mathcal{C}_1$ are called *homotopic* if there exists a path $\gamma_t \in \mathfrak{M}_{A,[v]}$ joining their centres and a family of charts $\mathcal{C}_t$ centred at γ_t such that the $\mathcal{C}_t$'s are defined by the corresponding decompositions (3.1) and the morphisms M^t which depend on t continuously. In other words, the corresponding spaces H_1^t, H_2^t, $\mathfrak{F}_1^t$, and $\mathfrak{F}_2^t$ form vector bundles over γ_t and the family $M^t : H_1^t \to \mathfrak{F}_1^t$ is a morphism of vector bundles. It is a simple exercise to show that for sufficiently close points on the moduli space there exist homotopic charts centred at them.

The notion of homotopy between charts is useful for the proof of Theorem 3.1. The first reason for this is that the corresponding operators $\mathscr{A}_{\mathcal{C}}(u, \mathsf{f})$, given by (3.3), can be joined by a family of linear isomorphisms. Besides, we also have the following assertion.

Lemma 3.4. *Let $\mathcal{C}_0$ and $\mathcal{C}_1$ be homotopic good charts of the described above (quasi finite-dimensional) atlas such that their intersection is not empty. Then these charts agree.*

The proof of the theorem. First, fix a good covering $\mathfrak{V} = \{\mathcal{C}_\alpha\}$ of the moduli space by charts of the quasi finite-dimensional atlas and denote by $G(\mathfrak{V})$ its nerve graph whose vertices are the centres of the $\mathcal{C}_\alpha$'s. For a proof of the theorem it is sufficient to show that the value of the first Stiefel-Whitney class on a closed curve γ in the graph $G(\mathfrak{V})$ is equal to zero if and only if the w_A-class on this curve vanishes. Since $\mathfrak{M}_{A,[v]}$ is weakly spectrally non-degenerate there exists a family of Fredholm operators satisfying the conditions in Definition 2.2; we denote it by $\widetilde{A}_*(u, \mathsf{f})$.

$\mathbf{w_A}(\gamma) = \mathbf{0} \Rightarrow \mathbf{w_1}(\gamma) = \mathbf{0}.$ Let γ be a closed path formed by the edges of $G(\mathfrak{V})$. Suppose that the class w_A vanishes on γ and let us show that so does w_1.

First, due to Lemma 2.1 the values $w_A(\gamma)$ and $w_{\widetilde{A}}(\gamma)$ coincide. Since γ is $\widetilde{A}$-spectrally non-degenerate, Corollary 2.6 implies that there exists a continuous $\widetilde{A}$-resolvent field $[\lambda]_{(u,\mathsf{f})}$ along γ. For any good chart $\mathcal{C}$, not necessarily from $\mathfrak{V}$, centred at a point (u, f) from γ and a representative λ of the class $[\lambda]_{(u,\mathsf{f})}$ let us consider the operator

$$D_{\mathcal{C}}(u, \mathsf{f}) = \left(\widetilde{A}_*(u, \mathsf{f}) - \lambda \operatorname{Id}\right)^{-1} \circ \mathscr{A}_{\mathcal{C}}(u, \mathsf{f}).$$

By [18, Lemma 5.9] it has the form $\operatorname{Id} + K$, where K is a compact operator, and it is clear that its L-S index does not depend on a representative $\lambda \in [\lambda]_{(u,\mathsf{f})}$. To such a chart one can assign a number $\chi_{\mathcal{C}} \in \mathbb{Z}_2$ by the following rule

$$\chi_{\mathcal{C}} = \begin{cases} 0, & \text{if L-S index of } D_{\mathcal{C}}(u, \mathsf{f}) = 1 \\ 1, & \text{if L-S index of } D_{\mathcal{C}}(u, \mathsf{f}) = -1. \end{cases}$$

We say that intersecting good charts $\mathcal{C}$ and $\mathcal{C}'$ χ-*agree* if they agree only when $\chi_{\mathcal{C}}$ and $\chi_{\mathcal{C}'}$ are equal; i.e. either $\chi_{\mathcal{C}} = \chi_{\mathcal{C}'}$ and the charts agree or $\chi_{\mathcal{C}} \neq \chi_{\mathcal{C}'}$ and the charts do not agree. Note that by the cocycle property if three charts intersect by a non-empty set and the first two χ-agree and the second χ-agrees with the third then the first also χ-agrees with the third. Further, Lemma 3.3 implies that any two charts centred at the same point χ-agree. For a family $\mathcal{C}_t$ given by a homotopy of good charts, centred at the points on γ, any two of them that intersect by a non-empty set also χ-agree. Indeed, since the $\widetilde{A}$-resolvent field $[\lambda]_{(u,\mathsf{f})}$ on γ is continuous, the L-S index of $D_{\mathcal{C}_t}(u_t, \mathsf{f}_t)$ and, hence, the functions $\chi_{\mathcal{C}_t}$ do not depend on t. On the other hand, due to Lemma 3.4 any two such intersecting charts agree.

For a proof of vanishing of the first Stiefel-Whitney class on γ it is sufficient to show that any two charts $\mathcal{C}_\alpha$ and $\mathcal{C}_\beta$ from the covering $\mathfrak{V}$, centred at points of γ, χ-agree. Indeed, in this case its Čech cocycle on the intersection $\mathcal{C}_\alpha \cap \mathcal{C}_\beta$ is equal to $\chi_{\mathcal{C}_\alpha} - \chi_{\mathcal{C}_\beta}$.

Let $\mathcal{C}_\alpha$ and $\mathcal{C}_\beta$ be two such charts centred at the vertices $(u_\alpha, \mathsf{f}_\alpha)$ and $(u_\beta, \mathsf{f}_\beta)$. Suppose that $\mathcal{C}_\alpha \cap \mathcal{C}_\beta \neq \varnothing$ and let (u^*, f^*) be a point on γ from this intersection. Let us denote by $\mathcal{C}^*$ a good chart centred at (u^*, f^*). To show that the charts $\mathcal{C}_\alpha$ and $\mathcal{C}_\beta$ χ-agree we, first, show that $\mathcal{C}_\alpha$ and $\mathcal{C}^*$ χ-agree. Let (u_t, f_t), where $t \in [0,1]$, be an arc of γ joining the points $(u_\alpha, \mathsf{f}_\alpha)$ and (u^*, f^*). For a point $t_0 \in [0,1]$ there exist its neighbourhood Λ along with a homotopy $\mathcal{C}_t(\Lambda)$ whose charts are good and centred at the points of γ_t; here t ranges over Λ. Without loss of generality, one can suppose that such charts $\mathcal{C}_t(\Lambda)$ are contained in $\mathcal{C}_\alpha$.

Let us cover $[0,1]$ by a finite number of such intervals Λ_i, $i = 0, \ldots, \ell$. Up to a renumbering of the Λ_i's, one can suppose that there exists a collection of points $t_0 < t_1 < \ldots < t_{\ell+1}$ such that

$$0 = t_0 \in \Lambda_0, \quad 1 = t_{\ell+1} \in \Lambda_{\ell+1}, \quad \text{and} \quad t_i \in \Lambda_{i-1} \cap \Lambda_i, \quad i = 1, \ldots, \ell.$$

Then by Lemma 3.3 the chart $\mathcal{C}_\alpha$ χ-agrees with the chart $\mathcal{C}_0(\Lambda_0)$. Applying Lemma 3.4 we see that $\mathcal{C}_0(\Lambda_0)$ χ-agrees with the chart $\mathcal{C}_{t_1}(\Lambda_0)$ which, again by Lemma 3.3, χ-agrees with $\mathcal{C}_{t_1}(\Lambda_1)$ and so on. Thus, by the cocycle property we see that the charts $\mathcal{C}_\alpha$ and $\mathcal{C}^*$ χ-agree. Similarly, one can show that the charts $\mathcal{C}^*$ and $\mathcal{C}_\beta$ also χ-agree. Hence, again by the cocycle property, the charts $\mathcal{C}_\alpha$ and $\mathcal{C}_\beta$ χ-agree and the sufficiency is demonstrated. ☐

$\mathbf{w_1}(\gamma) = \mathbf{0} \Rightarrow \mathbf{w_A}(\gamma) = \mathbf{0}.$ Suppose that the first Stiefel-Whitney class vanishes on a closed path γ in the nerve graph $G(\mathfrak{V})$. Hence, for a finite covering $\{\mathcal{C}_\alpha\}$ of γ by charts from $\mathfrak{V}$ centred at the vertices of $G(\mathfrak{V})$ there exists a 0-cochain $\{\chi_{\mathcal{C}_\alpha}\}$ such that any two charts that intersect by a non-empty set χ-agree. Take a family $\{\mathcal{C}\}$ of good charts, centred at points of γ, such that

- for any point $(u, \mathsf{f}) \in \gamma$ there exists a chart centred at (u, f);
- if a chart intersects with one of the $\mathcal{C}_\alpha$'s, then it is contained in this $\mathcal{C}_\alpha$.

The cochain $\{\chi_{\mathcal{C}_\alpha}\}$ can be extended to a 0-cochain $\{\chi_{\mathcal{C}}\}$ such that any two charts of the covering $\{\mathcal{C}\}$ that intersect by a non-empty set χ-agree. Since the values $w_A(\gamma)$ and $w_{\widetilde{A}}(\gamma)$ coincide, for a proof of the necessity it is sufficient, due to Corollary 2.6, to construct a continuous $\widetilde{A}$-resolvent field along the path γ.

First, suppose that there exists a point $(u_0, \mathsf{f}_0) \in \gamma$ such that all real eigenvalues of $\widetilde{A}^c_*(u_0, \mathsf{f}_0)$ have even multiplicities. Without loss of generality, one can assume that the 0-cochain $\{\chi_{\mathcal{C}}\}$ above is chosen such that for some chart $\mathcal{C}$ centred at (u_0, f_0) the following relation holds

$$\text{L-S index of } \left(\widetilde{A}_*(u_0, \mathsf{f}_0) - \lambda\,\mathrm{Id}\right)^{-1} \circ \mathscr{A}_{\mathcal{C}}(u_0, \mathsf{f}_0) = (-1)^{\chi_{\mathcal{C}}}, \tag{3.4}$$

where λ is a real resolvent point of $\widetilde{A}^c_*(u_0, \mathsf{f}_0)$. Note that relation (3.4) is invariant under the change of a chart centred at (u_0, f_0) and, hence, holds for any such a chart. Indeed, this follows from Lemma 3.3, since any two such charts $\mathcal{C}$ and $\mathcal{C}'$ χ-agree and the L-S indices of $D_{\mathcal{C}}(u_0, \mathsf{f}_0)$ and $D_{\mathcal{C}'}(u_0, \mathsf{f}_0)$ differ by the factor

$$\text{L-S index of } \mathscr{A}^{-1}_{\mathcal{C}'}(u_0, \mathsf{f}_0) \circ \mathscr{A}_{\mathcal{C}}(u_0, \mathsf{f}_0).$$

Now for any point (u, f) from the curve γ we define the equivalence class of the real resolvent points of the operator $\widetilde{A}^c_*(u, \mathsf{f})$ as follows. Take a chart $\mathcal{C}$ centred at $(u, \mathsf{f}) \in \gamma$ and define the class $[\lambda]_{(u,\mathsf{f})}$ by the corresponding relation (3.4). By the discussion above, if such a class exists, it does not depend on a chart $\mathcal{C}$ centred at (u, f). Its existence follows essentially from the $\widetilde{A}$-spectral non-degeneracy of γ. There are two possibilities:

(i) for any $(u, \mathsf{f}) \in \gamma$, except (u_0, f_0), there exists a real eigenvalue of $\widetilde{A}^c_*(u, \mathsf{f})$ whose multiplicity is odd.

(ii) there exists a point $(u_1, \mathsf{f}_1) \in \gamma$, different from (u_0, f_0), such that all real eigenvalues of $\widetilde{A}^c_*(u_1, \mathsf{f}_1)$ have even multiplicities.

In the case (i), we define $[\lambda]_{(u,\mathsf{f})}$ at any point simply by relation (3.4); its existence is obvious. To demonstrate the existence in the case (ii) we have to show that for any real λ from the resolvent set of $\widetilde{A}^c_*(u_1, \mathsf{f}_1)$ and some chart $\mathcal{C}$ centred at (u_1, f_1) the corresponding relation (3.4) holds.

Indeed, let (u_t, f_t), where $t \in [0, 1]$, be an arc of γ joining the points (u_0, f_0) and (u_1, f_1). Since the curve (u_t, f_t) is $\widetilde{A}$-spectrally non-degenerate, then, by Lemma 2.4, there exists a partition $t_0 < t_1 < \ldots < t_{\ell+1}$ of the interval $[0, 1]$, where t_0 and $t_{\ell+1}$ equal 0 and 1 respectively, and a collection of real numbers $\lambda_0, \ldots, \lambda_\ell$ such that:

- $\lambda_i \notin \sigma(\widetilde{A}^c_*(u_t, \mathsf{f}_t))$ for any $t \in [t_i, t_{i+1}]$, where $i = 1, \ldots, \ell$,
- the number of real eigenvalues (counting their multiplicities) of $\widetilde{A}^c_*(u_{t_i}, \mathsf{f}_{t_i})$ between λ_{i-1} and λ_i is even for any $i = 1, \ldots, \ell$.

Following the lines in the proof of the sufficiency, one can find two collections of charts $\mathcal{C}_i$ and $\mathcal{C}'_i$, $i = 1, \ldots, \ell$, centred at the points $(u_{t_i}, \mathsf{f}_{t_i})$, such that $\mathcal{C}_i$ and $\mathcal{C}'_{i+1}$ are homotopic. Now Lemmas 3.3 and 3.4 imply that for any $i = 0, \ldots, \ell + 1$ the identity

$$\text{L-S index of } \left(\widetilde{A}_*(u_{t_i}, \mathsf{f}_{t_i}) - \lambda \operatorname{Id}\right)^{-1} \circ \mathscr{A}_{\mathcal{C}_i}(u_{t_i}, \mathsf{f}_{t_i}) = (-1)^{\chi_{\mathcal{C}_i}}$$

holds, where λ equals λ_{i-1} or λ_i. Thus, λ_ℓ defines the resolvent class $[\lambda]_{(u_1,\mathsf{f}_1)}$ which satisfies (3.4) with $\mathcal{C} = \mathcal{C}_{\ell+1}$ and the claim is proved.

In the case when for any $(u, \mathsf{f}) \in \gamma$ there exists a real eigenvalue of $\widetilde{A}^c_*(u, \mathsf{f})$ whose multiplicity is odd, we define $[\lambda]_{(u,\mathsf{f})}$ simply by relation (3.4); its existence is obvious.

Thus, we obtain the well-defined resolvent field $[\lambda]_{(u,\mathsf{f})}$ on the curve γ. Now for a proof of the theorem we need to show that it is continuous. Since the statement is local, it is sufficient to show that for any point $(u_0, \mathsf{f}_0) \in \gamma$ and $\lambda \in [\lambda]_{(u_0,\mathsf{f}_0)}$ there exists a neighbourhood in γ of (u_0, f_0) such that for any (u, f) from this neighbourhood and any chart $\mathcal{C}$ centred at (u, f) the corresponding relation (3.4) holds with the same λ. This, in fact, can be done in the manner similar to that described above. Indeed, suppose that the neighbourhood of (u_0, f_0) is chosen such that for any (u, f) from it a real number λ is a resolvent point of $\widetilde{A}^c_*(u, \mathsf{f})$. Let $(u_1, \mathsf{f}_1) \in \gamma$ be an arbitrary point from this neighbourhood and (u_t, f_t), where $t \in [0,1]$, be an arc of γ joining the points (u_0, f_0) and (u_1, f_1). Let us take a partition $t_0 < t_1 < \ldots < t_{\ell+1}$ of the interval $[0,1]$ and two collections of charts $\mathcal{C}_i$ and $\mathcal{C}'_i$ centred at γ_{t_i} such that $\mathcal{C}_i$ and $\mathcal{C}'_{i+1}$ are homotopic. Now we see again that relation (3.4), with the same fixed λ, holds at every point t_i of the partition and, in particular, at the point $t_{\ell+1} = 1$. □

4. Orientation of the moduli space

As in Section 3 we suppose that the moduli space $\mathfrak{M}_{A,[v]}$ is connected (or consider only its connected component) and the conditions of Theorem 3.1 hold.

4.1. The construction of an oriented atlas

Here we would like to state explicitly an important consequence of the proof of Theorem 3.1 which gives a canonical way to endow the moduli space with an orientation using certain spectral data of the family of equations (0.5).

First, we describe a basic idea loosely. Suppose we are given a manifold along with some atlas and asked to determine whether this manifold is orientable and, in the positive case, to construct an oriented atlas. To answer the first question we consider the Čech cocycle (3.2). The manifold is orientable if and only if its cohomology class is trivial – the cocycle is the image of a 0-cochain χ_i under the coboundary operator. If this is the case an oriented atlas can be constructed in the following manner. Each chart $(\mathcal{C}_i, \phi_i)$ such that $\chi_i = 1$ we replace with a chart $(\mathcal{C}_i, -\phi_i)$, where the homeomorphism $-\phi_i$ is equal to the composition of ϕ_i with a linear isomorphism (of the model space) that changes an orientation. A new atlas obtained as a result of this procedure is oriented. The construction also shows that by an orientation we can understand the choice of such a cochain $\{\chi_i\}$.

Now let us consider the universal moduli space with an atlas given by the quasi finite-dimensional structure; see Section 3. Suppose it is weakly spectrally non-degenerate and denote by $\widetilde{A}_*(u, \mathsf{f})$ the corresponding family of Fredholm operators as in Definition 2.2. If the moduli space is orientable, then Theorem 3.1 implies that w_A vanishes. Due to Lemma 2.3, there exists a continuous $\widetilde{A}$-field of resolvent classes $[\lambda]_{(u,\mathsf{f})}$ on the universal moduli space.

Definition 4.1. A 0-cochain $\{\chi_{\mathcal{C}}\}$ on the moduli space (with its quasi finite-dimensional atlas) such that for any chart $\mathcal{C}$ the relation

$$\text{L-S index of } \Big(\widetilde{A}_*(u, \mathsf{f}) - \lambda\, \mathrm{Id}\Big)^{-1} \circ \mathscr{A}_{\mathcal{C}}(u, \mathsf{f}) = (-1)^{\chi_{\mathcal{C}}} \tag{4.1}$$

holds is called the *orienting cochain.* Here (u, f) is the centre of a chart $\mathcal{C}$, $\lambda \in [\lambda]_{(u,\mathsf{f})}$, and the operator $\mathscr{A}_{\mathcal{C}}(u, \mathsf{f})$ is given by formula (3.3).

The following lemma justifies this definition. Its proof goes essentially along the same lines as the sufficiency part in the proof of Theorem 3.1.

Lemma 4.1. *Suppose that the conditions of Theorem 3.1 hold and the universal moduli space is orientable. Then the Čech cocycle* (3.2)*, representing the first Stiefel-Whitney class, is the image of the orienting cochain under the coboundary operator.*

Oriented quasi-finite dimensional atlas on the moduli space. Now we construct an oriented quasi finite-dimensional atlas on the moduli space. Let $(\mathcal{C}_i, \phi_i)_i$ be an original quasi finite-dimensional atlas described in Section 3. Replace any chart $(\mathcal{C}_i, \phi_i)$ with $\chi_i = 1$ with a chart $(\mathcal{C}_i, -\phi_i)$, where $-\phi_i$ is the composition of ϕ_i and a linear isomorphism of the space $\mathcal{F}$ that changes an orientation. For example, if $\mathcal{F}$ is decomposed as $\mathbb{R} \oplus \mathcal{F}_0$ for some subspace $\mathcal{F}_0$ of codimension 1, then $(-\operatorname{Id}_{\mathbb{R}}) \oplus \operatorname{Id}_{\mathcal{F}_0}$ changes an orientation on $\mathcal{F}$ and can be chosen for such an isomorphism. Clearly, the obtained atlas is quasi finite-dimensional and due to Lemma 4.1 is oriented.

Remark 4.1. Let $\widetilde{A}_*$ and $\bar{A}_*$ be two families of Fredholm operators on the moduli space which satisfy conditions in Definition 2.2. Suppose that the moduli space is orientable and let $[\lambda_1]_{(u,\mathsf{f})}$ and $[\lambda_2]_{(u,\mathsf{f})}$ be the corresponding $\widetilde{A}$- and $\bar{A}$-resolvent classes; they exist due to Theorem 3.1 and Lemma 2.3. Then the corresponding orienting cochain (and, hence, the orientations) coincide if and only if there exists a point (u, f) from the moduli space and a real number that belongs to both $\widetilde{A}$- and $\bar{A}$-resolvent classes $[\lambda_1]_{(u,\mathsf{f})}$ and $[\lambda_2]_{(u,\mathsf{f})}$ at the point (u, f).

4.2. Index of a non-degenerate solution

In this section we discuss the notion of index for a non-degenerate solution and formulas for the degree $\deg \pi_A$.

Suppose that the moduli space is orientable and let us fix an orientation (or an orienting cochain) given by a continuous $\widetilde{A}$-resolvent field $[\lambda]_{(u,\mathsf{f})}$. Recall that a solution u of equations (0.5) is called non-degenerate if the linear operator $d\pi_A(u, \mathsf{f})$ is non-degenerate. By [18, Lemma 5.4] it is also equivalent to the non-degeneracy of the operator $A_*(u, \mathsf{f})$. In particular, there exists a chart $\mathcal{C}_*$ centred at (u, f), called the *canonical chart*, which corresponds to the unique decompositions (3.1) with the trivial spaces H_1 and $\mathcal{F}_1$.

Definition 4.2. For a non-degenerate solution u of equations (0.5) the value $\chi_{\mathcal{C}_*} \in \mathbb{Z}_2$ of the orienting cochain on the canonical chart $\mathcal{C}_*$ centred at (u, f) is called its index $\operatorname{ind}_2(u, \mathsf{f})$.

Lemma 4.2. *Suppose that the moduli space is spectrally non-degenerate and its orientation is given by an (A-)resolvent field $[\lambda]_{(u,\mathsf{f})}$. Then the index $\operatorname{ind}_2(u, \mathsf{f})$ is equal to the parity of the number of real eigenvalues of the operator $A_*^{\mathrm{c}}(u, \mathsf{f})$ between any representative $\lambda \in [\lambda]_{(u,\mathsf{f})}$ and zero; in other words, $\operatorname{ind}_2(u, \mathsf{f})$ is equal to zero, if $0 \in [\lambda]_{(u,\mathsf{f})}$, and to one otherwise.*

Proof. Since the operator $A_*(u, \mathsf{f})$ is non-degenerate, there exists a canonical chart $\mathcal{C}_*$ centred at (u, f). Now the statement follows from relation (4.1) with $\widetilde{A}_* = A_*$ and the observation that by [18, Lemma 5.9] the operator $\mathscr{A}_{\mathcal{C}_*}(u, \mathsf{f})$ coincides with $A_*(u, \mathsf{f})$. □

Lemma 4.3. *Suppose that the moduli space is oriented in accordance with the 0-cochain given by (4.1) and the projection π_A of the moduli space $\mathfrak{M}_{A,[v]}$ to the space of parameters $\mathcal{F}$ is proper. Then for any regular value f of the projection π_A the degree $\deg \pi_A$ is given by the formula*

$$\deg \pi_A = \sum_{(u,\mathsf{f}) \in \pi_A^{-1}(\mathsf{f})} (-1)^{\operatorname{ind}_2(u,\mathsf{f})}. \tag{4.2}$$

Proof. Suppose that the moduli space is endowed with the oriented quasi finite-dimensional atlas described above. For a proof of the lemma it is sufficient to show that in suitable local coordinates $d\pi_A(u, \mathsf{f})$ changes an orientation if and only if $\operatorname{ind}_2(u, \mathsf{f})$

equals 1. To show this, choose a canonical chart $\mathcal{C}_*$ centred at $(u, \mathsf{f}) \in \pi_A^{-1}(\mathsf{f})$. Clearly, in canonical charts of the 'non-oriented' atlas the projection π_A (and, hence, its differential) is equal to the identity. Therefore, due to Lemma 4.1, in canonical charts of the oriented atlas $d\pi_A(u, \mathsf{f})$ changes an orientation if and only if $\chi_{\mathcal{C}_*}$ equals 1. The combination of this with Definition 4.2 proves the lemma. □

Example 4.2. Suppose that the family of equations (0.5) is such as in Example 3.3. In this case the moduli space is spectrally non-degenerate and its w_A-class is equal to zero. Moreover, for any $(u, \mathsf{f}) \in \mathfrak{M}_{A,[v]}$ there exists a real number λ such that there is no eigenvalue of the operator $A_*^c(u, \mathsf{f})$ on the interval $(-\infty, \lambda]$. Clearly, the choice of such λ for any (u, f) from the moduli space defines a continuous field of resolvent classes on $\mathfrak{M}_{A,[v]}$; we suppose that the moduli space is oriented in accordance with the corresponding 0-cochain (4.1). Further, for any point $(u, \mathsf{f}) \in \mathfrak{M}_{A,[v]}$ we can define an integer valued index $\operatorname{ind}(u, \mathsf{f})$ as the number of real negative eigenvalues of $A_*^c(u, \mathsf{f})$. Due to Lemma 4.2, we have $\operatorname{ind}(u, \mathsf{f}) = \operatorname{ind}_2(u, \mathsf{f}) \bmod 2$ and formula (4.2) can be re-written in this notation as

$$\deg \pi_A = \sum_{(u,\mathsf{f}) \in \pi_A^{-1}(\mathsf{f})} (-1)^{\operatorname{ind}(u,\mathsf{f})}. \tag{4.3}$$

Under the conditions of Theorem 3.1, for an orientable moduli space of solutions, the following definition makes sense.

Definition 4.3. For two non-degenerate solutions u_0 and u_1 of equation (0.5) by the relative index $\operatorname{rel}_2(u_0, u_1)$ we call the spectral flow $\mu_2(A_*(\gamma))$ along a path γ in the moduli space joining (u_0, f) and (u_1, f).

Lemma 4.4. *Suppose that the conditions of Theorem 3.1 hold and the moduli space is orientable. Then for any two its non-degenerate points* (u_0, f_0) *and* (u_1, f_1) *we have*

$$\operatorname{ind}_2(u_0, \mathsf{f}_0) - \operatorname{ind}_2(u_1, \mathsf{f}_1) = \mu_2(A_*(\gamma)) \mod 2,$$

where γ *is a path in the moduli space joining these points. In particular, for the proper projection* π_A *the formula for the degree takes the form:*

$$|\deg \pi_A| = \left| \sum_{(u,\mathsf{f}) \in \pi_A^{-1}(\mathsf{f})} (-1)^{\operatorname{rel}_2(u,u_0)} \right|.$$

Here f *is a regular value of* π_A *and* u_0 *is one of the corresponding solutions of equation* (0.5).

Proof. The hypotheses of the lemma together with Lemma 2.3 imply the existence of a continuous $\widetilde{A}$-resolvent field $[\lambda]_{(u,\mathsf{f})}$ on the moduli space. Suppose $\mathfrak{M}_{A,[v]}$ is oriented in accordance with the corresponding 0-cochain. Due to Definitions 4.1 and 4.2 for a proof of the lemma it is sufficient to show that the L-S indices of the operators

$$D(u_i, \mathsf{f}_i) = \left(\widetilde{A}_*(u_i, \mathsf{f}_i) - \bar{\lambda} \operatorname{Id} \right)^{-1} \circ A_*(u_i, \mathsf{f}_i), \qquad i = 0, 1,$$

differ by the factor $(-1)^\mu$, where $\mu = \mu_2(A_*(\gamma))$. The operators above are defined up to a choice of $\bar{\lambda} \in [\lambda]_{(u_i,\mathsf{f}_i)}$, which does not affect the values of their L-S indices.

First, let us suppose that a path γ_t, parameterised by $t \in [0, 1]$ and joining (u_1, f_1) and (u_2, f_2), is such that the family $A_*(\gamma_t)$ does not have crossings. Then the spectral flow along γ is equal to zero and, due to the continuity of the resolvent field $[\lambda]_{(u,\mathsf{f})}$ and the L-S index, the relation for the indices holds trivially. For the general case we use the formula for the spectral flow due to Corollary 1.6. In more detail, choose

- a finite partition $t_0 < t_1 < \ldots < t_{\ell+1}$ of the interval $[0,1]$, where t_0 and $t_{\ell+1}$ are zero and one respectively,
- a collection of real numbers λ_i, where $i = 0, \ldots, \ell$,

such that λ_0 and λ_ℓ are equal to zero and the λ_i's do not belong to the spectrum $\sigma(A_*^c(\gamma_t))$ for any $t \in [t_i, t_{i+1}]$. Further, define the linear operator

$$\mathscr{D}(t,\lambda) = (\widetilde{A}_*(\gamma_t)) - \bar{\lambda}\,\mathrm{Id})^{-1} \circ (A_*(\gamma_t) - \lambda\,\mathrm{Id}), \qquad t \in [0,1],$$

where λ belongs to the real resolvent set of $A_*^c(\gamma_t)$. It depends on a choice $\bar{\lambda} \in [\lambda]_{\gamma_t}$, which, however, does not affect the value of its L-S index.

Firstly, we see that the L-S index of $\mathscr{D}(t, \lambda_{i+1})$ does not depend on $t \in [t_i, t_{i+1}]$. Secondly, due to the properties of the Leray-Schauder degree [26, Chapter 2], the L-S indices of $\mathscr{D}(t_i, \lambda_i)$ and $\mathscr{D}(t_i, \lambda_{i+1})$ differ by the factor $(-1)^{\nu_i}$, where ν_i stands for the number of real eigenvalues of $A_*^c(\gamma_{t_i})$ that lie between λ_i and λ_{i+1}. Combining these two observations we obtain that

$$\text{L-S index of } \mathscr{D}(1,0) = \text{L-S index of } \mathscr{D}(0,0)(-1)^{\sum \nu_i}.$$

Now using the trivial identities $D(u_0, \mathrm{f}_0) = \mathscr{D}(0,0)$ and $D(u_1, \mathrm{f}_1) = \mathscr{D}(1,0)$ and Corollary 1.6 we prove the formula for the indices. $\square$

5. Elements of Morse-Bott theory I

5.1. Infinitesimal structure of the spaces of solutions

Let A be a quasi-linear elliptic differential operator of order k defined on mappings $u : M \to M'$ whose coefficients are smooth, i.e. C^∞-smooth. Denote by $\mathfrak{S}_A$ the set formed by H^m-smooth solutions (which we simply call solutions) of the homogeneous equation

$$Au(x) = 0, \qquad x \in M. \tag{5.1}$$

By elliptic regularity theory these solutions are smooth. Below we regard $\mathfrak{S}_A$ as a topological subspace of $H^m(M, M')$.

Let u be a solution of equation (5.1). Recall that due to [20, Proposition 4.1] there exists its neighbourhood $\mathcal{O}(u)$ in the space $H^m(M, M')$ such that the set $\mathcal{O}(u) \cap \mathfrak{S}_A$ can be canonically identified[2] with a subset of a ball centred at zero in the space $\operatorname{Ker} A_*(u)$; see [31] for a similar statement for the harmonic map equation. In particular, if there exists a neighbourhood $\mathcal{O}(u)$ such that $\mathcal{O}(u) \cap \mathfrak{S}_A$ is identified with a ball in the linear space $\operatorname{Ker} A_*(u)$, then we say that the space of solutions $\mathfrak{S}_A$ is non-degenerate at a point u. We call the space $\mathfrak{S}_A$, or its connected component, *non-degenerate (in the sense of Morse-Bott)* if it is non-degenerate at any point. Alternatively, one can define this property in the following way.

Definition 5.1. The space of solutions $\mathfrak{S}_A$ is called *non-degenerate (in the sense of Morse-Bott)* if its each connected component is a smooth submanifold of $H^m(M, M')$ whose tangent space at a point u equals $\operatorname{Ker} A_*(u)$.

[2] here 'canonically' is in the sense of the implicit function theorem;

Example 5.1. It often happens that constant mappings are solutions of equation (5.1). In particular, if all constant mappings $u_y(x) \equiv y$, where $y \in M'$, are solutions and $\dim \operatorname{Ker} A_*(u_y) = \dim M'$ for any $y \in M'$, then the space of constant solutions is non-degenerate and is a connected component of $\mathfrak{S}_A$. We say that an elliptic operator A satisfies the *Liouville principle* if the space of its contractible solutions $\mathfrak{S}_{A,[pt]}$ coincides with the space formed by all constant mappings u_y, $y \in M'$. In the latter case the space $\mathfrak{S}_{A,[pt]}$ is diffeomorphic to M' under the evaluation map, which sends $u \mapsto u(y)$ for a fixed $y \in M'$.

Let $\mathfrak{S}_A$ be a non-degenerate space of solutions of equation (5.1). Then there is a well-defined vector bundle over any connected component of $\mathfrak{S}_A$ formed by the kernels $\operatorname{Ker} A_*(u)$, which coincides with its tangent bundle. Now suppose that the pull-back bundle u^*TM' for each $u \in \mathfrak{S}_A$ is endowed with an Euclidean metric which depends smoothly on u. In other words, we fix an Euclidean metric on the bundle obtained by pulling back TM' under a map $M \times \mathfrak{S}_A \to M'$, $(x, u) \mapsto u(x)$. This together with a smooth positive density on M gives rise to a natural L_2-scalar product on the space of smooth sections of u^*TM' for any $u \in \mathfrak{S}_A$. Denote by $A(u)^*$ the formally adjoint operator of $A_*(u)$ with respect to this scalar product, where $u \in \mathfrak{S}_A$. Since $\operatorname{ind} A_*(u)$ depends only on a homotopy class of u, see [18, 28], the vector bundle formed by $\operatorname{Ker} A(u)^*$ is well-defined over any connected component of the space $\mathfrak{S}_A$. (The fact that this bundle is smooth and locally trivial follows by arguments similar to those in [13, Chapter 3]). We denote the corresponding bundles formed by the kernels of linearised operator and its formally adjoint by $\operatorname{Ker} A$ and $\operatorname{Ker} A^*$ respectively.

Definition 5.2. The space of solutions $\mathfrak{S}_A$ is called *strongly non-degenerate* if it is non-degenerate and for any $u \in \mathfrak{S}_A$ the linearised operator $A_*(u)$ does not have adjoint vectors corresponding to the zero eigenvalue; i.e. the algebraic multiplicity of the zero eigenvalue is equal to the dimension of $\operatorname{Ker} A_*(u)$.

The following observation we state as a lemma; its proof is straightforward and, therefore, is omitted.

Lemma 5.1. *Suppose that the space od solutions $\mathfrak{S}_A$ is non-degenerate in the sense of Morse-Bott and the bundles* $\operatorname{Ker} A$ *and* $\operatorname{Ker} A^*$ *coincide. Then $\mathfrak{S}_A$ is strongly non-degenerate.*

Example 5.2. Suppose that the space $\mathfrak{S}_A$ is non-degenerate and the operator $A_*(u)$ commutes with its formally adjoint for any $u \in \mathfrak{S}_A$. It, then, follows straightforward that the bundles $\operatorname{Ker} A$ and $\operatorname{Ker} A^*$ coincide and, by Lemma 5.1, $\mathfrak{S}_A$ is strongly non-degenerate. This situation, for example, occurs when $A_*(u)$ is the Euler-Lagrange operator of a functional on mappings $u : M \to M'$ and $\mathfrak{S}_A$ is a non-degenerate set of its critical points.

Lemma 5.2. *Suppose that a connected component $\mathfrak{S}_A^\alpha$ of the space of solutions is strongly non-degenerate in the sense of Morse-Bott and the index of $A_*(u)$, where $u \in \mathfrak{S}_A^\alpha$, vanishes. Then the bundles* $\operatorname{Ker} A$ *and* $\operatorname{Ker} A^*$ *over $\mathfrak{S}_A^\alpha$ are isomorphic.*

Proof. Let u be an arbitrary point from $\mathfrak{S}_A^\alpha$ and denote by $\operatorname{Im} A_*(u)$ the image of the operator $A_*(u)$ acting on smooth sections of u^*TM'. Due to the Fredholm theorem [28, Chapter XI] the spaces $\operatorname{Ker} A^*(u)$ and $\operatorname{Im} A_*(u)$ are orthogonal with respect to the L_2-scalar product and span the space of all smooth sections. Note that a vector ν from $\operatorname{Ker} A_*(u)$ can not be orthogonal to $\operatorname{Ker} A^*(u)$. Indeed, otherwise ν belongs to $\operatorname{Im} A_*(u)$

and this contradicts to the strong non-degeneracy of $\mathfrak{S}_A^\alpha$. Thus, the orthogonal (with respect to the L_2-scalar product) projection of $\operatorname{Ker} A_*(u)$ to $\operatorname{Ker} A^*(u)$ is an isomorphism. Since it is constructed canonically in terms of objects which depend on $u \in \mathfrak{S}_A^\alpha$ smoothly, it is also smooth and defines an isomorphism of the corresponding vector bundles. □

We proceed with the following statement.

Lemma 5.3. *Suppose that a connected component $\mathfrak{S}_A^\alpha$ of the space of solutions of equation* (5.1) *is strongly non-degenerate and the index of $A_*(u)$, where $u \in \mathfrak{S}_A^\alpha$, vanishes. Then for any $u \in \mathfrak{S}_A^\alpha$ the linearised operator $A_*(u)$ has a non-empty resolvent set, the space of solutions $\mathfrak{S}_A^\alpha$ is spectrally non-degenerate and its w_A-class vanishes.*

Proof. Indeed, since the space $\mathfrak{S}_A^\alpha$ is strongly non-degenerate in the sense of Morse-Bott, the linearised operator $A_*(u)$ for any $u \in \mathfrak{S}_A^\alpha$ has the zero as its eigenvalue without adjoint vectors. By Lemma 1.2, such an operator of zero index has a non-empty resolvent set. Moreover, the zero eigenvalue has constant algebraic multiplicity as u ranges over $\mathfrak{S}_A^\alpha$. Hence, for any path γ in the space $\mathfrak{S}_A^\alpha$ we can find a small real number λ such that the spectral flow $\mu_2(A_*(\gamma) - \lambda \operatorname{Id})$ is equal to zero. This implies the lemma. □

Now return to the study of the perturbed equations (0.5). The following consequence of the lemma above shows that the strong non-degeneracy in the sense of Morse-Bott and the vanishing of the index imply that, at least locally near $\mathfrak{S}_A^\alpha$, the moduli space is orientable.

Corollary 5.4. *A strongly non-degenerate connected component $\mathfrak{S}_A^\alpha$ such that the index of $A_*(u)$, where $u \in \mathfrak{S}_A^\alpha$, equals zero has a neighbourhood in the moduli space which is spectrally non-degenerate and its w_A-class vanishes.*

Proof. Since, by Lemma 1.2, for any $u \in \mathfrak{S}_A^\alpha$ the linearised operator $A_*(u, 0)$ has a non-empty resolvent set, so does $A_*(u, \mathsf{f})$, where (u, f) belongs to some neighbourhood of $\mathfrak{S}_A^\alpha$ in the moduli space. Moreover, Lemmas 2.2 and 5.3 also imply that this neighbourhood can be chosen spectrally non-degenerate. Now we show that one can also choose it such that its w_A-class vanishes.

Since $\mathfrak{S}_A^\alpha$ is a submanifold, there is its tubular neighbourhood which has $\mathfrak{S}_A^\alpha$ as its deformation retract. Then the value of the w_A-class on a loop in the tubular neighbourhood is equal to the value of w_A on a homotopic loop in the space $\mathfrak{S}_A^\alpha$. This implies the statement, since, due to Lemma 5.3, the class $w_A(\mathfrak{S}_A^\alpha)$ is trivial. □

5.2. Index of a connected strongly non-degenerate space of solutions

Suppose that the conditions of Theorem 3.1 hold and the universal moduli space is orientable. Let us fix an orientation (or an orienting cochain) given by a corresponding continuous $\widetilde{A}$-resolvent field $[\lambda]_{(u,\mathsf{f})}$. By $\mathfrak{S}_A^\alpha$ we denote a connected component of the space of solutions corresponding to equation (5.1). We always suppose that $\mathfrak{S}_A^\alpha$ is strongly non-degenerate in the sense of Morse-Bott, unless there is an explicit statement to the contrary. To define the index of $\mathfrak{S}_A^\alpha$ we, first, discuss the notion of a canonical chart centred at a point $(u, 0) = u \in \mathfrak{S}_A^\alpha$.

Consider the decompositions (3.1) such that:

- the space H_1 is the kernel of the operator $A_*(u)$;

- the space H_2 is the closure in the H^m-topology of the image of the operator $A_*(u)$ acting on smooth vector fields along u;
- the space $\mathcal{F}_1$ is equal to $(\Psi'_{\mathfrak{f}})^{-1}\operatorname{Ker} A_*(u)$; here $(\Psi'_{\mathfrak{f}})^{-1}$ stands for the right inverse operator which exists due to [18, Lemma 5.9]. The proof of this statement also gives a canonical construction of the right inverse operator.

Since $\mathfrak{S}^\alpha_A$ is strongly non-degenerate the spaces H_1 and H_2, chosen as above, are indeed complement. There is a freedom in the choice of an isomorphism $M : H_1 \to \mathcal{F}_1$ which together with $\Psi'_{\mathfrak{f}}$ defines the isomorphism

$$\Psi'_{\mathfrak{f}}(u,0)\circ M : \operatorname{Ker} A_*(u) \to \operatorname{Ker} A_*(u).$$

To fix this we suppose that an isomorphism M is chosen such that the operator above preserves an orientation.

A chart $\mathcal{C}_*$ corresponding to these choices of the spaces H_1, H_2, and $\mathcal{F}_1$ and the isomorphism M is called the *canonical chart* centred at $u \in \mathfrak{S}^\alpha_A$. By the index $\operatorname{ind}_2(u,0) \in \mathbb{Z}_2$ of a solution $u \in \mathfrak{S}^\alpha_A$ we mean the value $\chi_{\mathcal{C}_*}$ of the element of an orienting cochain corresponding to a canonical chart $\mathcal{C}_*$. By relation (4.1), the ambiguity in the choice of the subspace $\mathcal{F}_2$, complement to $\mathcal{F}_1$, does not affect the value of $\chi_{\mathcal{C}_*}$.

Clearly, due to this definition, $\operatorname{ind}_2(u,0)$ does not depend on a canonical chart centred at u. Moreover, for different solutions from $\mathfrak{S}^\alpha_A$ and a path joining them there exists a homotopy through canonical charts centred at the points of the path (here we use the fact that the right inverse operator $(\Psi'_{\mathfrak{f}})^{-1}$ can be constructed canonically). This together with relation (4.1) implies that $\operatorname{ind}_2(u,0)$ does not also depend on $u \in \mathfrak{S}^\alpha_A$. Thus, the following definition makes sense.

Definition 5.3. For a strongly non-degenerate connected component $\mathfrak{S}^\alpha_A$ of the space of solutions corresponding to equation (5.1) the value $\operatorname{ind}_2(u,0)$ for any $u \in \mathfrak{S}^\alpha_A$ is called the index $\operatorname{ind}_2(\mathfrak{S}^\alpha_A)$ of a component $\mathfrak{S}^\alpha_A$.

The following assertion is an analogue of Lemma 4.2; it reduces to the latter when $\mathfrak{S}^\alpha_A$ is a point.

Lemma 5.5. *Suppose that the moduli space (or its subdomain which contains $\mathfrak{S}^\alpha_A$) is spectrally non-degenerate and its orientation is given by an (A-)resolvent field $[\lambda]_{(u,\mathfrak{f})}$. Then the index $\operatorname{ind}_2(\mathfrak{S}^\alpha_A)$ is equal to the parity of the number of real eigenvalues of the operator $A^c_*(u)$, where $u \in \mathfrak{S}^\alpha_A$, between any representative $\lambda \in [\lambda]_{(u,0)}$ and zero (without counting the multiplicity of zero) exactly in the following cases:*

- *the dimension of the manifold $\mathfrak{S}^\alpha_A$ is even;*
- *the dimension of $\mathfrak{S}^\alpha_A$ is odd and the resolvent class $[\lambda]_{(u,0)}$ can be represented by a positive number.*

Proof. Due to the definition of the index $\operatorname{ind}_2(u,0)$ we have

$$(-1)^{\operatorname{ind}_2(u,0)} = \text{L-S index of } (A_*(u) - \lambda \operatorname{Id})^{-1}\circ \mathscr{A}_{\mathcal{C}_*}(u,0), \tag{5.2}$$

where λ is a representative of the class $[\lambda]_{(u,\mathfrak{f})}$ and the operator $\mathscr{A}_{\mathcal{C}_*}(u,0)$ is defined by formula (3.3). Due to the definition of a canonical chart the corresponding subspaces H_1 and H_2 are invariant under the operator in the right-hand side of (5.2). Therefore, by the multiplicity of the L-S index, see [26], $(-1)^{\mathrm{ind}_2}$ is equal to the product of the L-S indices of the operators

$$(A_*(u) - \lambda\,\mathrm{Id})^{-1}\circ(-\Psi'_{\mathsf{f}}\circ M)\Big|_{H_1} \text{ and } (A_*(u) - \lambda\,\mathrm{Id})^{-1}\circ A_*(u)\Big|_{H_2}.$$

Thus, for a proof of the lemma it is sufficient to show that in the enumerated cases the L-S index of the former operator equals 1. Indeed, since H_1 is the kernel of $A_*(u)$ this operator is equal to $(-1/\lambda)(-\Psi'_{\mathsf{f}}\circ M)$. Due to the choice of M for a canonical chart the L-S index of this equals 1 precisely in the cases stated in the lemma. □

Under the conditions of Theorem 3.1, for an orientable moduli space one can also (as in the case of non-degenerate solutions) define a relative index for a pair of strongly non-degenerate connected components of $\mathfrak{S}_A$ and show an analogue of Lemma 4.4. We end this section with the corresponding definition.

Definition 5.4. For two strongly non-degenerate components $\mathfrak{S}^0_A$ and $\mathfrak{S}^1_A$ of the space of solutions and an appropriately small real λ by the relative index $\mathrm{rel}_2(\mathfrak{S}^0_A, \mathfrak{S}^1_A, \lambda)$ we call the spectral flow $\mu_2(A_*(\gamma) - \lambda\,\mathrm{Id})$ along a path γ in the moduli space joining two points $u_0 \in \mathfrak{S}^0_A$ and $u_1 \in \mathfrak{S}^1_A$, where $\lambda \neq 0$ is a real resolvent point of the operators $A^c_*(u_0)$ and $A^c_*(u_1)$ such that there is no real eigenvalue of those operators between λ and zero.

Since the w_A-class of an orientable moduli space vanishes, it is clear that the definition above does not depend on a curve γ joining the points $u_0 \in \mathfrak{S}^0_A$ and $u_1 \in \mathfrak{S}^1_A$ as well as, due to the strong non-degeneracy, on these points.

6. Elements of Morse-Bott theory II

6.1. Statements of the results

Starting from this section we use the notation $\mathfrak{S}_{A,[v]}$ and $\mathfrak{S}^\alpha_{A,[v]}$ for the space of smooth solutions of equation (5.1) in a homotopy class $[v]$ and its components respectively. Below for quasi-linear elliptic equations we develop ideas analogous to the Morse-Bott theory. For this we perturb equation (5.1) by C^s-smooth ($s > \dim M/2 + 2$) small non-autonomous vector fields on M', i.e. in the notation above, we suppose that the parameter f takes values in a neighbourhood $\mathcal{U}$ of the origin in the space $\mathcal{F}$ formed by such vector fields. Besides, assume that the following hypotheses are satisfied:

- *Compactness.* There exists a neighbourhood $\mathcal{U}$ of the origin in the reflexive Banach space $\mathcal{F}$ formed by sufficiently smooth non-autonomous vector fields such that the projection
$$\mathfrak{M}_{A,[v]} \supset \pi_A^{-1}(\mathcal{U}) \xrightarrow{\pi_A} \mathcal{U} \subset \mathcal{F}, \tag{6.1}$$
given by the rule $(u, \mathsf{f}) \mapsto \mathsf{f}$, is proper.

- *Non-degeneracy (Morse-Bott).* The space of solutions $\mathfrak{S}_{A,[v]}$ is non-degenerate in the sense of Morse-Bott and the evaluation map
$$M \times \mathfrak{S}_{A,[v]} \ni (x, u) \overset{\mathrm{ev}}{\longmapsto} (x, u(x)) \in M \times M' \tag{6.2}$$
is an immersion.

The supposition that the evaluation map (6.2) is an immersion is equivalent to the supposition that non-trivial solutions of the linear differential equation $A_*(u)\nu = 0$ do not have zeroes for any $u \in \mathfrak{S}_{A,[v]}$. Indeed, denote by ev_x the restriction of the evaluation map (6.2) on the submanifold $x \times \mathfrak{S}_{A,[v]}$. The map ev is an immersion if and only if ev_x is an immersion for any $x \in M$. Now the statement follows from the rule

$$\operatorname{Ker} A_*(u) \ni \nu \mapsto \nu(x) \in T_{u(x)}M'$$

for the action of the differential $d\,\mathrm{ev}_x(u)$, where $u \in \mathfrak{S}_{A,[v]}$.

We start the discussion of Morse-Bott theory with recalling the following result in [20, Chapter 4].

Theorem 6.1. *Let A be a quasi-linear elliptic differential operator and $[v]$ be a homotopy class of mappings such that $\operatorname{ind}_{[v]} A = 0$. Suppose that the hypotheses of compactness and non-degeneracy are satisfied. Then for a generic non-autonomous vector field $\mathsf{f} \in \mathfrak{U}$ the number of solutions of the equation*

$$Au(x) = \mathsf{f}(x, u(x)), \qquad x \in M,$$

in $[v]$ is finite and equal mod 2 to the sum over all connected components of $\mathfrak{S}_{A,[v]}$ of the $\mathbb{Z}_2$-Euler numbers of the bundles $\operatorname{Ker} A^*$.

Now we turn to the computation of the integer-valued $\deg \pi_A$. We have the following oriented version of the previous result.

Theorem 6.2. *Let A be a quasi-linear elliptic differential operator and $[v]$ be a homotopy class of mappings such that $\operatorname{ind}_{[v]} A = 0$. Suppose that the hypotheses of compactness and non-degeneracy are satisfied. In addition, suppose also that the space of solutions $\mathfrak{S}_{A,[v]}$ is strongly non-degenerate. Then there exists a neighbourhood $\mathfrak{U}$ of the origin in the space $\mathfrak{F}$ of non-autonomous vector fields such that the subdomain $\pi_A^{-1}(\mathfrak{U})$ in the moduli space is orientable and the degree of the projection π_A restricted to $\pi_A^{-1}(\mathfrak{U})$ is given by the formula*

$$\deg \pi_A|_{\pi_A^{-1}(\mathfrak{U})} = \sum_\alpha \chi(\mathfrak{S}^\alpha_{A,[v]})(-1)^{\operatorname{ind}_2 \mathfrak{S}^\alpha_{A,[v]}},$$

where $\chi(\mathfrak{S}^\alpha_{A,[v]})$ denotes the Euler characteristic of the connected component $\mathfrak{S}^\alpha_{A,[v]}$ and the sum is taken over all connected components of $\mathfrak{S}_{A,[v]}$.

Theorem 6.2 gives a formula for the integer-valued degree under the strong non-degeneracy hypothesis, which (due to Lemma 5.2) implies that the bundles $\operatorname{Ker} A$ and $\operatorname{Ker} A^*$ are isomorphic. It seems plausible that there should be an oriented version of Theorem 6.1 for the non-isomorphic bundles $\operatorname{Ker} A$ and $\operatorname{Ker} A^*$.

6.2. Discussion of the proofs

In this subsection we discuss the proofs of Theorems 6.1 and 6.2 for some special case when an additional transversality supposition is satisfied. In this case we are able to compute the degree $\deg \pi_A$ by a direct construction of regular perturbations of equation (5.1) by vector fields. The use of this construction in a general case meets some technical difficulties connected with the properties of the evaluation map (6.2). In the case when the evaluation map is an immersion the way to overcome these difficulties, by enlarging the space of parameters – introducing the so-called multi-valued vector fields

– is described in [20, Appendix 4.B]. For the sake of simplicity we also assume that the space of solutions $\mathfrak{S}_{A,[v]}$ is connected.

Suppose that the space of solutions is non-degenerate in the sense of Morse-Bott. Let f be a smooth non-autonomous vector field on M'. For any map $u : M \to M'$ it determines the vector field $M \ni x \mapsto \mathsf{f}(x, u(x)) \in T_{u(x)}M'$ along the map u. Let us denote by $\mathfrak{A}_{A,[v]}$ the vector space formed by smooth non-autonomous vector fields f on M' such that for any $u \in \mathfrak{S}_{A,[v]}$ the vector field $\mathsf{f}(\cdot, u(\cdot))$ belongs to $\operatorname{Ker} A(u)^*$. In particular, any $\mathsf{f} \in \mathfrak{A}_{A,[v]}$ determines the section

$$\mathfrak{S}_{A,[v]} \ni u \longmapsto \mathsf{f}(\cdot, u(\cdot)) \in \operatorname{Ker} A(u)^* \tag{6.3}$$

of the bundle $\operatorname{Ker} A^*$ over the space $\mathfrak{S}_{A,[v]}$. In general, not every section of $\operatorname{Ker} A^*$ is induced by a vector field on M'. Moreover, it is straightforward to construct an example of an operator A and a homotopy class $[v]$ such that the hypotheses of compactness and non-degeneracy are satisfied and any vector field from $\mathfrak{A}_{A,[v]}$ induces the zero section of $\operatorname{Ker} A^*$ in accordance with (6.3); see [20, Appendix 4.A]. Nevertheless, below we shall adopt the following version of the non-degeneracy hypothesis.

Non-degeneracy & Transversality hypothesis. The space of solutions $\mathfrak{S}_{A,[v]}$ is non-degenerate in the sense of Morse-Bott (non-degeneracy) and the vector space $\mathfrak{A}_{A,[v]}$ contains a vector field f such that the section of the bundle $\operatorname{Ker} A^*$ given by (6.3) has only non-degenerate zeroes (transversality).

It is clear that the transversality part of the hypothesis is satisfied if the evaluation map (6.2) is an embedding. (This is, for example, so if $[v] = [pt]$ and the operator A satisfies the Liouville principle, see Example 5.1). Moreover, in this case any section of the bundle $\operatorname{Ker} A^*$ is induced by a vector field on M'. Indeed, a given section of $\operatorname{Ker} A^*$ we can regard as a map

$$M \times \mathfrak{S}_{A,[v]} \ni (x, u) \mapsto \mathsf{f}(x, u) \in T_{u(x)}M'.$$

Since the evaluation map is an embedding, setting $\mathsf{f}(x, u(x)) = \mathsf{f}(x, u)$ for all $x \in M$ we obtain a well-defined vector field f on the image $\operatorname{ev}(M \times \mathfrak{S}_{A,[v]})$. Alternatively, one can say that the vector f is given by pushing forward the vector $\mathsf{f}(x, u)$ under the map ev_x for any $x \in M$. Since this image is a submanifold the obtained vector field can be extended on $M \times M'$.

The proofs of Theorems 6.1 and 6.2 are based on a construction of a regular perturbation f of equation (5.1) such that all solutions of the perturbed equation within a given homotopy class $[v]$ belong to the space $\mathfrak{S}_{A,[v]}$ and are non-degenerate zeroes of some section of the vector bundle $\operatorname{Ker} A^*$. For this we need the following lemmas due to [20, Chapter 4]. To make our exposition more self-contained we include their proofs in Appendix A.2.

Lemma 6.3. *Let A be a quasi-linear elliptic differential operator and $[v]$ be a given homotopy class of mappings $M \to M'$. Suppose that the space of solutions $\mathfrak{S}_{A,[v]}$ is non-degenerate and the hypothesis of compactness is satisfied. Let f be a vector field from the space $\mathfrak{A}_{A,[v]}$. Then there exists $\varepsilon' > 0$ such that for any $0 < \varepsilon < \varepsilon'$ any homotopic to v solution of the equation*

$$Au(x) = \varepsilon \mathsf{f}(x, u(x)), \qquad x \in M, \tag{6.4}$$

is a solution of homogeneous equation (5.1).

Lemma 6.4. *Let A be a quasi-linear elliptic differential operator and $[v]$ be a homotopy class of mappings such that $\operatorname{ind}_{[v]} A = 0$. Suppose that the hypotheses of compactness and non-degeneracy & transversality are satisfied and let f be a vector field from the space $\mathfrak{A}_{A,[v]}$ such that the zeroes of the section given by (6.3) are non-degenerate. Then there exists $\varepsilon'' > 0$ such that for any $0 < \varepsilon < \varepsilon''$ and any solution u of equation (6.4) the linearised equation*

$$A_*(u)(x)\mathsf{v}(x) = \varepsilon \mathsf{f}_*(x, u(x))\mathsf{v}(x), \qquad x \in M, \tag{6.5}$$

has only the trivial solution.

Proof of Theorem 6.1. By the transversality hypothesis there exists a smooth vector field $\mathsf{f} \in \mathfrak{A}_{A,[v]}$ such that the zeroes of the induced section (6.3) of the bundle $\operatorname{Ker} A^*$ are non-degenerate. By Lemma 6.3 for a sufficiently small $\varepsilon > 0$ all solutions of equation (6.4) in a homotopy class $[v]$ belong to the space $\mathfrak{S}_{A,[v]}$. In particular, relation (6.4) implies that for any such solution u we have that $\mathsf{f}(x, u(x)) \equiv 0$, $x \in M$. Conversely, if for a map $u \in \mathfrak{S}_{A,[v]}$ the relation $\mathsf{f}(x, u(x)) \equiv 0$ holds, then u is a solution of equation (6.4). Thus, solutions of (6.4) for $0 < \varepsilon < \varepsilon'$ (where ε' is such as in Lemma 6.3) are exactly zeroes of the section $u \mapsto \mathsf{f}(\cdot, u(\cdot))$ of the vector bundle $\operatorname{Ker} A^*$. In particular, since these zeroes are non-degenerate, the number of the solutions is equal to the $\mathbb{Z}_2$-Euler number of $\operatorname{Ker} A^*$ modulo 2.

Now for a proof of the theorem it remains to show that there exists a sufficiently small $\varepsilon > 0$ such that the vector field $\varepsilon\mathsf{f}$ is a regular perturbation of equation (5.1). In fact, it is so for any sufficiently small $\varepsilon > 0$. Indeed, this follows by the combination of Lemma 6.4 and [18, Lemma 5.4]. □

Proof of Theorem 6.2. The existence of a neighbourhood $\mathcal{U}$ (of the origin in the space $\mathcal{F}$) such that the domain $\pi_A^{-1}(\mathcal{U})$ is orientable follows from Corollary 5.4 and Theorem 3.1. Making, if necessary, the neighbourhood $\mathcal{U}$ smaller one can suppose, since $\mathfrak{S}_{A,[v]}$ is compact, that there exists $\lambda > 0$ which is not an eigenvalue of the operator

$$A_*(u, \mathsf{f}) = A_*(u) - \mathsf{f}_*(\cdot, u(\cdot))$$

for any $(u, \mathsf{f}) \in \pi_A^{-1}(\mathcal{U})$, and the domain $\pi_A^{-1}(\mathcal{U})$ is oriented in accordance with its resolvent class $[\lambda]_{(u,\mathsf{f})}$.

Due to the transversality hypothesis we can pick up a smooth vector field $\mathsf{f} \in \mathfrak{S}_{A,[v]}$ such that all zeroes of the induced section (6.3) are non-degenerate. Following the lines of the proof of Theorem 6.1, we see that there exists $\bar\varepsilon > 0$ such that for any $0 < \varepsilon \leqslant \bar\varepsilon$ the vector field $\varepsilon\mathsf{f} \in \mathcal{U}$ is a regular value of the projection π_A. Moreover, solutions of the corresponding perturbed equation (6.4) belong to the space $\mathfrak{S}_{A,[v]}$ and are exactly non-degenerate zeroes of the section (6.3) of $\operatorname{Ker} A^*$. By $\mathsf{f}_*(u_0) : \operatorname{Ker} A_*(u_0) \to \operatorname{Ker} A^*(u_0)$ we denote the linearisation of this section at a zero $u_0 \in \mathfrak{S}_{A,[v]}$.

Now let $\ell(u) : \operatorname{Ker} A_*(u) \to \operatorname{Ker} A^*(u)$, where $u \in \mathfrak{S}_{A,[v]}$, be an isomorphism of the corresponding vector bundles given by the orthogonal projection with respect to the L_2-scalar product; see the proof of Lemma 5.2. Then $\ell^{-1}(u)\mathsf{f}(\cdot, u(\cdot))$ is a section of $\operatorname{Ker} A$, whose zeroes are non-degenerate, and its linearisation at a zero u_0 is given by the operator $\ell^{-1}\mathsf{f}_*(u_0)$. Suppose also that the number $\bar\varepsilon > 0$ is such that for any solution u of equation (6.4) the absolute value of any eigenvalue of the linear operator $\bar\varepsilon\ell^{-1}\mathsf{f}_*(u)$ is smaller than λ. Since (under our assumptions) the space of solutions $\mathfrak{S}_{A,[v]}$ is connected, then for a proof of the theorem we are to show that

$$\deg \pi_A|_{\pi_A^{-1}(\mathcal{U})} = \chi(\mathfrak{S}_{A,[v]})(-1)^{\operatorname{ind}_2 \mathfrak{S}_{A,[v]}}. \tag{6.6}$$

For this it is sufficient to prove that for any solution u of equation (6.4) the L-S index of the operator

$$D(u, \bar\varepsilon \mathsf{f}) = (A_*(u, \bar\varepsilon \mathsf{f}) - \lambda \operatorname{Id})^{-1} \circ A_*(u, \bar\varepsilon \mathsf{f})$$

is equal to $\operatorname{sgn}\det \ell^{-1} \mathsf{f}_*(u)(-1)^{\operatorname{ind}_2 \mathfrak{S}_{A,[v]}}$. Indeed, due to Lemma 4.2 this L-S index is equal to $(-1)^\mu$, where $\mu = \operatorname{ind}_2(u, \bar\varepsilon \mathsf{f})$. Thus, equality (6.6) would follow from Lemma 4.3 and the Poincaré-Hopf formula for the Euler characteristic.

Consider the operator $A_*(u)$ acting on smooth sections of u^*TM' and denote by H_2 and H_2' the closures of its image in the H^m- and H^{m-k}-topologies respectively. Define the isomorphism

$$L = \ell(u) \oplus \operatorname{Id} : \operatorname{Ker} A_*(u) \oplus H_2' \to \operatorname{Ker} A^*(u) \oplus H_2'.$$

Since $\ell(u)$ is the projection of $\operatorname{Ker} A_*(u)$ to $\operatorname{Ker} A^*(u)$, the operator L can be deformed to the identity through the linear isomorphisms L_t of the form $\ell_t \oplus \operatorname{Id}$;

$$\ell_t = (1-t)\ell(u) + t \operatorname{Id}_{\operatorname{Ker} A_*(u_0)}, \qquad t \in [0,1].$$

Now we assert that, maybe after making $\bar\varepsilon$ smaller, the operator

$$L_t \circ L^{-1} \circ A_*(u, \bar\varepsilon \mathsf{f}) - \lambda \operatorname{Id}$$

is an isomorphism for any $t \in [0,1]$. Indeed, firstly, it maps H_2 isomorphically onto H_2'. Secondly, its restriction to $\operatorname{Ker} A_*(u)$ is equal to $\ell_t(-\bar\varepsilon \ell^{-1} \mathsf{f}_*(u)) - \lambda \operatorname{Id}$; this operator has a trivial kernel (due to the choice of $\bar\varepsilon > 0$) and its image does not intersect H_2' (maybe after making $\bar\varepsilon$ smaller). This implies that the family of quasi finite-dimensional isomorphisms

$$\big(L^{-1} \circ A_*(u, \bar\varepsilon \mathsf{f}) - \lambda L_t\big)^{-1} \circ \big(L^{-1} \circ A_*(u, \bar\varepsilon \mathsf{f})\big)$$

defines the homotopy between the operators $D(u, \bar\varepsilon \mathsf{f})$ and

$$\big(L^{-1} \circ A_*(u, \bar\varepsilon \mathsf{f}) - \lambda \operatorname{Id}\big)^{-1} \circ \big(L^{-1} \circ A_*(u, \bar\varepsilon \mathsf{f})\big);$$

in particular, they have the same L-S index. We compute this L-S index now.

Denote the latter operator by $D_L(u, \bar\varepsilon \mathsf{f})$. Note that the kernel of $A_*(u)$ is invariant under it and we have

$$D_L(u, \varepsilon \mathsf{f})|_{\operatorname{Ker} A_*(u)} = -(-\varepsilon \ell^{-1} \mathsf{f}_*(u) - \lambda \operatorname{Id})^{-1} \circ \varepsilon \ell^{-1} \mathsf{f}_*(u).$$

Due to the choice of $\bar\varepsilon$ the latter operator is non-degenerate, and by elementary properties of the Leray-Schauder degree [26] we obtain

$$\text{L-S index of } \ D_L(u, \bar\varepsilon \mathsf{f})|_{\operatorname{Ker} A_*(u)} = \operatorname{sgn}\det \ell^{-1} \mathsf{f}_*(u). \tag{6.7}$$

Further, the space H_2 is invariant under the operator $D_L(u, 0)$ and the image $D_L(u, \varepsilon \mathsf{f}) H_2$ is transversal to $\operatorname{Ker} A_*(u)$ for any ε such that $0 \leqslant \varepsilon \leqslant \bar\varepsilon$. This means that the parameter $\varepsilon \in [0, \bar\varepsilon]$ establishes the deformation in the class of quasi finite-dimensional isomorphisms between the operators $D_L(u, \bar\varepsilon \mathsf{f})$ and

$$D_L(u, 0) H_2 \oplus D_L(u, \bar\varepsilon \mathsf{f})|_{\operatorname{Ker} A_*(u)}.$$

Hence, due to the multiplicity property of the Leray-Schauder degree [26], the L-S index of $D(u, \bar\varepsilon \mathsf{f})$ is equal to the product of the L-S indices of the operators in the direct sum above. Finally, since the operator $D_L(u, 0)|_{H_2}$ is given by

$$\Big[(A_*(u) - \lambda \operatorname{Id})^{-1} \circ A_*(u)\Big]\Big|_{H_2},$$

wee see that, due to Lemma 5.5, its L-S index is equal to $(-1)^{\operatorname{ind}_2 \mathfrak{S}_{A,[v]}}$. Combining this with relation (6.7) we finish the proof. □

7. Example 0. Introduction

In this section we consider the periodic boundary value problem for an ordinary differential equation on a compact manifold M'. Using our techniques we sharpen a classical estimate for the number of periodic solutions of a non-autonomous system of differential equations. The results are not new, but it seems that some of them exist only in the mathematical folklore.

On the Lefschetz formula for the number of 1-periodic orbits

Introduction. Let M' be a closed manifold and let us consider the periodic boundary value problem

$$\frac{d}{dt}u(t) + \mathsf{f}(t, u(t)) = 0, \quad t \in S^1 = \mathbb{R}/\mathbb{Z}. \tag{7.1}$$

Here f is a C^3-smooth non-autonomous vector field on M' and u is a C^4-smooth map $\mathbb{R}/\mathbb{Z} \to M'$. It is known that *for a generic vector field* f *the number of solutions of equation* (7.1) *is finite.* Indeed, due to the general theory, for example in [18, 20], the moduli space $\mathfrak{M}$ formed by pairs (u, f), where f is an arbitrary vector field as above and u is a solution of the corresponding equation (7.1), is a (non-connected) C^2-smooth Banach manifold and its projection π, $(u, \mathsf{f}) \mapsto \mathsf{f}$, is a C^2-smooth Fredholm map of zero index. Hence, for any regular value f of the projection π the space of solutions $\pi^{-1}(\mathsf{f})$ consists of isolated points. Now the Arzela-Ascoli theorem implies that the projection π is proper and the statement is demonstrated; see details in [18, 20].

Following the familiar lines we could estimate the number of these solutions in the following manner. First, for such a generic f let us consider the succession map S_{f} which sends $u_0 \in M'$ to the point $u(1) \in M'$, where $u(t)$, $0 \leqslant t \leqslant 1$, solves $\dot{u}(t) + \mathsf{f}(t, u(t)) = 0$, $u(0) = u_0$. Then solutions of equation (7.1) correspond to the fixed points of $S_{\mathsf{f}} : M' \to M'$ and, since the latter map is homotopic to the identity, by the Lefschetz theorem the algebraic number of solutions equals the Euler characteristic $\chi(M')$. Moreover, each solution u in this sum is counted with the sign $\operatorname{sgn}\det(\mathbf{F} - \mathrm{Id})$, where $\mathbf{F}$ is the Floquet map (the linearised succession map $dS_{\mathsf{f}}(u(0))$):

$$\mathbf{F} : T_{u(0)}M' \to T_{u(0)}M', \qquad \nu_0 \mapsto \nu(1), \tag{7.2}$$

where $\nu(t)$, $0 \leqslant t \leqslant 1$, solves $\dot{\nu} + \mathsf{f}'_u(\cdot, u(\cdot))\nu = 0$, $\nu(0) = \nu_0$.

Counting of 1-periodic solutions within a homotopy class. Now we fix a homotopy class $[v]$ of free loops in M' and denote by $\mathfrak{M}_{[v]}$ the corresponding universal moduli space, formed by solutions of equation (7.1) within $[v]$. The operator given by the linearisation of the left-hand side in (7.1) satisfies the hypotheses of Example 3.3 (see also Example 3.2) and, hence, Theorem 3.1 applies – the moduli space is orientable. By the same symbol π as above we denote the projection $(u, \mathsf{f}) \mapsto \mathsf{f}$ restricted to the moduli space $\mathfrak{M}_{[v]}$. This map is proper (see the discussion above) and, hence, its degree $\deg \pi$ – the algebraic number of solutions in $[v]$ – is well-defined.

In more detail, since the operator d/dt is anti self-adjoint, then due to perturbation theory [17] the number of real positive (and also negative) eigenvalues of the linearised operator

$$\nu \longmapsto \frac{d}{dt}\nu + \mathsf{f}'_u(\cdot, u(\cdot))\nu, \qquad \nu \text{ is a section of } u^*TM', \tag{7.3}$$

is finite. For a non-degenerate $(u, \mathsf{f}) \in \mathfrak{M}_{[v]}$ we define its index $\operatorname{ind}(u, \mathsf{f})$ as this integer. Moreover, for any (u, f) from the moduli space the resolvent class of a real number λ

that is greater than all these real positive eigenvalues defines a continuous field. If the moduli space is oriented in accordance with this field of resolvent classes, then, due to Lemmas 4.2 and 4.3, we have the following formula for the degree:

$$\deg \pi = \sum_{\pi^{-1}(\mathsf{f})} (-1)^{\mathrm{ind}(u,\mathsf{f})} \tag{7.4}$$

for any regular value f. We compute this quantity now.

First, consider the case when a homotopy class $[v]$ is non-trivial. Then the space $\pi^{-1}(0)$ is empty and, hence, zero is a regular value of the projection π. Thus, the degree (7.4) is equal to zero. Now suppose that $[v]$ is trivial. Consider the space of contractible solutions of the equation $\dot{u}(t) = 0$, where $t \in S^1 = \mathbb{R}/\mathbb{Z}$; it is formed by constant mappings and, hence, is diffeomorphic to M'. The kernel of the linearised operator

$$\left(\frac{du_y}{dt}\right)_* \nu = \frac{d}{dt}\nu, \qquad \nu : S^1 \to T_y M' \simeq \mathbb{R}^{\dim M'},$$

at any such solution $u_y(\cdot) \equiv y$, $y \in M'$, consists only of constant vector-functions. This implies that the space of contractible solutions is non-degenerate in the sense of Morse-Bott. Besides, the operator $(du_y/dt)_*$ does not have adjoint vectors corresponding to the zero eigenvalue and, hence, the space of constant solutions is strongly non-degenerate. Combining this with Theorem 6.2 we obtain the following statement.

Theorem 7.1. *For a generic vector field f on a closed manifold M' the number of solutions of equation* (7.1) *is finite and the algebraic sum* (7.4) *of these in a fixed homotopy class $[v]$ is equal to the Euler characteristic $\chi(M')$ or zero with respect to the cases $[v]$ is trivial or not. Moreover, for any non-autonomous vector field there exists a contractible* 1*-periodic solution of* (7.1) *provided $\chi(M') \neq 0$.*

As a consequence we have the following application.

Corollary 7.2. *The evaluation map ev_u from the group of diffeomorphisms Diff(M') to M' given by*

$$\mathit{Diff}(M') \ni \varphi \longmapsto \varphi(u) \in M', \qquad u \in M',$$

induces the trivial homomorphism $\mathrm{ev}_u^\natural : \pi_1(\mathit{Diff}(M')) \to \pi_1(M')$ for any $u \in M'$ provided $\chi(M') \neq 0$.[3]

Proof. Let $\varphi_t : \mathbb{R}/\mathbb{Z} \to \mathit{Diff}(M')$ be a loop in the group of diffeomorphisms. It defines a vector field $\mathsf{f}(t, \varphi_t(u)) = \dot{\varphi}_t(u)$ whose 1-periodic orbits are precisely the closed paths $\mathbb{R}/\mathbb{Z} \ni t \mapsto \varphi_t(u) \in M'$. Due to Theorem 7.1 at least one of these is contractible and, hence, so are all of them. □

Due to Lemma 7.3 below the solutions in the algebraic sum (7.4) within an orientable homotopy class (i.e. such that the pull-back v^*TM' is the trivial bundle) are counted with the same sign as in the Lefschetz formula for the algebraic number of all 1-periodic solutions; if the homotopy class is non-orientable the solutions are counted with the sign opposite to this in the Lefschetz formula. Thus, Theorem 7.1 indeed sharpens the classical result.

[3]Corollary 7.2 is a specific case of a more general result, which can be proved using the Nielsen-Wecken theory of fixed point classes, see [10, Theorem IV.1]: the image of $\pi_1(\mathit{Maps}(M', M'))$ under the evaluation map, called the Gottlieb group, vanishes provided $\chi(M')$ does not equal to zero. It is also easy to see that the statement is sharp in the sense that it no longer holds if $\chi(M') = 0$. Indeed, all orbits of the action of $S^1 = \mathbb{R}/\mathbb{Z}$ on the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ given by $S^1 \ni t : \mathbb{T}^n \to \mathbb{T}^n$, $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n + t)$, are non-contractible.

Lemma 7.3. *For an* 1*-periodic non-degenerate solution* u *of equation* (7.1) *let us denote by* $\mathbf{F}$ *its Floquet map* (7.2)*. Then the following relation holds:*

$$\operatorname{sgn}\det(\mathbf{F}-\mathrm{Id}) = \begin{cases} (-1)^{\mathrm{ind}(u,\mathsf{f})}, & \text{if } u^*TM' \text{ is trivial,} \\ (-1)^{\mathrm{ind}(u,\mathsf{f})-1}, & \text{if } u^*TM' \text{ is non-trivial.} \end{cases}$$

Proof. First, suppose that the bundle u^*TM' is trivial. Then its sections can be regarded as 1-periodic vector-functions and the linearised differential operator has the form $d/dt + \mathsf{G}(t)$, where $\mathsf{G}(t)$ is a 1-periodic matrix-function. Direct calculations show that a complex number λ is an eigenvalue of $d/dt + \mathsf{G}(t)$ with periodic boundary conditions if and only if $\exp(-\lambda)$ is an eigenvalue of its Floquet map $\mathbf{F}$. Moreover, they have the same algebraic multiplicity; i.e. the dimensions of the spaces formed by the corresponding eigenvectors together with the adjoint vectors coincide. Thus, we see that in this case the number of real positive eigenvalues of the linearised operator ($=\mathrm{ind}(u,\mathsf{f})$) is equal to the number of the real negative eigenvalues of the linear operator $\mathbf{F}-\mathrm{Id}$. This proves the lemma when u^*TM' is trivial.

Now let u^*TM' be a non-trivial bundle. Then its sections can be regarded as vector-functions $\nu : [0,1] \to \mathbb{R}^{n'}$ such that $\nu(0) = E^-\nu(1)$, where E^- is a diagonal matrix with the components $-1, 1, \ldots, 1$. Without loss of generality one can suppose that the Floquet map $\mathbf{F} : T_{u(0)}M' \to T_{u(1)}M'$, as a matrix in $\mathbb{R}^{n'}$, has a canonical Jordan form. As above one can show that a complex number λ is an eigenvalue of the linearised operator with the boundary condition $\nu(0) = E^-\nu(1)$ if and only if $\exp(-\lambda)$ is an eigenvalue of the matrix $E^-\mathbf{F}$ of the same algebraic multiplicity. The collections of eigenvalues corresponding to $E^-\mathbf{F}$ and $\mathbf{F}$ are easy to compare: if $\exp(-\lambda_1), \ldots, \exp(-\lambda_\ell)$ are eigenvalues of $E^-\mathbf{F}$ with multiplicities $k_1, \ldots, k_\ell$, then $-\exp(-\lambda_1), \exp(-\lambda_1), \ldots, \exp(-\lambda_\ell)$ are eigenvalues of $\mathbf{F}$ with multiplicities $1, k_1-1, \ldots, k_\ell$ respectively. Thus, we see that the difference between the number of negative eigenvalues of $\mathbf{F}-\mathrm{Id}$ and the number of positive eigenvalues of $d/dt + \mathsf{G}(t)$ is equal to one. This demonstrates the lemma. □

Finally, note that for the case of time-independent vector field the formula for the algebraic number of the contractible solutions of (7.1) can be regarded as the generalised Poincaré-Hopf formula. Indeed, all non-degenerate zeroes of a vector field $-\mathsf{f}$ are contractible solutions (= constant mappings) which enter into sum (7.4) with the same sign as in the formula of Poincaré-Hopf (also a consequence of Lemma 7.3). Thus, in this case the algebraic number of contractible solutions which are not constant is also equal to zero.

8. Example I. Perturbed harmonic map equation

Here we consider quasi-linearly perturbed harmonic map equation. As results in [14, 22] show, the equations of this type can be useful in applications. A semi-linear perturbation of the harmonic map equation can be also considered as an equation for maps with prescribed tension field. In view of the compactness results [15, 19] our discussion below is basically concerned with the case when the target manifold is non-positively curved.

After the description of a set-up we apply our Morse-Bott theory to compute the algebraic number of maps with prescribed tension field within a fixed homotopy class. This covers the cases when a target manifold is negatively curved or is a non-positively curved locally symmetric space; see Theorem 8.2. For negatively curved manifolds the corresponding results were previously announced in [15] and were also treated in [16]. The $\mathbb{Z}_2$-version of Theorem 8.2 is due to [20].

Notation. Let M and M' be closed Riemannian manifolds. Their Riemannian metrics give rise to a natural metric on the bundle $J^1(M, M')$ of 1-jets over the space $M \times M'$ and for maps $u : M \to M'$ we consider the energy functional

$$E(u) = \frac{1}{2} \int\limits_M \|du(x)\|^2 \, dVol_g(x), \qquad x \in M.$$

The Euler-Lagrange equation for this functional

$$-\tau(u)(x) = 0, \qquad x \in M,$$

is called the *harmonic map equation* and its solutions are *harmonic mappings*. In local coordinates on M and M' the operator $\tau(u)$ has the form

$$\tau^i(u) = \Delta_M u^i + g^{\alpha\beta} \Gamma'^{\,i}_{jk} \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta},$$

where $g^{\alpha\beta}$ and $\Gamma'^{\,i}_{jk}$ denote the tensor inverse to the metric on M and the Cristoffel symbols of the Levi-Civita connection on M' respectively, and Δ_M is the Laplace-Beltrami operator on M.[4]

For maps $u : M \to M'$ in a fixed homotopy class $[v]$ let us consider the quasi-linearly perturbed harmonic map equation

$$-\tau(u)(x) = \mathsf{G}(x, u(x)) \cdot du(x) + \mathsf{g}(x, u(x)), \qquad x \in M. \tag{8.1}$$

Here G stands for a morphism of the vector bundles $J^1(M, M')$ and $(\Pi')^* TM'$, where $\Pi' : M \times M' \to M'$ is a natural projection onto the Cartesian factor, and g is a non-autonomous vector field on M'. We suppose that the G and g are $W^{p,s+1}$-smooth ($p > \dim M + \dim M'$, $s > \dim M/2 + 2$), and hence C^s-smooth, and denote by $\mathcal{F}$ the vector space of pairs (G, g). Recall that the universal moduli space $\mathfrak{M}_{\tau,[v]}$ is defined as the set of triples $(u, \mathsf{G}, \mathsf{g})$ which satisfy (8.1) and such that $u \in [v]$ is H^{s-1}-smooth and $(\mathsf{G}, \mathsf{g}) \in \mathcal{F}$. We also have the natural projection

$$\pi : \mathfrak{M}_{\tau,[v]} \to \mathcal{F}, \qquad (u, \mathsf{G}, \mathsf{g}) \mapsto (\mathsf{G}, \mathsf{g}),$$

whose fibers are spaces of solutions with a fixed right-hand side.

The following important assertion is due to [15, 19].

Compactness theorem. *Suppose that a Riemannian manifold M' has non-positive sectional curvature. Then for any homotopy class $[v]$ of maps from M to M' there exists a constant $C_{[v]} > 0$ such that for the domain*

$$\mathcal{U} = \{(\mathsf{G}, \mathsf{g}) \in \mathcal{F} : \ \|\mathsf{G}(x, u)\| < C_{[v]} \ \textit{for any } x \in M, u \in M'\}. \tag{8.2}$$

the map $\pi|_{\pi^{-1}(\mathcal{U})}$ is proper.

Orientability of the moduli space and its consequences. Recall the definition of the linearised operator for equation (8.1). For an arbitrary map u and a vector field ν along it take a family of maps $u_t : M \to M'$, where $t \in (-\varepsilon, \varepsilon)$, such that $u_0 = u$ and $(\partial/\partial t)|_{t=0} u_t = \nu$. Let ∇' be a Levi-Civita connection on M'; by the same symbol we

[4] There are different conventions for the choice of the sign of the Laplace-Beltrami operator. Due to our definition this operator is non-positive.

also denote its pull-back on the bundle U^*TM', where $U(t,\cdot) = u_t(\cdot)$. The linearised operator $L_*(u, \mathsf{G}, \mathfrak{g})$ acts on a vector field ν along u by the formula

$$L_*(u, \mathsf{G}, \mathfrak{g})\nu = \nabla'_{\frac{\partial}{\partial t}}\Big|_{t=0}(-\tau(u_t) - \mathsf{G}_t \cdot du_t - \mathfrak{g}_t)\,;$$

here G_t and $\mathfrak{g}_t$ denote the morphism $\mathsf{G}(\cdot, u_t(\cdot))$ and the vector field $\mathfrak{g}(\cdot, u_t(\cdot))$ respectively. Direct calculations, see [18, 20], show that this operator has the form $J(u)+\{$first-order differential operator$\}$, where

$$J(u)\nu = -trace_g(\nabla^2\nu + R'(du, \nu)du) \tag{8.3}$$

is the linearisation of $-\tau(u)$, called the Jacobi operator.

Since the Laplacian $-trace_g\nabla^2$ is self-adjoint and non-negative, then due to elementary perturbation theory [17] for any $(u, \mathsf{G}, \mathfrak{g})$ there exists a real constant C such that the spectrum of $L_*^c(u, \mathsf{G}, \mathfrak{g})$ lies in the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z \geqslant C\}$. Moreover, the statement in Example 3.3 (see also Example 3.2) applies, and the moduli space $\mathfrak{M}_{\tau,[v]}$ is spectrally non-degenerate and has the vanishing w_A-class. Now we summarise the consequences of this due to the results in Sections 3 and 4:

- First, due to Theorem 3.1, the universal moduli space is orientable. For any $(u, \mathsf{G}, \mathfrak{g}) \in \mathfrak{M}_{\tau,[v]}$ we can choose a real number λ such that there is no real eigenvalue of $L_*^c(u, \mathsf{G}, \mathfrak{g})$ less than λ, and its resolvent class $[\lambda]_{(u,\mathsf{G},\mathfrak{g})}$ defines a continuous field on the moduli space. In sequel we suppose that $\mathfrak{M}_{\tau,[v]}$ is oriented in accordance with this field of resolvent classes. For a non-degenerate point $(u, \mathsf{G}, \mathfrak{g})$ from the moduli space (i.e. a point such that $L_*(u, \mathsf{G}, \mathfrak{g})$ is non-degenerate) we define the index $\operatorname{ind}(u, \mathsf{G}, \mathfrak{g})$ as the number of real negative eigenvalues of $L_*^c(u, \mathsf{G}, \mathfrak{g})$; see Lemma 4.2.

- Suppose that the target manifold M' has non-positive sectional curvature. Then the compactness theorem and Lemma 4.3 (see also Example 4.2) imply that for a generic right-hand side $(\mathsf{G}, \mathfrak{g}) \in \mathfrak{U}$ the number of solutions in a fixed homotopy class is finite and the sum

$$\deg \pi|_{\pi^{-1}(\mathfrak{U})} = \sum_{\pi^{-1}(\mathsf{G},\mathfrak{g})} (-1)^{\operatorname{ind}(u,\mathsf{G},\mathfrak{g})} \tag{8.4}$$

  does not depend on such a right-hand side. Further, non-degenerate solutions $(u_1, \mathsf{G}, \mathfrak{g})$ and $(u_2, \mathsf{G}, \mathfrak{g})$ enter into the algebraic sum (8.4) above with the same sign if and only if the spectral flow along a curve in the moduli space joining them is equal to zero.

Elements of the Morse-Bott theory. Now we use the results in Section 6 to compute the degree. First, we discuss when the space of harmonic maps in a fixed homotopy class is non-degenerate in the sense of Morse-Bott. Throughout we suppose that a target manifold is endowed with a Riemannian metric of non-positive sectional curvature. The following general result is due to [29].

Proposition 8.1. *Let M and M' be closed Riemannian manifolds and suppose that M' has non-positive sectional curvature. Then the space of harmonic mappings $\mathfrak{S}_{\tau,[v]}$ in a homotopy class $[v]$ is a compact connected manifold (possibly with a Lipschitz boundary) which is smooth outside of its boundary. Moreover, $\mathfrak{S}_{\tau,[v]}$ admits a Riemannian metric of non-positive sectional curvature such that for any $x \in M$ the evaluation map ev_x is an isometric immersion into M'. In addition, if M' is real-analytic, then $\mathfrak{S}_{\tau,[v]}$ is closed.*

Let us consider the case when $[v]$ is trivial in more detail. Since M' has non-positive sectional curvature, then due to [12, Corollary of Statement (F)] the space of contractible harmonic mappings is formed by all constant mappings $u_y \equiv y$, $y \in M'$. The Jacobi operator $J(u_y)$ at a constant solution coincides with the Laplace-Beltrami operator on vector-valued functions $\nu : M \to T_y M' \simeq \mathbb{R}^{n'}$ and its kernel is formed by all constant vector functions. Thus, the space $\mathfrak{S}_{\tau,[pt]}$ is diffeomorphic to M' and is non-degenerate in the sense of Morse-Bott; cf. Example 5.1.

Now suppose that a given homotopy class $[v]$ is non-trivial. If the sectional curvature of M' is strictly negative, then the corresponding space $\mathfrak{S}_{\tau,[v]}$ can be also explicitly described. Due to [12] we have the following alternative – either

- a homotopy class contains a harmonic map u onto a closed geodesic γ in M' and harmonic maps in $[v]$ coincide with 'rotations of u' (i.e., maps obtained by moving each point $u(x)$ a fixed oriented distance along γ); or
- a homotopy class contains a unique harmonic representative.

Thus, under the strict curvature hypothesis $\mathfrak{S}_{\tau,[v]}$ is a circle or a point and, by [20, Lemma 4.17] is also non-degenerate in the sense of Morse-Bott.

There are other situations when $\mathfrak{S}_{\tau,[v]}$ is non-degenerate. More precisely, this is the case when M' is a locally symmetric space of non-positive sectional curvature; see [31, Proposition 3.2]. Note that, since the Jacobi operator (8.3) is formally self-adjoint, in all these cases the space $\mathfrak{S}_{\tau,[v]}$ is, in fact, strongly non-degenerate.

Finally, recall that due to the curvature hypothesis the Jacobi operator is non-negative. This together with Lemma 5.5 implies that when the dimension of the space $\mathfrak{S}_{\tau,[v]}$ is even, the index $\operatorname{ind}_2 \mathfrak{S}_{\tau,[v]}$ is equal to zero. (The case when the dimension is odd does not affect the formulas below, since the corresponding Euler characteristics vanish.) As a result of the discussion above we arrive at the following consequence of Theorem 6.2.

Theorem 8.2. *Suppose that a Riemannian manifold M' has non-positive sectional curvature. Then for any homotopy class $[v]$ of mappings from M to M' there exists a constant $C_{[v]} > 0$ such that for a generic right-hand side (G, g) from the domain $\mathfrak{U}$ given by (8.2) the number of solutions of equation (8.1) in $[v]$ is finite and their algebraic number (8.4) does not depend on (G, g). Moreover,*

(i) if the homotopy class $[v]$ is trivial, then $\deg \pi|_{\pi^{-1}(\mathfrak{U})} = \chi(M')$.

(ii) if the sectional curvature of M' is negative and the homotopy class is not trivial, then the degree $\deg \pi|_{\pi^{-1}(\mathfrak{U})}$ is equal to one or zero with respect to the cases $[v]$ contains a unique harmonic representative or not. In particular, if the first case holds, then equation (8.1) has a solution in $[v]$ for any right-hand side $(\mathsf{G}, \mathsf{g}) \in \mathfrak{U}$.

(iii) if M' is locally symmetric, then for any homotopy class $[v]$ the manifold $\mathfrak{S}_{\tau,[v]}$ is closed and $\deg \pi|_{\pi^{-1}(\mathfrak{U})} = \chi(\mathfrak{S}_{\tau,[v]})$.

In the recent preprint [22], using different arguments, it is shown that $\deg \pi$ is still equal to the Euler characteristic of the space $\mathfrak{S}_{\tau,[v]}$, formed by homotopic harmonic maps, even when the hypotheses providing its non-degeneracy are dropped. Moreover, when the perturbation of the harmonic map equation is variational, the Morse inequalities for the number of homotopic solutions hold.

Example. The classical example of equation (8.1) is given by the Euler-Lagrange equations of the multi-valued functional

$$\frac{1}{2}\int_0^1 |\dot{u}|^2\, dt + \int_D \tilde{u}^*\Omega + \int_0^1 V(t,u(t))dt,$$

which describes the movement of a charged particle on a Riemannian manifold in the electro-magnetic field. Here D is a unit ball in $\mathbb{R}^2$ and $\tilde{u} : D \to M'$ is such that $\tilde{u}(\exp(2\pi it)) = u(t)$; the choice of such a map $\tilde{u}$ is parameterised by the elements of $\pi_2(M')$. The closed 2-form Ω is called the strength of a magnetic field and V is the potential of an electric field. In the case when the magnetic field is potential, $\Omega = dA$, or more generally $\Omega|_{\pi_2} = 0$ the corresponding functional becomes single-valued. The existence of 1-periodic critical points of this functional is a popular question. The essential problem here is the lack of compactness when the magnetic field becomes too strong.

Theorem 8.2 implies the existence of a contractible 1-periodic magnetic extremal provided that the strength of a magnetic field is not too large, the Euler-Poincare number $\chi(M')$ does not vanish, and M' has non-positive sectional curvature.

9. Example II. Semi-linear Cauchy-Riemann equations

Here we discuss the Cauchy-Riemann equations perturbed by a semi-linear term for mappings of a 2-dimensional torus with a given conformal structure into a symplectic manifold. Various versions of the perturbed Cauchy-Riemann equations arise naturally in symplectic topology and Floer's theory [11, 25]; see also [23]. Our Theorem 9.1 can be regarded as an analogue of the Lefschetz formula for a dynamical system with complex time. It sharpens the results established in [23] for the Kähler case and in [3] for the case of monotone symplectic manifolds with equations perturbed by a gradient term.

Basic notation and facts. Let $\mathbb{T}^2$ be a complex torus, viewed as the quotient $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, where τ is an element of the fundamental domain for the action of $\mathrm{PSL}(2,\mathbb{Z})$ on the upper half-plane. We suppose that $\mathbb{T}^2$ is endowed with a Riemannian metric from the conformal class determined by the complex structure and its volume equals 1. Let M' be an almost complex manifold, i.e. endowed with an operator field J such that $J^2 = -\,\mathrm{Id}$. For a mapping $u : \mathbb{T}^2 \to M'$ let us consider the vector field

$$\bar{\partial}u(z) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + J(u)\frac{\partial u}{\partial y}\right) \in T_{u(z)}M', \quad z = x+iy \in \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}).$$

This defines the differential operator

$$\mathit{Maps}(\mathbb{T}^2, M') \ni u \longmapsto \bar{\partial}u \in \mathit{Sections}(u^*TM'),$$

which is, obviously, elliptic; see [18, 20]. Recall that a map $u : \Sigma \to M'$, where Σ is a Riemannian surface, is called J-holomorphic if the complex anti-linear part $(1/2)(du + J \circ du \circ i_\Sigma)$ of the differential du vanishes. It is clear that $u : \mathbb{T}^2 \to M'$ is J-holomorphic if and only if $\bar{\partial}u = 0$.

We study the following perturbed Cauchy-Riemann equations

$$\bar{\partial}u(z) = \mathsf{f}(z, u(z)), \qquad z \in \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}) = \mathbb{T}^2, \tag{9.1}$$

where f is a $W^{p,5}$-smooth ($p > \dim M' + 2$), and hence C^4-smooth, non-autonomous vector field on M' and u is a H^3-smooth map. For a homotopy class $[v]$ of mappings

$\mathbb{T}^2 \to M'$ denote by $\mathfrak{M}_{\bar\partial,[v]}$ the universal moduli space, formed by pairs (u,f) satisfying (9.1) with $u \in [v]$. Solutions of (9.1) we also call *perturbed J-holomorphic tori.*

Suppose that M' is symplectic and its symplectic form ω is J-invariant and tames J, i.e. $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$ is a Hermitian metric. Such a triple (g, J, ω) is called an *almost Kähler structure* on M'. It is a known (and easily proved) fact [25, Chapter 4] that the quantity

$$S = \inf\{E(u) : u \text{ is a non-constant } J\text{-holomorphic sphere in } M'\}$$

is positive. Here the infimum over the empty set is supposed to be equal to infinity, e.g. when $\omega|_{\pi_2} = 0$, and $E(u)$ denotes the energy of a map u with respect to the metric g and any metric on a sphere. Note that the energy of such a J-holomorphic sphere (as well as any J-holomorphic curve) is equal to its symplectic area

$$\langle u^*[\omega], S^2\rangle = \int_{S^2} u^*\omega,$$

i.e. the evaluation of the class $u^*[\omega]$ on the fundamental cycle. More generally, for solutions of equation (9.1) the following energy inequality holds:

$$E(u) \leqslant 2\int_{\mathbb{T}^2} \max_{u\in M'} \|\mathsf{f}(z,u)\|_g^2 \, dvol(z) + \langle u^*[\omega], \mathbb{T}^2\rangle. \tag{9.2}$$

At hand with this, the standard rescaling argument implies the following statement.

Compactness theorem. *Let M' be a closed symplectic manifold and denote by $\mathscr{C}$ any set formed by homotopy classes $[v]$ of mappings $\mathbb{T}^2 \to M'$ such that*

$$V_{\mathscr{C}} = \sup\left\{\langle v^*[\omega], \mathbb{T}^2\rangle : [v] \in \mathscr{C}\right\} < S.$$

Then the natural projection

$$\pi : \bigcup_{[v]\in\mathscr{C}} \mathfrak{M}_{\bar\partial,[v]} \to \left\{C^4\text{-smooth right-hand side } \mathsf{f}\right\}, \qquad (u,\mathsf{f}) \longmapsto \mathsf{f},$$

restricted on the domain $\pi^{-1}(\mathfrak{U})$ is proper, where

$$\mathfrak{U} = \left\{C^4\text{-smooth } \mathsf{f} : \int_{\mathbb{T}^2} \max_u \|\mathsf{f}(z,u)\|_g^2 \, dvol(z) < \frac{1}{2}\,(S - V_{\mathscr{C}})\right\}.$$

In sequel, we call a homotopy (homology) class $[v]$ of mappings *non-positive* if the value $\langle v^*[\omega], \mathbb{T}^2\rangle$ is non-positive. In particular, the compactness theorem implies that the space formed by solutions of equation (9.1) within the set $\mathscr{C}$ of non-positive homotopy classes is always compact provided the right-hand side f satisfies $\int \max_u \|\mathsf{f}(\cdot,u)\|_g^2 < S/2$.

There is also somewhat different version of the compactness theorem, due to [23], where the domain $\mathfrak{U}$ is given by the inequality $\max_{z,u} \|\mathsf{f}(z,u)\|_g^2 < (S - V_{\mathscr{C}})$. Further, instead of homotopy classes in the theorem one can consider homology classes of mappings, since the condition $\langle u^*[\omega], \mathbb{T}^2\rangle < S$ is, in fact, (co)homologous.

Orientation on the universal moduli space. Now we discuss the orientability of the moduli space. Denote by ∇ the Levi-Civita connection of the metric g and by $\bar\partial_*(u,\mathsf{f})$ the operator obtained by the linearisation of equation (9.1) with respect to ∇. In natural isothermal coordinates $x+iy$ on the torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ the linearised operator has the following form

$$\frac{1}{2}\left(\nabla_{\partial/\partial x}\mathsf{v}+J(u)\nabla_{\partial/\partial y}\mathsf{v}+(\nabla_{\mathsf{v}}J(u))\frac{\partial u}{\partial y}\right)-\nabla_{\mathsf{v}}\mathsf{f}(\cdot,u(\cdot)),$$

where v is a vector field along u. Due to the Riemann-Roch theorem, see e.g. [25], the index of $\bar\partial_*(u,\mathsf{f})$ is equal to $2\langle u^*[c_1],\mathbb{T}^2\rangle$, where $[c_1]$ is the first Chern class. Recall that, due to Proposition 1.1, it vanishes, when the linearised operator has a non-empty resolvent set. The hypothesis on the resolvent set to be non-empty is always supposed to hold in sequel. This does not rule out interesting cases: under the vanishing of $\langle u^*[c_1],\mathbb{T}^2\rangle$ the emptiness of the resolvent sets implies that there exists no solution of (9.1) for a generic right-hand side f. (This is a consequence of Proposition 1.1 and [18, Lemma 5.4].) For example, the operator $\bar\partial_*(u,\mathsf{f})$ for any contratible map u has a non-empty resolvent set.

Let $\widetilde\nabla$ be a connection on M' which preserves the almost complex structure J. To each point (u,f) from the moduli space we assign the operator

$$\mathsf{v}\longmapsto\frac{1}{2}\left(\widetilde\nabla_{\partial/\partial x}\mathsf{v}+J(u)\widetilde\nabla_{\partial/\partial y}\mathsf{v}\right), \tag{9.3}$$

which acts on vector fields v along u. It commutes with the almost complex structure and, hence, its any eigenvalue has even multiplicity. Thus, the spectral flow along any path γ in the moduli space with respect to operators (9.3) is equal to zero. Since these differ from $\bar\partial_*(\gamma)$ by a zero-order term, we obtain that the moduli space is weakly spectrally non-degenerate. Moreover, the corresponding families of the operators $\bar\partial_*(\gamma)$ and (9.3) are homotopic and Lemma 2.1 implies that they determine the same w_A-class which is trivial. Now Theorem 3.1 implies that the moduli space is orientable. Any two resolvent points of operator (9.3) define the same resolvent class $[\lambda]_{(u,\mathsf{f})}$, where $(u,\mathsf{f})\in\mathfrak{M}_{\bar\partial,[v]}$, and we suppose that the moduli space is oriented in accordance with it. This defines the index function $\mathrm{ind}_2(u,\mathsf{f})\in\mathbb{Z}_2$ and the degree of the projection π, restricted to the moduli space of solutions within $[v]$, is given by the familiar sum $\sum(-1)^{\mathrm{ind}_2(u,\mathsf{f})}$, where f is a regular value and (u,f) ranges over $\pi^{-1}(\mathsf{f})$.

The Lefschetz-type formula for a dynamical system with complex time. First, we discuss the degree of π for a non-trivial homotopy class $[v]$. Suppose $[v]$ does not contain J-holomorphic tori. This, for example, occurs when the homotopy class is non-positive, $\langle v^*[\omega],\mathbb{T}^2\rangle\leqslant 0$; see inequality (9.2). If, in addition, the pulled-back first Chern class vanishes, then the degree of π restricted to $\pi^{-1}(\mathcal{U})$ is well-defined and, obviously, equals zero.

Now we suppose that $[v]$ is trivial. Then, due to (9.2), it contains only constant J-holomorphic tori. Thus, the space of solutions $\mathfrak{S}_{\bar\partial,[pt]}$ of the homogeneous equation $\bar\partial u=0$ is formed by all constant mappings $u_y\equiv p$, $p\in M'$. The linearised operator $\bar\partial_*(u_p,0)$ at the constant map u_p is the Cauchy-Riemann operator on vector-functions $\mathsf{v}:\mathbb{T}^2\to T_pM'\simeq\mathbb{C}^{n'/2}$. By the Liouville theorem its kernel is formed by all constant vector-functions and the space of solutions $\mathfrak{S}_{\bar\partial,[pt]}\simeq M'$ is non-degenerate in the sense of Morse-Bott. Moreover, the operator $\bar\partial_*(u_p,0)$ commutes with its formally self-adjoint and the space of solutions $\mathfrak{S}_{\bar\partial,[pt]}$ is strongly non-degenerate; see Example 5.2.

Due to Corollary 5.4 one can also orient the neighbourhood of $\mathfrak{S}_{\bar\partial,[pt]}$ by a family of resolvent classes corresponding to the operator $\bar\partial_*(u,\mathsf{f})$. Any such an orientation coincides with the one above, since any two resolvent points of operator (9.3) for $u \in \mathfrak{S}_{\bar\partial,[pt]}$ belong to the same resolvent class; see Remark 4.1.

Summarising all this we arrive at the following consequence of Theorem 6.2; a slightly different version of this for the Kähler case is due to [23].

Theorem 9.1. *Let M' be a symplectic manifold with an almost Kähler structure (g, J, ω). Then for a generic non-autonomous vector field f on M' such that*

$$\int_{\mathbb{T}^2} \max_u \|\mathsf{f}(z,u)\|_g^2 \, dvol(z) < S/2, \tag{9.4}$$

the number of contractible solutions of equation (9.1) is finite and their algebraic number is equal to $\chi(M')$. Moreover, if $\chi(M') \neq 0$, then equation (9.1) has a contractible solution for any right-hand side satisfying (9.4).

For a non-trivial homotopy class $[v]$ without J-holomorphic tori and such that $\langle v^*[c_1], \mathbb{T}^2\rangle$ vanishes the number of solutions of equation (9.1) for a generic right-hand side f is finite and their algebraic number is equal to zero provided

$$\int_{\mathbb{T}^2} \max_u \|\mathsf{f}(z,u)\|_g^2 \, dvol(z) < \frac{1}{2}\left(S - \langle v^*[\omega], \mathbb{T}^2\rangle\right). \tag{9.5}$$

Note that, since the compactness of the space of solutions and the vanishing of index are determined only by the values $\langle v^*[\omega], \mathbb{T}^2\rangle$ and $\langle v^*[c_1], \mathbb{T}^2\rangle$, the theorem above has an alternative version concerned with counting of solutions within homology classes.

Further, by straightforward calculations one can show that one-point solutions, i.e. non-degenerate zeroes of time-independent vector fields, are counted with respect to its usual index (as a non-degenerate zero). It seems also plausible that our index $\mathrm{ind}_2(u, \mathsf{f})$ of solutions agrees with the sign given by the coherent orientations as in Floer's theory; see [3, 9] for the further details and references.

As a consequence of Theorem 9.1 we also have the following statement; compare this (and examples below) with the Lefschetz formula – Theorem 7.1.

Corollary 9.2. *For a generic right-hand side f such that $\int \max_u \|\mathsf{f}(\cdot,u)\|_g^2 < S/2$ the number of isolated perturbed J-holomorphic tori within all non-positive homotopy (homology) classes is finite and their algebraic number within a fixed homotopy (homology) class is equal to the Euler characteristic of M' or zero with respect to the cases the class is trivial or not.*

Example 9.1. Suppose that the dimension of a symplectic manifold M' is equal to four. Recall that a map of a Riemannian surface into such a symplectic manifold is called Lagrangian if the pull-back of a symplectic form vanishes. We have the following assertion due to Gromov and Lees [24]: *a homotopy class $[v]$ such that $v^*[\omega] = 0$ satisfies $\langle v^*[c_1], \mathbb{T}^2\rangle = 0$ if and only if it can be represented by a Lagrangian immersion.* Combining this with Corollary 9.2 we arrive at the following statement.

For a generic right-hand side f such that $\int \max_u \|\mathsf{f}(\cdot,u)\|_g^2 < S/2$ the number of perturbed J-tori in a non-trivial homology class that can be represented by a Lagrangin immersion is finite and the algebraic number of these is equal to zero.

Example 9.2 (Generalised Einstein condition). Suppose that a symplectic manifold M' with a fixed almost Kähler structure satisfies the *generalised Einstein condition*: the first Chern class is a non-zero multiple of ω, $[c_1] = k[\omega]$. This is a symplectic analogue of the Kähler-Einstein condition; in the Kähler case $[c_1] = k[\omega]$ can be interpreted geometrically: *Ricci* $= R\omega$, where *Ricci* denotes the Ricci 2-form and R is the scalar curvature. For such symplectic manifolds the classes $v^*[c_1]$ and $v^*[\omega]$ vanish simultaneously, and due to Corollary 9.2 we see that *for a generic right-hand side which satifies* (9.4) *the number of isolated perturbed J-tori is finite and their algebraic number in a fixed homotopy (homology) class is equal to $\chi(M')$ or zero with respect to the cases the class is trivial or not.* This sharpens one of the results in [3] which states that for a monotone symplectic manifold[5] the total number of isolated perturbed J-holomorphic tori is equal to the Euler characteristic of M'.

Example 9.3 ($c_1(A) \neq 0$ for any non-trivial $A \in \pi_2(M')$). Suppose M' does not contain spheres with the vanishing first Chern class c_1. Then any non-trivial toroidal homology class A with $c_1(A) = 0$ can not be represented by a J-torus for a generic almost complex structure J compatible with ω. Recall that a J-curve is called simple if it is not a branched covering of degree greater than one of another J-curve. For a generic J the class A can not contain simple J-tori: due to the Riemann-Roch theorem the family of unparameterised simple J-tori representing A forms a manifold of negative dimension. Hence, there can be only non-simple J-tori, which have to be branched coverings of 2-spheres. However, since $c_1(A) = 0$, this contradicts to the hypothesis on the first Chern class. Thus, we see that *for such symplectic manifolds and for a generic almost complex structures J the algebraic number of isolated perturbed J-tori in any non-trivial homology class is equal to zero, under the condition that a generic right-hand side satisfies* (9.5).

The examples above show that in many situations we have a version of the Lefschetz formula for the algebraic number of the homotopic (homologous) solutions, which is close to its version for dynamical systems. We end with two applications.

Example 9.4 (Gromov-type invariant counting tori). First, recall how to define, following the idea in [11], an almost complex structure J^{f} on the product $\mathbb{T}^2 \times M'$ such that J^{f}-holomorphic sections of

$$\mathbb{T}^2 \times M' \to \mathbb{T}^2, \qquad (z, u) \longmapsto z, \tag{9.6}$$

are precisely the graphs of solutions of (9.1). Let f be a non-autonomous vector field on M' and denote by $X^{0,1}$ a field of complex anti-linear operators

$$X^{0,1}_{(z,u)} : T_z\mathbb{T}^2 (\simeq \mathbb{C}) \to T_u M'$$

such that for any $z \in \mathbb{T}^2$ and $u \in M'$ the value of $X^{0,1}_{(z,u)}$ on the vector $\partial/\partial x (\simeq 1)$ is equal to $\mathsf{f}(z, u)$. Then we define an almost complex structure J^{f} on the product by the following relation:

$$J^{\mathsf{f}}_{(z,u)}(\xi, \eta) = \left(i\xi, J(u)\eta + 2J(u) \circ X^{0,1}_{(z,u)}(\xi)\right), \qquad (\xi, \eta) \in \mathbb{C} \times T_u M'.$$

It follows straightforward that a map $u : \mathbb{T}^2 \to M'$ is a solution of equation (9.1) if and only if its graph $\mathbb{T}^2 \to \mathbb{T}^2 \times M'$, $z \mapsto (z, u(z))$, is a J^{f}-holomorphic section of (9.6). Theorem 9.1 can be viewed as a calculation of a Gromov invariant that counts the number of J-sections of the bundle (9.6) with a fixed complex structure on the base $\mathbb{T}^2$; see [25, Chapter 8] for more details. Indeed, one can show that the complex structure J^{f} is regular (among the so-called vertical almost complex structures) if and only if the vector field f is regular as a right-hand side of (9.1). Thus, we obtain that this

[5] i.e. such that $[c_1] = k[\omega]$, $k > 0$.

invariant corresponding to the homology class of a fibre, i.e. $\mathbb{T}^2 \times [pt]$, is equal to the Euler characteristic of M'. Further, suppose that $\dim M = 4$ and $v : \mathbb{T}^2 \to M'$ is a non-contractible Lagrangian immersion. Then, the Gromov invariant corresponding to the homology class of $\mathbb{T}^2 \times v(\mathbb{T}^2)$ vanishes.

Example 9.5 (Non-linear eigenvalue problems [23]). For compex-valued vector-functions $U : \mathbb{T}^2 \to \mathbb{C}^{n'+1}\backslash\{0\}$ consider the following non-linear homogeneous eigenvalue problem

$$\bar{\partial}U(z) - \mathsf{F}(z, U(z)) = \lambda U(z), \qquad z \in \mathbb{T}^2,\ \lambda \in \mathbb{C}; \tag{9.7}$$

here $\mathsf{F} : \mathbb{T}^2 \times \mathbb{C}^{n'+1} \to \mathbb{C}^{n'+1}$ is a 1-homogeneous with respect to U vector-function (i.e. $\mathsf{F}(z, \mu U) = \mu \mathsf{F}(z, U)$ for any $\mu \in \mathbb{C}$) which is C^4-smooth, maybe, except the points $(z, 0)$, $z \in \mathbb{T}^2$. For simplicity, we also suppose that $\mathbb{T}^2$ is endowed with the standard complex structure. A solution of this problem is a class of pairs $(U(z), \lambda)$ – the vector-function U is considered up to a non-zero factor. Given a 1-homogeneous vector-function F the canonical projection $p : \mathbb{C}^{n'+1}\backslash\{0\} \to \mathbb{C}P^{n'}$ defines, by pushing forward, the C^4-smooth vector field f on $\mathbb{C}P^{n'}$. Conversely, any vector-field on $\mathbb{C}P^{n'}$ can be pulled-back to a 1-homogeneous vector-function. It is a simple exercise to show that a pair $(U(z), \lambda)$ solves equation (9.7) if and only if the map $u = p \circ U : \mathbb{T}^2 \to \mathbb{C}P^{n'}$ is a contractible solution of (9.1). Moreover, two solutions $(U_1(z), \lambda_1)$ and $(U_2(z), \lambda_2)$ correspond to the same solution of equation (9.1) if and only if there exist integers $m, n \in \mathbb{Z}$ such that

$$U_2(x + iy) = e^{2\pi i(mx+ny)} U_1(x + iy), \qquad \lambda_2 = \lambda_1 + \pi(im - n);$$

see [23]. If we endow the complex projective space with the Fubini-Studi metric, then relation (9.4) can be re-written in terms of F as

$$\int_{\mathbb{T}^2} \max_{|W|=1} \Big(|\mathsf{F}(z, W)|^2 - |\langle \mathsf{F}(z, W), W\rangle|^2 \Big)\, dvol(z) < \frac{\pi}{2}, \tag{9.8}$$

where the brackets $\langle\cdot,\cdot\rangle$ denote the standard Hermitian scalar product on $\mathbb{C}^{n'+1}$. Thus, Theorem 9.1 implies that *for a non-autonomous vector-function* F *which satisfies* (9.8) *there exists a family of solutions*

$$e^{2\pi i(mx+ny)} U(x + iy), \qquad \lambda + \pi(im - n), \qquad m, n \in \mathbb{Z}.$$

of non-linear eigenvalue problem (9.7). *Moreover, for a generic* F *which satisfies* (9.8) *there exist at least* $(n + 1)$ *such families.*

A. Appendix: proofs of some auxiliary lemmas

A.1. Lemmas in Section 3

Proof of Lemma 3.3. For the sake of simplicity and in order to make the calculations more transparent we shall prove the lemma for the case when the point (u_0, f_0) is a regular point of the projection π_A. The proof for the general case follows along similar lines.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two charts of the quasi finite-dimensional atlas both centred at a fixed point (u_0, f_0). First, by the transitivity argument, it is sufficient to consider the case when $\mathcal{C}_1$ and $\mathcal{C}_2$ are given by decompositions

$$H^m(u_0^* TM') = 0 \oplus H, \qquad \mathcal{F} = 0 \oplus \mathcal{F},$$

and

$$H^m(u_0^*TM') = H^\triangle \oplus \widetilde{H}, \qquad \mathcal{F} = \mathcal{F}^\triangle \oplus \widetilde{\mathcal{F}},$$

satisfying properties (1) and (2) on p. 18. In this situation the linear operators $\Psi'_u(u_0, \mathsf{f}_0)$ and

$$d\Psi(u_0, \mathsf{f}_0)|_{\widetilde{H}\times\mathcal{F}^\triangle} : \widetilde{H} \times \mathcal{F}^\triangle \to H^{m-k}(u_0^*TM')$$

are isomorphisms. Adopting the notation

$$Z^\triangle = \Psi'_{\mathsf{f}}(\mathcal{F}^\triangle), \qquad \widetilde{Z} = \Psi'_u(\widetilde{H}), \tag{A.1}$$

we have the decomposition $H^{m-k}(u_0^*TM') = Z^\triangle \oplus \widetilde{Z}$. Let us also denote by $\Pi^\triangle$ the corresponding projection onto $Z^\triangle$ along $\widetilde{Z}$ and define the maps

$$\Psi^\triangle = \Pi^\triangle \circ \Psi, \qquad \widetilde{\Psi} = (\mathrm{Id} - \Pi^\triangle) \circ \Psi.$$

Here Ψ denotes the map $W(u_0) \times \mathcal{F} \to H^{m-k}(u_0^*TM')$ defined in Section 3.2, where $W(u_0)$ is a small neighbourhood of u_0 in the space $H^m(M, M')$; we identify $W(u_0)$ with a ball in the space $H^m(u_0^*TM')$ by means of the exponential map. Due to the relations in (A.1) we have the following identities:

$$\frac{\partial \Psi^\triangle}{\partial \tilde{u}}(u_0, \mathsf{f}_0) = 0, \qquad \frac{\partial \widetilde{\Psi}}{\partial \tilde{u}}(u_0, \mathsf{f}_0) = \Psi'_u(u_0, \mathsf{f}_0)|_{\widetilde{H}}\,, \tag{A.2}$$

$$\frac{\partial \Psi^\triangle}{\partial \mathsf{f}^\triangle}(u_0, \mathsf{f}_0) = \Psi'_{\mathsf{f}}(u_0, \mathsf{f}_0)|_{\mathcal{F}^\triangle}\,, \qquad \frac{\partial \widetilde{\Psi}}{\partial \mathsf{f}^\triangle}(u_0, \mathsf{f}_0) = 0, \tag{A.3}$$

where $\tilde{u}$ and $\mathsf{f}^\triangle$ stand for the vectors from the spaces $\widetilde{H}$ and $\mathcal{F}^\triangle$ respectively.

Consider the transition functions from the chart $\mathcal{C}_1$ to $\mathcal{C}_2$. They have the following form

$$\mathcal{F}^\triangle \oplus \widetilde{\mathcal{F}} \ni (\mathsf{f}^\triangle, \tilde{\mathsf{f}}) \mapsto (M \circ u^\triangle, \tilde{\mathsf{f}}) \in \mathcal{F}^\triangle \oplus \widetilde{\mathcal{F}}$$

and, hence, the L-S index of their differential is given by $\operatorname{sgn}\det(M \circ (\partial u^\triangle/\partial f^\triangle))$. On the other hand, differentiating the equality $\Psi^\triangle(u, \mathsf{f}) = 0$, where (u, f) belongs to the moduli space, we obtain

$$\frac{\partial \Psi^\triangle}{\partial u^\triangle}\frac{\partial u^\triangle}{\partial \mathsf{f}^\triangle} + \frac{\partial \Psi^\triangle}{\partial \tilde{u}}\frac{\partial \tilde{u}}{\partial \mathsf{f}^\triangle} + \frac{\partial \Psi^\triangle}{\partial \mathsf{f}^\triangle} = 0.$$

Combining this with the first identity in (A.2) we see that

$$\frac{\partial u^\triangle}{\partial \mathsf{f}^\triangle} = -\left(\frac{\partial \Psi^\triangle}{\partial u^\triangle}\right)^{-1} \frac{\partial \Psi^\triangle}{\partial \mathsf{f}^\triangle}\bigg|_{(u_0, \mathsf{f}_0)},$$

and, thus, the L-S index of the differential of the transition functions at (u_0, f_0) is given by the following expression

$$\operatorname{sgn}\det\left(-M\left(\frac{\partial \Psi^\triangle}{\partial u^\triangle}\right)^{-1}\frac{\partial \Psi^\triangle}{\partial \mathsf{f}^\triangle}\right)\bigg|_{(u_0, \mathsf{f}_0)}. \tag{A.4}$$

Now we calculate the L-S index of $\mathscr{A}_{\mathcal{C}_1}^{-1} \circ \mathscr{A}_{\mathcal{C}_2}|_{(u_0, \mathsf{f}_0)}$. The combination of formula (3.3) and relations in (A.2) and (A.3) yields

$$\mathscr{A}_{\mathcal{C}_2}(u_0, \mathsf{f}_0) = \left[\left(-\frac{\partial \Psi^\triangle}{\partial \mathsf{f}^\triangle} M\right) \oplus \frac{\partial \widetilde{\Psi}}{\partial \tilde{u}}\right]\bigg|_{(u_0, \mathsf{f}_0)}.$$

Further, the operator $\mathscr{A}_{\mathcal{C}_1}(u_0, \mathsf{f}_0)$ is equal to $\Psi'_u(u_0, \mathsf{f}_0)$. Now contracting $\partial\widetilde{\Psi}/\partial u^{\triangle}(u_0, \mathsf{f}_0)$ to zero and using the first identity in (A.2) we see that $\mathscr{A}_{\mathcal{C}_1}^{-1} \circ \mathscr{A}_{\mathcal{C}_2}\big|_{(u_0,\mathsf{f}_0)}$ is homotopic (in the class of linear isomorphisms of the form Id+{compact}) to the operator

$$\left(\frac{\partial\Psi^{\triangle}}{\partial u^{\triangle}} \oplus \frac{\partial\widetilde{\Psi}}{\partial \tilde{u}}\right)^{-1}\left[\left(-\frac{\partial\Psi^{\triangle}}{\partial \mathsf{f}^{\triangle}}M\right) \oplus \frac{\partial\widetilde{\Psi}}{\partial \tilde{u}}\right]\Bigg|_{(u_0,\mathsf{f}_0)} = \left(-\left(\frac{\partial\Psi^{\triangle}}{\partial u^{\triangle}}\right)^{-1}\frac{\partial\Psi^{\triangle}}{\partial \mathsf{f}^{\triangle}}M\right)\Bigg|_{(u_0,\mathsf{f}_0)} \oplus \mathrm{Id}\,.$$

Thus, its L-S index is equal to an expression in (A.4) and the lemma is demonstrated. □

Proof of Lemma 3.4. Let $\gamma_t = (u_t, \mathsf{f}_t)$ be a path joining the centres of given charts $(\mathcal{C}_0, \phi_0)$ and $(\mathcal{C}_1, \phi_1)$, and let $(\mathcal{C}_t, \phi_t)$ be a family of charts, centred at γ_t, given by a homotopy. Dividing the curve into small arcs and considering these one can suppose that the points γ_0 and γ_1 are close such that $\gamma_0 \in \mathcal{C}_t$ for any t. One can also suppose that the families of corresponding spaces $\mathfrak{F}_1^t$ and $\mathfrak{F}_2^t$ do not depend on t. Now consider the differentials $d(\phi_t \circ \phi_0^{-1})$ evaluated at the point $\phi_0(\gamma_0)$. Due to the construction of the homotopy they have the form $L_t \oplus \mathrm{Id}_{\mathfrak{F}_2}$, where $L_t : \mathfrak{F}_1 \to \mathfrak{F}_1$. Moreover, the family of the L_t's depend on the parameter continuously. Indeed, they coincide with the composition of the following maps:

$$M_0^{-1} : \mathfrak{F}_1 \to H_1^0, \qquad H_1^0 \to H_1^0 \oplus H_2^0 = H^m(u_0^*TM'),$$
$$d_{u_0}\exp_{u_t}^{-1} : H^m(u_0^*TM') \to H^m(u_t^*TM'),$$
$$H^m(u_t^*TM') = H_1^t \oplus H_2^t \to H_1^t, \qquad M_t : H_1^t \to \mathfrak{F}_1.$$

Here the first two maps do not depend on t and the fourth one denotes the projection along the subspace H_2^t.

Since the L-S index is a homotopy invariant and $L_0 = \mathrm{Id}$, we see that the L-S index of $L_1 \oplus \mathrm{Id}$ is equal to one. Finally, since the intersection of the charts $\mathcal{C}_0$ and $\mathcal{C}_1$ is connected, we obtain that the L-S index of the differential at any point of $\mathcal{C}_0 \cap \mathcal{C}_1$ is equal to one. Thus, the lemma is demonstrated. □

A.2. Lemmas in Section 6

We start with giving a proof of Lemma 6.3 for the case when a homotopy class is trivial and the space of contractible solutions $\mathfrak{S}_{A,[pt]}$ of the homogeneous equation $Au(x) = 0$ is formed by all constant mappings $u_y \equiv y$, where $y \in M'$; i.e. A satisfies the Liouville principle, see Example 5.1. For any constant map-solution u_y the linearised operator $A_*(u_y)$ acts on vector-functions $M \to T_yM' \simeq \mathbb{R}^{n'}$ and depends continuously (and even smoothly) on $y \in M'$. Under our suppositions, the space $\mathfrak{S}_{A,[pt]}$ is non-degenerate and, hence, the kernel $\operatorname{Ker} A_*(u_y)$ is formed by constant vector functions. Further, it follows from the standard theory that there exists a constant $C_1 > 0$ such that

$$\|\mathsf{v}\|_m \leqslant C_1 \|A_*(u_y)\mathsf{v}\|_{m-k} \quad \text{for any } y \in M', \tag{A.5}$$

and for any vector-function v with zero mean value; here $\|\cdot\|_\ell$ denotes the Sobolev H^ℓ-norm.

Proof of Lemma 6.3 (for the trivial homotopy class). Assume the contrary to the statement of the lemma. Then a sequence $\varepsilon_j > 0$ and a sequence of smooth mappings $u_j : M \to M'$ can be found such that $\varepsilon_j \to 0$ as $j \to \infty$, u_j is not a constant mapping for any j and

$$Au_j(x) = \varepsilon_j \mathsf{f}(x, u_j(x)), \qquad x \in M. \tag{A.6}$$

By the compactness hypothesis one can suppose that u_j converges to a (constant) map $u_0 \in \mathfrak{S}_{A,[pt]}$ in the H^m-topology. In particular, this means that u_j converges uniformly and, since $u_0 \equiv \mathit{const}$, we can assume that the image $u_j(M)$ belongs to some chart $\mathcal{C} \subset \mathbb{R}^{n'}$ on M' for any j. Hence, one can regard u_j as a vector-valued function.

For any j one can decompose u_j into the sum $u_j = \hat{u}_j + \nu_j$, where $\hat{u}_j$ is a constant and ν_j has zero mean value. Since $u_j \to u_0 \equiv \mathit{const}$, we see that $\hat{u}_j \to u_0$ and $\nu_j \to 0$ in the H^m-topology. In particular, the constants $\hat{u}_j$'s lie in the same chart as u_0. Linearising the operator A at $\hat{u}_j$ we have

$$Au_j = A_*(\hat{u}_j)\nu_j + O(|\nu_j|^2),$$

where $O(|\nu_j|^2)$ denotes the quantity such that $\|O(|\nu_j|^2)\|_{m-k} \leqslant C_2 \left\|\nu_j\right\|_m^2$. Analogously, we have

$$\mathsf{f}(\cdot, u_j(\cdot)) = \mathsf{f}(\cdot, \hat{u}_j) + O(|\nu_j|),$$

where $\|O(|\nu_j|)\|_{m-k} \leqslant C_3 \left\|\nu_j\right\|_m$; the constants C_2 and C_3 above do not depend on the j's. Combining the last two relations with equation (A.6) we arrive at the following

$$A_*(\hat{u}_j)\nu_j = \varepsilon_j \mathsf{f}(\cdot, \hat{u}_j) + O(\varepsilon_j |\nu_j| + |\nu_j|^2). \tag{A.7}$$

Recall, see [28, Chapter XI], that the space of smooth vector-functions on M can be decomposed as the orthogonal sum of the image of $A_*(\hat{u}_j)$ and $\operatorname{Ker} A(\hat{u}_j)^*$ with respect to the L_2-scalar product induced by the metric on M'. Denote by Π_j the corresponding orthogonal projection onto the image of $A_*(\hat{u}_j)$. Now due to the suppositions of the lemma, $\mathsf{f}(\cdot, \hat{u}_j)$ belongs to the space $\operatorname{Ker} A(\hat{u}_j)^*$. Thus, applying Π_j to relation (A.7) we have

$$A_*(\hat{u}_j)\nu_j = O_{\Pi_j}(\varepsilon_j |\nu_j| + |\nu_j|^2). \tag{A.8}$$

Here the right-hand side stands for the image of the quantity $O(\varepsilon_j |\nu_j| + |\nu_j|^2)$ in (A.7) under Π_j. Note that, since $A(\hat{u}_j)^* = A(\hat{u}_j)^* \circ \Pi_j$ on smooth vector-functions, the following inequalities hold

$$C_4 \left\|\Pi_j w\right\|_{m-k} \leqslant \left\|A(\hat{u}_j)^* \Pi_j w\right\|_{m-2k} = \left\|A(\hat{u}_j)^* w\right\|_{m-2k} \leqslant C_5 \left\|w\right\|_{m-k}$$

for any smooth vector-function w. The positive constants C_4 and C_5 here can be also chosen independently of the j's. Thus, we obtain

$$\left\|O_{\Pi_j}(\varepsilon_j |\nu_j| + |\nu_j|^2)\right\|_{m-k} \leqslant C' \left(\varepsilon_j \left\|\nu_j\right\|_m + \left\|\nu_j\right\|_m^2\right).$$

Now combining this and inequality (A.5) with relation (A.8) we get that

$$\left\|\nu_j\right\|_m \leqslant C \left(\varepsilon_j \left\|\nu_j\right\|_m + \left\|\nu_j\right\|_m^2\right).$$

Since the sequences ε_j and $\left\|\nu_j\right\|_m$ tend to zero as $j \to \infty$, the last inequality implies that $\nu_j \equiv 0$ for sufficiently large indices j. In particular, we see that the mappings $u_j = \hat{u}_j$ are constant, thus, arriving to a contradiction. □

Remarks on the proof in a general case. In a general setting the space of solutions $\mathfrak{S}_{A,[v]}$ has a tubular neighbourhood whose fibre over u can be identified via the exponential map with an appropriate subspace in $H^m(u^*TM')$ complement to $\operatorname{Ker} A_*(u)$; for example, with its orthogonal complement. Thus, for any map u from this neighbourhood there exists a unique pair $(\hat{u}, \mathsf{v})$, where $\hat{u} \in \mathfrak{S}_{A,[v]}$ and $\mathsf{v} \in \operatorname{Ker} A_*(u)^\perp$, such that $u(x) = \exp_{\hat{u}(x)} \mathsf{v}(x)$. Using this decomposition the proof in a general case goes essentially along the similar lines. However, more care should be taken to show that the analogous constants C_2, C_3, C_4, and C_5 do not depend on the large j's. Relation (A.5) above should be also replaced by the analogous one with u_y replaced by $\hat{u} \in \mathfrak{S}_{A,[v]}$ and v by $\mathsf{v} \in \operatorname{Ker} A_*(\hat{u})^\perp$. This is the only place where the Morse-Bott non-degeneracy is used.

Proof of Lemma 6.4. Assume the contrary. Then there exist a sequence of positive numbers $\varepsilon_j > 0$, a sequence of solutions u_j of equation (6.4), and a sequence of vector fields $\mathsf{v}_j \neq 0$ along u_j such that $\varepsilon_j \to 0$ as $j \to \infty$ and

$$A_*(u_j)\mathsf{v}_j = \varepsilon_j \mathsf{f}_*(u_j)\mathsf{v}_j, \tag{A.9}$$

where we denoted $\mathsf{f}(u_j) = \mathsf{f}(\cdot, u_j(\cdot))$. First, it is sufficient to consider only ε_j such that $\varepsilon_j < \varepsilon'$, where ε' is taken as in Lemma 6.3. In this case all solutions u_j belong to the space $\mathfrak{S}_{A,[v]}$ and are zeroes of the section $u \mapsto \mathsf{f}(\cdot, u(\cdot))$ of the bundle $\operatorname{Ker} A^*$. Since, by the suppositions of the lemma, these zeroes are non-degenerate, they are isolated and, since $\mathfrak{S}_{A,[v]}$ is compact, there is only a finite number of them. Therefore, without loss of generality one can suppose that all solutions u_j coincide with some solution u_0. Hence, instead of relation (A.9) we have

$$A_*(u_0)\mathsf{v}_j = \varepsilon_j \mathsf{f}_*(u_0)\mathsf{v}_j. \tag{A.10}$$

Further, since the vector fields v_j are smooth, then by the standard embedding theorem one can also suppose that the sequence v_j converges with respect to the H^m-norm to some vector field $\mathsf{v}_0 \in \operatorname{Ker} A_*(u_0)$. Let us decompose $\mathsf{v}_j = \hat{\mathsf{v}}_j + w_j$ such that $\hat{\mathsf{v}}_j \in \operatorname{Ker} A_*(u_0)$ and w_j are orthogonal to the space $\operatorname{Ker} A_*(u_0)$ with respect to the L_2-scalar product. In particular, it follows that $\hat{\mathsf{v}}_j \to \mathsf{v}_0$ and $\mathsf{w}_j \to 0$. Then equation (A.10) takes the form

$$A_*(u_0)\mathsf{w}_j = \varepsilon_j \mathsf{f}_*(u_0)\hat{\mathsf{v}}_j + \varepsilon_j \mathsf{f}_*(u_0)\mathsf{w}_j. \tag{A.11}$$

Let us show that for any $\hat{\mathsf{v}} \in \operatorname{Ker} A_*(u_0)$ the vector field $\mathsf{f}_*(u_0)\hat{\mathsf{v}}$ is orthogonal to the image of the operator $A_*(u_0)$ (acting on the space of smooth vector fields along u_0) with respect to the L_2-scalar product on the sections of ${u_0}^*TM'$ induced by the metric on M'. Indeed, since the space $\mathfrak{S}_{A,[v]}$ is non-degenerate, for any $\hat{\mathsf{v}} \in \operatorname{Ker} A_*(u_0)$ there exists a family $u_t \in \mathfrak{S}_{A,[v]}$, where $t \in (-\delta, \delta)$ for a small $\delta > 0$, such that $u_t|_{t=0} = u_0$ and $(\partial/\partial t)|_{t=0}\, u_t = \hat{\mathsf{v}}$. For any smooth vector field w along u_0 let us denote by $\mathsf{w}_t(\cdot)$, $t \in (-\delta, \delta)$, its extension – a smooth vector field along the map $U : (-\delta, \delta) \times M \to M'$ that is given by $U(t, \cdot) = u_t(\cdot)$ such that $\mathsf{w}_t|_{t=0} = \mathsf{w}$. Since $\mathsf{f}(u_t) \in \operatorname{Ker} A(u_t)^*$, we have the following relation

$$\int_M \langle A_*(u_t)\mathsf{w}_t, \mathsf{f}(u_t)\rangle_{g'}\, d\mu = 0.$$

Differentiating the left-hand side with respect to t we have

$$\frac{d}{dt}\bigg|_{t=0} \int_M \langle A_*(u_t)\mathsf{w}_t, \mathsf{f}(u_t)\rangle_{g'}\, d\mu = \int_M \langle \nabla'_{\frac{\partial}{\partial t}}\Big|_{t=0} A_*(u_t)\mathsf{w}_t, \mathsf{f}(u_0)\rangle_{g'}\, d\mu$$
$$+ \int_M \langle A_*(u_0)\mathsf{w}, \mathsf{f}_*(u_0)\hat{\mathsf{v}}\rangle_{g'}\, d\mu = \int_M \langle A_*(u_0)\mathsf{w}, \mathsf{f}_*(u_0)\hat{\mathsf{v}}\rangle_{g'}\, d\mu.$$

Here ∇' stands for the Levi-Civita connection of g' pulled-back on the bundle U^*TM' and the last equality holds due to $\mathsf{f}(u_0) = 0$. Combining the last two relations we obtain the claim.

Following the proof of Lemma 6.3, let us denote by Π the orthogonal in the sense of the L_2-scalar product projection onto the image of $A_*(u_0)$ in the space of smooth sections of ${u_0}^*TM'$. Since, by the preceding discussion, $\mathsf{f}_*(u_0)\hat{\nu}_j$ is orthogonal to the image of $A_*(u_0)$, then applying Π to relation (A.11) we have that

$$A_*(u_0)w_j = \varepsilon_j \Pi(\mathsf{f}_*(u_0)w_j).$$

Here $\Pi(\mathsf{f}_*(u_0)w_j)$ satisfies the following estimate with a positive constant C' that does not depend on j:

$$\|\Pi(\mathsf{f}_*(u_0)w_j)\|_0 \leqslant \|\mathsf{f}_*(u_0)w_j\|_0 \leqslant C' \|w_j\|_0 .$$

Since the w_j's are L_2-orthogonal to $\operatorname{Ker} A_*(u_0)$, then by the standard elliptic theory we have

$$\|w_j\|_m \leqslant C'' \|A_*(u_0)w_j\|_0 = \varepsilon_j C'' \|\Pi(\mathsf{f}_*(u_0)w_j)\|_0 \leqslant \varepsilon_j C'C'' \|w_j\|_0 .$$

Thus, we see that

$$\|w_j\|_0 \leqslant \varepsilon_j C \|w_j\|_0 .$$

Since $\varepsilon_j \to 0$ as $j \to \infty$, this implies that $w_j = 0$ for sufficiently large indices j. In other words, we have that $\nu_j \in \operatorname{Ker} A_*(u_0)$. Combining this with relation (A.10) and the fact that $\mathsf{f}_*(u_0)$ is non-degenerate we see that $\nu_j = 0$ and, hence, arrive at a contradiction. □

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