A note on Morse inequalities for harmonic maps with potential and their applications

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Abstract

We discuss Morse inequalities for homotopic critical maps of the energy functional with a potential term. For a generic potential this gives a lower bound on the number of homotopic critical maps in terms of the Betti numbers of the moduli space of harmonic maps. Other applications include sharp existence results for maps with prescribed tension field and pseudo-harmonic maps. Our hypotheses are that the domain and target manifolds are closed and the latter has non-positive sectional curvature.

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1. Introduction

The purpose of this note is two-fold: first, we point out that Morse inequalities for harmonic maps with potential hold – a direct consequence of the recent compactness results [11, 14] and old results of Uhlenbeck [22], which seems to have been unnoticed until now. More precisely, suppose that M and M' are closed Riemannian manifolds and the latter has non-positive sectional curvature. Then for a generic potential the number of homotopic critical maps $M \to M'$ of the energy functional with the corresponding potential term is finite and is at least the sum of the Betti numbers of the moduli space *Harm* formed by harmonic mappings within the same homotopy class. As an application we give a general version of the Hopf-type rigidity principle for variational problems, see [2, 3], in the framework of harmonic maps with potential.

Second, we apply the above to sharpen our previous results in [16], as well as the results by Kappeler and Latschev [12], concerning the algebraic number of maps with prescribed tension field. More precisely, Theorem 5 below implies that if the sectional curvature of M' is non-positive, then for a generic non-autonomous vector field f on M' the number of homotopic maps $u : M \to M'$ with the tension field $\mathsf{f}(x, u(x))$, where $x \in M$, is finite and their algebraic number equals the Euler characteristic of *Harm*. The important consequence of this is the existence of maps with prescribed tension field when $\chi(Harm)$ does not vanish.

As another application we discuss the existence of (so-called) pseudo-harmonic maps within fixed homotopy classes. These are solutions of elliptic quasi-linear systems for maps of manifolds, which, for example, include the Hermitian harmonic map equation, introduced by Jost and Yau [10]. As is shown in [10] such maps do not always exist if

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the domain manifold M is closed. Our approach gives a sharp existence result of pseudo-harmonic maps in certain situations, thus, complementing the work by other authors, which as a rule involves considerable analysis for the same purpose. We apply this to show an existence of harmonic maps equivariant under certain actions of discrete groups by homotheties.

The present note complements the circle of results discussed in our collaborative work [16] and can be regarded as an appendix to the latter paper. The author is grateful to Sergei Kuksin for the comments on an earlier version of this note.

Conventions. Throughout the text smoothness means C^∞-smoothness unless there is an explicit statement to the contrary. We also adopt the summation convention for the repeated indices.

2. Morse inequalities for harmonic maps with potential

2.1. Notation and the main statement

Let (M,g) and (M',g') be closed Riemannian manifolds of dimensions n and n' respectively. Their Riemannian metrics give rise to a natural metric on the jet bundle $J^1(M,M')$ over the space $M\times M'$; the latter induces a norm on linear operators $T_xM\to T_uM'$, where $x\in M$, $u\in M'$, which coincides with the Hilbert-Scmidt norm. For maps $u:M\to M'$ we consider the energy functional

$$E(u)=\frac{1}{2}\int\limits_M \|du(x)\|^2\, dVol_g(x),\qquad x\in M.$$

The Euler-Lagrange equation for this functional

$$-\tau(u)(x)=0,\qquad x\in M,$$

is called the *harmonic map equation* and its solutions are *harmonic mappings*. In local coordinates on M and M' the operator $\tau(u)$ has the form

$$\tau^i(u)=\Delta_M u^i+g^{\alpha\beta}\Gamma'^{\,i}_{jk}\frac{\partial u^j}{\partial x^\alpha}\frac{\partial u^k}{\partial x^\beta},$$

where $g^{\alpha\beta}$ and $\Gamma'^{\,i}_{jk}$ denote the tensor inverse to the metric on M and the Cristoffel symbols of the Levi-Civita connection ∇' on M' respectively, and Δ_M is the Laplace-Beltrami operator on M.[1]

For a given smooth function V on the product $M\times M'$, called the potential, consider the energy functional with the potential term

$$E_V(u)=E(u)+\int_M V(x,u(x))dVol_g(x).$$

The corresponding Euler-Lagrange equation has the form

$$-\tau(u)(x)+\nabla' V(x,u(x))=0,\qquad x\in M,\tag{1}$$

and its solutions are called *harmonic maps with potential*; see [4, 7] and [19]. The linearisation of the Euler-Lagrange operator with respect to the Levi-Civita connection ∇' yields

[1] There are different conventions for the choice of the sign of the Laplace-Beltrami operator. Due to our definition this operator is non-positive.

a second order linear elliptic differential operator on the pull-back bundle u^*TM'. It has the form

$$J(u)+\nabla'^2V(\cdot,u), \qquad (2)$$

where $J(u)$ is the Jacobi operator,

$$J(u)v=-trace_g(\nabla'^2v+R'(du,v)du), \quad v \text{ is a section of } u^*TM', \qquad (3)$$

and $\nabla'^2V(\cdot,u)$ is regarded as an endomorphism of T_uM', $u\in M'$. In particular, the operator (2) is formally self-adjoint, its spectrum is real, and the number of its negative eigenvalues is finite. The latter is called the index of a critical point.

Now fix a homotopy class $[v]$ of mappings $M\to M'$ and suppose that M' has non-positive sectional curvature. Then due to the results of Schoen and Yau [20], the moduli space of harmonic mappings within $[v]$, in sequel denoted by *Harm*, is a compact connected manifold (possibly with a Lipschitz boundary) which is smooth outside of its boundary. If, in addition, the target manifold M' is real-analytic, then the manifold *Harm* is closed. Recall that the moduli space *Harm* has the same homotopy type as the space of smooth mappings from M to M' in the homotopy class $[v]$ endowed with the compact-open topology; the harmonic map heat flow retracts the latter onto the former, see [6] and [21, Lem. 4.1]. Denote by b_k the Betti numbers (with respect to any coefficient field) of the space *Harm* and by $m_k(V)$ the number of harmonic maps with a potential V homotopic to v whose index is equal to k.

The general Morse theory [18] for variational problems requires the Palais-Smale condition, which fails for the functional E_V when $\dim M>1$. Nevertheless, under the curvature hypothesis on M' the following statement holds.

Theorem 1. *Suppose that a Riemannian manifold M' has non-positive sectional curvature. Then for a generic smooth potential V on the product $M\times M'$ the number of the critical maps of the functional E_V within a given homotopy class $[v]$ is finite and*

$$m_k(V)-m_{k-1}(V)+\cdots+(-1)^k m_0(V)\geqslant b_k-b_{k-1}+\cdots+(-1)^k b_0$$

for every integer k. In particular, there are at least b_k critical maps of index k, $m_k(V)\geqslant b_k$, and

$$\sum(-1)^k m_k(V)=\chi(Harm).$$

Such generic potentials form an open and dense set in the space of all smooth potentials.

This theorem can be viewed as a computation of the 'Morse homology' of E_V with a generic potential via the homology of the degenerate set of critical maps of E – the space of harmonic maps. The remarkable point is that we do not require any hypotheses on the structure of the critical set *Harm* (e.g., Morse-Bott non-degeneracy) whatsoever. The explanation for this is that the curvature hypothesis of Theorem 1 already makes the space *Harm* special – the latter is formed by absolute minimisers in a fixed homotopy class.

2.2. Appendix I: on the homology of Harm

First, note that under the above curvature hypothesis on M' homotopically trivial harmonic mappings are precisely constant mappings; the corresponding space *Harm* is diffeomorphic to M' and, hence, has the same homology as M'. In fact, the space *Harm* (and its homology) for other homotopy classes can be described in many situations. Recall that due to Hartman [9] the moduli space *Harm* is a point or a circle provided $[v]\neq[pt]$ and M' carries a metric of negative sectional curvature. Further, we mention the following assertion.

Lemma 2. *Suppose that a Riemannian manifold M' has non-positive sectional curvature. Then the fundamental group of the space Harm, formed by harmonic mappings in $[v]$, coincides with the centraliser of the image $v_*(\pi_1(M))$ in $\pi_1(M')$. Conversely, for any homomorphism $v_* : \pi_1(M) \to \pi_1(M')$ the centraliser of its image is the fundamental group of a totally geodesic immersed submanifold (maybe with a boundary) in M', which in particular admits a metric of non-positive sectional curvature.*

Proof. By [8, Lem. 2] the fundamental group of the space of continuous mappings of M to M' homotopic to v (with the compact-open topology) is the centraliser of $v_*(\pi_1(M))$ in $\pi_1(M')$. As is known this space has the same homotopy type as the space of smooth mappings homotopic to v. The latter, by [21, Lem. 4.1], has *Harm* as its strong deformation retract. This proves the first statement.

The second statement follows from the properties of *Harm*, see [20, Sect. 2], and the fact that, since M' is a $K(\pi,1)$-space, any homomorphism $\pi_1(M) \to \pi_1(M')$ is induced by a smooth map from M to M'. □

As a consequence we have a purely algebraic way of computing the homology of *Harm*.

Corollary 3. *Suppose that a Riemannian manifold M' has non-positve sectional curvature. Then for any homotopy class $[v]$ the homology of Harm coincide with the homology of the centaliser of the image $v_*(\pi_1(M))$ in $\pi_1(M')$.*

Proof. By the results in [20, Sect. 2], the moduli space *Harm* is geodesically locally convex and has non-positive sectional curvature. Hence, the distance function (induced by the Riemannian metric) makes it into a complete length space of non-positive curvature in the sense of Alexandrov. By the version of the Cartan-Hadamard theorem [1, II.4.1], the moduli space *Harm* has a contractible universal cover; i.e. is a $K(\pi,1)$-space. The homology of such a space coincide with the homology of its fundamental group. Now Lemma 2 implies the statement. □

2.3. Appendix II: on the Hopf-type rigidity

In the papers [2, 3] the Hopf-type rigidity for variational problems corresponding to the functional E_V has been studied. Loosely speaking, it states that in certain situations a potential V for which all critical maps of E_V are absolute minima has to be of a very restrictive type. We describe a general related principle.

Suppose a potential V^* is such that all critical maps of E_{V^*} in a given homotopy class are stable. Then, in many situations it is rigid in the sense that for almost any its perturbation V the new functional E_V picks up unstable critical maps. More precisely, we have the following assertion.

Corollary 4. *Suppose that a Riemannian manifold M' has negative sectional curvature or is real-analytic and has non-positive sectional curvature. Then for a given homotopy class $[v]$ of mappings from M to M' the following alternative holds – either*

- *$[v]$ has a unique harmonic representative; or*
- *the set formed by potentials V such that all critical maps of E_V within $[v]$ are stable belongs to the complement of an open and dense set in the space of all smooth potentials.*

For an arbitrary Riemannian manifold M' of non-positive sectional curvature the trivial homotopy class always satisfies the latter.

Proof. Suppose that a given homotopy class $[v]$ does not have a unique harmonic representative. Then due to [9, 20] the suppositions of the corollary imply that the moduli space *Harm* is a closed connected manifold of dimension $\ell > 0$. By Theorem 1, for a generic potential V there are at least b_ℓ critical maps of index ℓ and none of them can be stable. Since the rank b_ℓ of $H_\ell(Harm, \mathbf{Z}_2)$ equals 1, we are done. □

3. Algebraic number of maps with prescribed tension field

Now we describe other direct consequences of Theorem 1. Consider the quasi-linearly perturbed harmonic map equation

$$-\tau(u)(x) = \mathsf{G}(x, u(x)) \cdot du(x) + \mathsf{g}(x, u(x)), \qquad x \in M. \tag{4}$$

Here the symbol G stands for a morphism of the vector bundles $J^1(M, M')$ and $(\Pi')^*TM'$, where $\Pi' : M \times M' \to M'$ is a natural projection onto the Cartesian factor, and g is a non-autonomous vector field on M'.

We adopt the framework in [16] and suppose that maps u under consideration are $W^{2,s-1}$-smooth in the Sobolev sense, where s is a sufficiently large integer ($s > n/2 + 2$), and the G and g in the right-hand side are $W^{p,s+1}$-smooth, where $p > n + n'$. By standard embedding theorems and elliptic regularity, any solution of equation (4) is, in fact, C^{s+1}-smooth under these conditions. Theorem 1 implies the following assertion.

Theorem 5. *Suppose that a Riemannian manifold M' has non-positive sectional curvature. Then for any homotopy class $[v]$ of mappings from M to M' there exists a constant $C_{[v]} > 0$ such that for a generic right-hand side (G, g) such that*

$$\|\mathsf{G}(x, u)\| < C_{[v]} \qquad \text{for any } x \in M, \quad u \in M', \tag{5}$$

the number of solutions of equation (4) *within the homotopy class $[v]$ is finite and its algebraic number is equal to the Euler characteristic of Harm. Moreover, if $\chi(Harm)$ is not equal to zero, then for any right-hand side (G, g) which satisfies* (5) *there exists a solution of equation* (4) *within a homotopy class $[v]$.*

Remark 1. In inequality (5) the norm of the linear operator $\mathsf{G}(x, u) : \mathscr{L}(T_x M, T_u M') \to T_u M'$ is induced by the Riemannian metrics on M and M'. The constant $C_{[v]}$ is related to an optimal constant in the isoperimetric-type inequality for the L_2-widths of geodesic homotopies; see [11, Th. 0.3].

For the case when the moduli space of harmonic mappings within a fixed homotopy class is non-degenerate in the sense of Morse-Bott the statement of Theorem 5 is independently treated in [16] and [12]. Thus, we see that the Morse-Bott condition is unnecessary.

When the linear part G vanishes, the solutions of equation (4) can be regarded as maps with prescribed tension field; the latter coincides with the mean curvature vector if the corresponding solution is a Riemannian immersion. Theorem 5 gives a formula for the algebraic number of such maps within a fixed homotopy class.

The other important consequence of Theorem 5 is an existence of solutions of equation (4) in a fixed homotopy class for any admissable right-hand side provided that the Euler characteristic $\chi(Harm)$ does not vanish. This is a topological condition – it does not depend on metrics on M and M'. In applications a given homotopy class often contains a unique harmonic map and the hypothesis on $\chi(Harm)$ is trivially satisfied. This, for example, occurs when:

- a target manifold M' is negatively curved and a given homotopy class does not contain a map onto a closed geodesic, see [9];
- a target manifold M' is non-positively curved and a homotopy class does not contain a map v for which the pull-back bundle v^*TM' has a non-trivial parallel section, see [20].

In particular, in such homotopy classes there exists a map with a prescribed tension field f without any assumption on a given vector field f whatsoever; compare this with the results by Chen and Jost [5] for the corresponding Dirichlet problem. Note also that these conditions are much stronger than $\chi(Harm) \neq 0$ and do not illustrate the complete statement. Simple examples show that the latter is necessary.

The proofs of Theorems 1 and 5 appear in Sect. 5. First, we discuss some other applications.

4. Pseudo-harmonic mappings

In this section we describe a specific class of equations of the form (4), which are of geometric interest, and discuss the existence of their solutions; for other existence results as well as geometric applications we refer to [17].

Let $\nabla^\dagger$ be an arbitrary linear connection on the domain manifold M and ∇' be the Levi-Civita connection of a fixed Riemannian metric g' on M'. Consider the second fundamental form $\mathscr{D}^2u$ of a map u, given by

$$\mathscr{D}^2u(X,Y) = \widetilde{\nabla}_X(du)(Y).$$

Above X and Y are vector fields on M and $\widetilde{\nabla}$ denotes the connection on the tensor product $T^*M \otimes u^*TM'$ induced by the connections $\nabla^\dagger$ and ∇'. Taking the trace with respect to a Riemannian metric g on M we see that the correspondence

$$Maps(M,M') \ni u \longmapsto \tau^\dagger(u) = trace_g \mathscr{D}^2u \in Sections(u^*TM')$$

defines a second order elliptic differential operator, which differs from $\tau(u)$ by a linear first order term $-du(trace_g(\nabla^\dagger - \nabla))$; here ∇ is the Levi-Civita connection of the metric g. We call solutions of the equation $\tau^\dagger(u) = 0$ *pseudo-harmonic mappings*.

Now we explain the general ideology for introducing pseudo-harmonic maps. Suppose connections $\nabla^\dagger$ and ∇' preserve (or, more generally, are related to) given structures on M and M' respectively. Then we expect the morphisms of these structures be corresponding pseudo-harmonic maps. For example, if M and M' are Riemannian manifolds and $\nabla^\dagger = \nabla$ and ∇' are their Levi-Civita connections then totally geodesic maps are, of course, harmonic. Analogously, if M and M' are complex manifolds and $\nabla^\dagger$ and ∇' are complex connections, then holomorphic and anti-holomorphic maps solve the Hermitian harmonic map equation, introduced by Jost and Yau [10]. We discuss this equation in Example I in more detail. First, we state a new existence result for general pseudo-harmonic maps.

Choosing a Riemannian metric g on M and its Levi-Civita connection ∇ as a background connection, we regard the space of all linear connections on M as a vector space with the pseudo-norm

$$\left\|\nabla^\dagger\right\|_g = \sqrt{n'} \max_{x\in M} \left|trace_g(\nabla^\dagger - \nabla)\right|_g, \qquad (6)$$

where n' is the dimension of M'.

The following statement is a direct consequence of Theorem 5.

Corollary 6. *Let (M,g) and (M',g') be closed Riemannian manifolds and suppose M' has non-positive sectional curvature. Suppose also that a given homotopy class $[v]$ of mappings $M \to M'$ has a non-vanishing Euler characteristic $\chi(Harm)$. Then there exists a positive constant $C_{[v]}$, which depends on the geometry of M and M', and the homotopy class $[v]$, such that for any linear connection $\nabla^\dagger$ on the domain M whose pseudo-norm* (6) *is less than $C_{[v]}$ the equation $-\tau^\dagger(u) = 0$ has a solution within $[v]$.*

Remark 2. In [17], following the method of Jost and Yau [10], under the same curvature hypothesis we show the existence of general pseudo-harmonic maps without any restriction on a linear connection $\nabla^\dagger$, but under a stronger constraint on the homotopy class which, in particular, implies the uniqueness of a harmonic representative.

Example I: Hermitian harmonic map equation [10]. Suppose a domain manifold M is endowed with a complex structure J and let $\nabla^\dagger$ be a torsion-free connection which preserves J. Denote also by g a Hermitian metric on M; by this we mean a J-homogeneous Riemannian metric. Let $\tau^\dagger(u)$ be an operator constructed in the way described above with respect to the connection $\nabla^\dagger$. Direct calculations [15] show that $(1/2)\tau^\dagger(u)$ coincides with the Hermitian harmonic map operator, studied in [10], which in local coordinates has the form

$$\frac{1}{2}\tau^\dagger(u)^i = \bar{g}^{\alpha\bar{\beta}}\left(\frac{\partial^2 u^i}{\partial z^\alpha \partial z^{\bar{\beta}}} + {\Gamma'}^i_{jk}\frac{\partial u^j}{\partial z^\alpha}\frac{\partial u^k}{\partial z^{\bar{\beta}}}\right).$$

Here $\bar{g}$ is the corresponding Hermitian metric on the holomorphic tangent bundle of M. In particular, $\tau^\dagger(u) = 0$ coincides with the harmonic map equation when M is Kähler. More precisely, straightforward calculations yield

$$\tau^\dagger(u) - \tau(u) = -du(J \cdot \delta J), \tag{7}$$

where the vector field δJ is given by the formula

$$\delta J = -trace_g(\nabla J);$$

the symbol ∇ denotes the Levi-Civita connection of the Hermitian metric g. The difference field $J \cdot \delta J$ is is called the *Lee field* of a Hermitian structure (g,J). From relation (7) we also conclude that the Hermitian harmonic maps coincide with harmonic maps when the domain manifold M is co-symplectic (i.e. when δJ vanishes).

When M' has non-positive sectional curvature, Jost and Yau [10] showed an existence of Hermitian harmonic maps in certain homotopy classes, which have unique harmonic representatives. Corollary 6 translates into the following assertion.

Corollary 6$'$. *Let (M,g) be a closed Hermitian manifold and (M',g') be a closed Riemannian manifold of non-positive sectional curvature. Suppose that a given homotopy class $[v]$ has a non-vanishing Euler characteristic, i.e. non-vanishing $\chi(Harm)$. Then it contains a Hermitian harmonic map provided the C^0-norm of the Lee field,* $\max_M |J\delta J|_g$*, is sufficiently small.*

Remark 3. Due to an example in [10, Sect. 2] the hypothesis on $\chi(Harm)$ above is necessary.

Example II: Weyl harmonic map equation [17]. A Weyl structure on a conformal manifold (M,c) is a torsion-free linear connection ∇^W preserving the conformal structure c. This implies that for any Riemannian metric $g \in c$ there exists an 1-form Θ such that $\nabla^W g = \Theta \otimes g$. Alternatively, one can define ∇^W by the following formula

$$\nabla^W_X Y = \nabla_X Y - \frac{1}{2}\Theta(X)Y - \frac{1}{2}\Theta(Y)X + \frac{1}{2}g(X,Y)\Theta^\sharp,$$

where $\Theta^\sharp$ is a vector field dual to Θ with respect to g and ∇ is the Levi-Civita connection of g. A Weyl structure is called closed (exact) if the form Θ is closed (exact); this property does not depend on a metric $g \in c$. Note also that an exact Weyl structure, $\Theta = dV$, coincides with the Levi-Civita connection of $\exp(-V)g$. More generally, a closed Weyl structure is locally the Levi-Civita connection of a compatible metric; it does not need to be a global metric connection unless M is simply connected.

Let us fix a Riemannian metric $g \in c$ and consider the equation constructed as above with respect to a Weyl connection and a metric g on the domain M,

$$\tau^W(u)(x) = 0, \qquad x \in M. \tag{8}$$

Its solutions are called *Weyl harmonic maps*. Clearly, the property of being a Weyl harmonic map does not depend on a metric $g \in c$ used to define $\tau^W(u)$. Straightforward calculations yield

$$\tau^W(u) - \tau(u) = -\left(\frac{n-2}{2}\right) du(\Theta^\sharp).$$

In particular, in dimension 2 equation (8) coincides with the harmonic map equation. Further, if Θ is exact, $\Theta = dV$, then Weyl harmonic maps coincide with harmonic maps with respect to the metric $\exp(-V)g$ on the domain. More generally, for a closed Weyl structure solutions of (8) are locally harmonic maps with respect to a compatible metric on M and do not need to be global harmonic maps. This means that solutions of (8) are harmonic maps from the covering M^* (endowed with a compatible metric) such that the lifted form Θ^* becomes exact. The fundamental group $\pi_1(M)$ acts by deck transformations on M^* and these harmonic maps are $\pi_1(M)$-equivariant.

The Eells-Sampson existence theorem [6] for harmonic maps into manifolds of non-positive sectional curvature shows that equation (8) has a solution within any homotopy class provided $\dim M = 2$ or Θ is exact. In the case when Θ is not necessarily exact and $\dim M > 2$ we have the following existence theorem due to Corollary 6.

Corollary 6$''$. *Let (c, ∇^W) be a Weyl structure on a closed manifold M and (M', g') be a closed Riemannian manifold of non-positive sectional curvature. Suppose that a given non-trivial homotopy class $[v]$ has a non-vanishing Euler characteristic, i.e. non-vanishing $\chi(Harm)$. Then it contains a solution of equation* (8) *provided the norm $\left|\Theta^\sharp\right|_g$, with respect to some $g \in c$, is sufficiently small.*

Finally, mention that for closed Weyl structures this corollary implies the existence of harmonic maps equivariant under actions of discrete groups by homotheties. In more detail, let (M^*, g^*) be a Riemannian manifold, $\dim M > 2$, endowed with a free, cocompact, and proper discontinuous actions of a discrete group Γ by homotheties,

$$\gamma^* g^* = C_\gamma g^*, \qquad \gamma \in \Gamma.$$

This induces a homomorphism

$$\rho : \Gamma \to \mathbf{R}, \qquad \gamma \mapsto -\ln C_\gamma,$$

which, in turn, can be viewed as a cohomology class in $H^1(M^*/\Gamma, \mathbf{R})$. Let Θ be an 1-closed form representing this class and Θ^* be its lift to M^*. The latter form is exact, $\Theta^* = dV$, and the potential function V is ρ-equivariant;

$$V(\gamma \cdot x) - V(x) = \int_x^{\gamma \cdot x} \Theta^* = -\ln C_\gamma,$$

where the integral does not depend on a path joining points $\gamma \cdot x$ and x in M^*. This implies that the metric $g = \exp(V)g^*$ is Γ-invariant and, in particular, descends to the quotient M^*/Γ.

Combining this construction with Corollaries 3 and 6″ we arrive at the following statement:

Let (M', g') be a closed Riemannian manifold of non-positive sectional curvature. Suppose that a discrete group Γ acts by homotheties on (M^*, g^*) in the described above fashion. Let $v_* : \Gamma \to \pi_1(M')$ be a homomorphism such that the centraliser of its image has a non-zero Euler characteristic. Then the homomorphism v_* is induced by a Γ-equivariant harmonic map $(M^*, g^*) \to (M', g')$ provided the norm $\left|\Theta^\sharp\right|_g$ is sufficiently small.

5. Proofs of Theorems 1 and 5

Consider the quasi-linearly perturbed harmonic map equation (4),

$$-\tau(u)(x) = \mathrm{G}(x, u(x)) \cdot du(x) + \mathrm{g}(x, u(x)), \qquad x \in M. \tag{4}$$

As in Sect. 3 we suppose that the coefficients in its right-hand side, given by a morphism G (coefficients of the linear part of a perturbation) and a vector field g (the semi-linear part), are $W^{p,s+1}$-smooth in the Sobolev sense, where $p > n + n'$ and $s > n/2 + 2$. Solutions u of the equation are supposed to be $W^{2,s-1}$-smooth.

Fix a homotopy class $[v]$ of mappings $M \to M'$ and denote by $\mathscr{M}$ the space formed by the triples $(u, \mathrm{G}, \mathrm{g})$ which satisfy equation (4), are of described smoothness, and such that u belongs to the homotopy class $[v]$. Due to the standard embedding theorem and elliptic regularity theory, the inclusion

$$\mathscr{M} \subset C^{s+1}(M, M') \times \{C^s\text{-smooth } (\mathrm{G}, \mathrm{g})\}$$

holds and we suppose that $\mathscr{M}$ is endowed with the corresponding induced topology. Denote also by π the projection

$$\pi : \mathscr{M} \to \left\{W^{p,s+1}\text{-smooth } (\mathrm{G}, \mathrm{g})\right\}, \quad (u, \mathrm{G}, \mathrm{g}) \mapsto (\mathrm{G}, \mathrm{g}).$$

Thus, the space $\pi^{-1}(\mathrm{G}, \mathrm{g})$ is simply the moduli space of solutions in $[v]$ of equation (4) with the fixed right-hand side.

5.1. Background I (Compactness).

Recall that a continuous map between topological spaces is called proper if the preimage of a compact set is compact. All our results are based on the following principal assertion due to [11, 14].

Compactness theorem. *Let M and M' be closed Riemannian manifolds and suppose that M' has non-positive sectional curvature. Then for any homotopy class $[v]$ of maps from M to M' there exists a constant $C_{[v]} > 0$ such that for the domain*

$$\mathcal{U} = \{(\mathrm{G}, \mathrm{g}) : \ \|\mathrm{G}(x, u)\| < C_{[v]} \ \text{ for any } \ x \in M, \ u \in M'\}$$

the map $\pi|_{\pi^{-1}(\mathcal{U})}$ is proper.

The norm of the linear operator $\mathrm{G}(x, u) : \mathscr{L}(T_x M, T_u M') \to T_u M'$ in the theorem is induced by the Riemannian metrics on M and M'.

5.2. Background II (Transversality and the degree of π).

Now we linearise equation (4). First, for a map u and a vector field v along it choose a family of maps $u_t : M \to M'$, where $t \in (-\varepsilon, \varepsilon)$, such that

$$u_0 = u \quad \text{and} \quad (\partial u_t/\partial t)|_{t=0} = v.$$

By the symbol ∇' we denote a Levi-Civita connection on M' as well as its pull-back on the bundle U^*TM', where $U(t,\cdot) = u_t(\cdot)$. The linearised operator $L_*(u, \mathrm{G}, \mathrm{g})$ acts on a vector field v according to the formula

$$L_*(u,\mathrm{G},\mathrm{g})v = \nabla'_{\frac{\partial}{\partial t}}\Big|_{t=0} \left(-\tau(u_t) - \mathrm{G}(\cdot,u_t)\cdot du_t - \mathrm{g}(\cdot,u_t)\right).$$

Direct calculations show that this operator has the form

$$J(u) + \{\text{first order differential operator}\},$$

where $J(u)$ is the Jacobi operator (3) and the second term depends on (G, g) and their derivatives; for the details we refer to [13, 15]. In particular, the number of its real negative eigenvalues (counting their multiplicities) is finite and called the *index* of a point $(u, \mathrm{G}, \mathrm{g})$ from the space $\mathscr{M}$. We denote this integer by the symbol $\operatorname{ind}(u, \mathrm{G}, \mathrm{g})$.

Definition. A solution u of equation (4) is called *non-degenerate* if the kernel of the operator $L_*(u, \mathrm{G}, \mathrm{g})$ is trivial. The pair (G, g) is called *regular* if any solution of equation (4) with the corresponding G and g in the right-hand side is non-degenerate.

Lemma 7. *Under the hypotheses of Theorem 5, the set of regular pairs is of the second category in the sense of Baire and, in particular, is dense in the space of all $W^{p,s+1}$-smooth pairs (G, g). Moreover, the set of regular pairs in $\mathcal{U}$, where the set $\mathcal{U}$ is from the compactness theorem, is open.*

For a proof we refer to [13, 15]. The idea is to show that the space $\mathscr{M}$ is a C^2-smooth Banach manifold and the projection π is a C^2-smooth Fredholm map of zero index. The regular pairs are identified with regular values of π and Lemma 7 follows from the Sard-Smale theorem.

Now we apply the degree theory to the projection π. The following specific version is due to [16], where similar results for quite general elliptic operators on mappings of manifolds are discussed.

The degree theorem. *Let M and M' be closed Riemannian manifolds and let $[v]$ be a homotopy class of mappings from M to M'. Let $\mathcal{U}$ be an open subset in the space of $W^{p,s+1}$-smooth pairs (G, g) such that the projection $\pi|_{\pi^{-1}(\mathcal{U})}$ is proper. Then (i) for any regular pair $(\mathrm{G}, \mathrm{g}) \in \mathcal{U}$ the number of solutions of the corresponding equation (4) is finite and (ii) their algebraic number*

$$\deg \pi|_{\pi^{-1}(\mathcal{U})} = \sum_{\pi^{-1}(\mathrm{G},\mathrm{g})} (-1)^{\operatorname{ind}(u,\mathrm{G},\mathrm{g})}, \tag{9}$$

does not depend on $(\mathrm{G}, \mathrm{g}) \in \mathcal{U}$. Moreover, if this number does not equal zero, then equation (4) has a solution for any pair $(\mathrm{G}, \mathrm{g}) \in \mathcal{U}$.

5.3. Harmonic maps with potential.

Now we consider solutions of equation (1), the Euler-Lagrange equation for the perturbed energy functional $E_V(u)$; see Sect. 2. In accordance with the lines above, such a solution (critical map of E_V) is called *non-degenerate* if the corresponding linearised operator $J(u)+\nabla'^2V$ is non-degenerate. A potential V is called *regular* if all critical maps of E_V are non-degenerate. Clearly, we have the relation

$$L_*(u,0,-\nabla'V) = J(u)+\nabla'^2V(\cdot,u), \tag{10}$$

and, hence, the potential V is regular if and only if the pair $(0,-\nabla'V)$ is regular.

The following statement is an analogue of Lemma 7 for harmonic maps with potential; the proof essentially follows the same idea and is outlined at the end of this section.

Lemma 8. *Under the hypotheses of Theorem 1, the set of regular potentials is open and dense in the space of smooth functions on the product $M\times M'$.*

Denote by $\mathscr{H}_c$ the space formed by C^2-smooth mappings $u: M\to M'$ which lie in a fixed homotopy class $[v]$ and such that $E_V(u)<c$. The proof of Theorem 1 is based on the following statement, which is a reformulation of the result due to Uhlenbeck [22, Th. 3.3].

Uhlenbeck's theorem. *Let M and M' be closed Riemannian manifolds and V be a smooth function on their product. Suppose that M' has non-positive sectional curvature. If $a<b$ are not critical values of E_V on the space $\mathscr{H}_\infty$ and all smooth critical maps $u: M\to M'$ with $a<E_V(u)<b$ are non-degenerate, then the space $\mathscr{H}_a$ with handles adjoined corresponding to these critical maps of E_V, the dimensions of the handles correspond to indices of the critical maps, is a deformation retract of $\mathscr{H}_b$.*

5.4. Proof of Theorem 1.

Recall that due to [21, Lem. 4.1] the curvature hypothesis implies that the space $\mathscr{H}_\infty$ has the subspace *Harm*, formed by harmonic mappings, as its strong deformation retract; the corresponding deformation is given by the heat flow. In particular, its homology coincide with the homology of *Harm*.

Now let V be a regular smooth potential. Then, by the combination of the compactness theorem and the part (i) of the degree theorem, the number of critical maps of E_V homotopic to v is finite. Choose a real number c such that $E_V(u)<c$ for any such a critical point u. Now, since E_V is bounded from below, Uhlenbeck's theorem says that the cell complex formed by adjoined cells corresponding to the critical points of E_V (the dimensions of cells correspond to the indices of the critical points) is a deformation retract of the space $\mathscr{H}_c$. This implies the Morse inequalities for the number of critical points and the Betti numbers of $\mathscr{H}_c$; see [18]. Since there are no critical points in $\mathscr{H}_\infty\backslash\mathscr{H}_c$, then passing to the limit as $c\to+\infty$ we see that the homology of $\mathscr{H}_c$ and $\mathscr{H}_\infty$ coincide and, in particular, equal the homology of *Harm*. This proves the first statement of the theorem. The assertion on the set of regular potentials is provided by Lemma 8. $\square$

5.5. Proof of Theorem 5.

For a proof of the theorem we have to show that $\deg\pi$ in the degree theorem is equal to the Euler characteristic of *Harm*. Note that harmonic maps with a potential V can be regarded as solutions of equation (4) with the right-hand side corresponding to the pair $(0,-\nabla'V)$. As was mentioned above a potential V is regular if and only if this pair is regular. The index $\operatorname{ind}(u,0,-\nabla'V)$ of any solution u – the number of real negative eigenvalues of the operator (10) – coincides with the index of a harmonic map with potential as defined in Sect. 2. Now let V be a regular potential, then the Morse inequalities in Theorem 1 imply that the algebraic sum (9) is equal to the Euler characteristic $\chi(Harm)$. Thus, the statement follows from the degree theorem. □

5.6. Outline of the proof of Lemma 8

First, it is sufficient to prove that regular $W^{p,s+2}$-smooth potentials are dense and open in the space of $W^{p,s+2}$-smooth functions, where $p > n+n'$ and the integer $s > n/2+2$ can be arbitrary large. We also suppose below that critical maps of the functional E_V are $W^{2,s-1}$-smooth.

Denote by $\mathscr{M}^\star$ the space formed by the pairs (u,V), where u is a critical map for E_V in a given homotopy class $[v]$ and V is a potential function; both satisfy the smoothness conditions above. Let $\pi^\star$ be a projection defined as the map

$$\pi^\star : \mathscr{M}^\star \to \left\{W^{p,s+2}\text{-smooth } V\right\}, \quad (u,V) \mapsto V.$$

The proof of the statement on the density of regular potentials is based on identifying these with regular values of $\pi^\star$ and using the Sard-Smale theorem. For this one requires to show that the space $\mathscr{M}^\star$ is a C^2-smooth Banach manifold and the projection $\pi^\star$ is a C^2-smooth Fredholm map of zero index. By the standard implicit function theorem argument, see e.g. [13, 15], the latter reduces to the *transversality claim* below. The openness of regular potentials follows then from the compactness theorem.

Recall that a fixed Riemanninan metric on M' together with the volume form on M define a natural L_2-scalar product on the space of sections of u^*TM', where $u : M \to M'$ is an arbitrary map. We denote this scalar product by the brackets $\langle\cdot,\cdot\rangle$.

Transversality claim. *Let u be a critical map for the functional E_V and suppose that a given $W^{2,s-1}$-smooth vector field w along u is such that*

(i) $\left\langle w, \left(J(u)+\nabla'^2V(\cdot,u)\right)v\right\rangle = 0$ *for any section v of the bundle u^*TM',*

(ii) $\langle w, \nabla'U(\cdot,u)\rangle = 0$ *for any $W^{p,s+2}$-smooth function U on the product $M\times M'$.*

Then the vector field w vanishes identically.

Sketch of the proof. The first condition implies that w belongs to the kernel of operator (10) and, hence, is C^ℓ-smooth, where $\ell > 0$ depends on the integer s. If the vector field w is non-trivial, $w \neq 0$, then there exists a point $x_0 \in M$ together with its neighbourhood in which w does not have zeroes. Thus, for a proof of the claim it is sufficient for a given non-zero vector field w to construct a smooth function U such that the condition (ii) is violated. This can be done explicitly in a neighbourhood of the point x_0. We refer to [23, p. 75-77] where the construction is carried out in detail for the case when the dimension of M equals 1. □

References

[1] Bridson, M., Haefliger, A. *Metric spaces of non-positive curvature.* Grundlehren der Mathematischen Wissenschaften, 319. Springer-Verlag, Berlin, 1999. xxii+643 pp.

[2] Bialy, M., Polterovich, L. *Hopf-type rigidity for Newton equations.* Math. Res. Lett. **2** (1995), 695–700.

[3] Bialy, M., MacKay, R. S. *Variational properties of a nonlinear elliptic equation and rigidity.* Duke Math. J. **102** (2000), 391–401.

[4] Chen, Q. *Maximum principles, uniqueness and existence for harmonic maps with potential and Landau-Lifshitz equations.* Calc. Var. Partial Differential Equations **8** (1999), 91–107.

[5] Chen, W., Jost, J. *Maps with prescribed tension fields.* Comm. Anal. Geom. **12** (2004), 93–109.

[6] Eells, J., Sampson, J. H. *Harmonic mappings of Riemannian manifolds.* Amer. J. Math. **86** (1964), 109–160.

[7] Fardoun, A., Ratto, A., Regbaoui, R. *On the heat flow for harmonic maps with potential.* Ann. Global Anal. Geom. **18** (2000), 555–567.

[8] Gottlieb, D. H. *Covering transformations and universal fibrations.* Illinois J. Math. **13** (1969), 432–437.

[9] Hartman, P. *On homotopic harmonic maps.* Canad. J. Math. **19** (1967), 673–687.

[10] Jost, J., Yau, S.–T. *A nonlinear elliptic system for maps from Hermitian to Riemannian manifolds and rigidity theorems in Hermitian geometry.* Acta Math., **170** (1993), 221–254.

[11] Kappeler, T., Kuksin, S., Schroeder, V. *Perturbations of the harmonic map equation.* Commun. Contemp. Math. **5** (2003), 629–669.

[12] Kappeler, T., Latschev, J. *Counting solutions of perturbed harmonic map equations.* Enseign. Math. (2) **51** (2005), 47–85.

[13] Kokarev, G., Kuksin, S. *Quasilinear elliptic differential equations for mappings between manifolds, I.* (Russian) Algebra i Analiz **15** (2003), 1–60; translation in St. Petersburg Math. Journal **15** (2004), 469–505.

[14] Kokarev, G. *On the compactness property of the quasilinearly perturbed harmonic map equation.* Sbornik: Mathematics **194** (2003), 1055-1068.

[15] Kokarev, G. *Elements of qualitative theory of quasilinear elliptic partial differential equations for mappings valued in manifolds.* PhD Thesis, Heriot-Watt University, 2003.

[16] Kokarev, G., Kuksin, S. *Quasi-linear elliptic differential equations for mappings of manifolds, II.* Ann. Global Anal. Geom. **31** (2007), 59–113.

[17] Kokarev, G. *On pseudo-harmonic maps in conformal geometry.* Preprint, 2007.

[18] Palais, R. S., Smale, S. *A generalized Morse theory.* Bull. Amer. Math. Soc. **70** (1964), 165–172.

[19] Peng, X., Wang, G. *Harmonic maps with a prescribed potential.* C. R. Acad. Sci. Paris Sr. I Math. **327** (1998), 271–276.

[20] Schoen, R., Yau, S.-T. *Compact group actions and the topology of manifolds with non-positive sectional curvature.* Topology **18** (1979), 361–380. Errata in **21** (1982), 483.

[21] Sunada, T. *Rigidity of certain harmonic mappings.* Invent. Math. **51** (1979), 297–307.

[22] Uhlenbeck, K. *Morse theory by perturbation methods with applications to harmonic maps.* Trans. Amer. Math. Soc. **267** (1981), 569–583.

[23] Weber, J. *Perturbed closed geodesics are periodic orbits: index and transversality.* Math. Z. **241** (2002), 45–82.