

On the topology of the evaluation map and rational curves

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Abstract

We explore a relationship between topological properties of orbits of 2-cycles in the symplectomorphism group $Symp(M)$ and the existence of rational curves in M . Under the absence of rational curves hypothesis, we show that evaluation map vanishes on π_2 and obtain a Gottlieb-type vanishing theorem for toroidal cycles in $Symp(M)$.

Mathematics Subject Classification (2000): 22F50, 32J27, 35J60.

1. Statement and discussion of results

1.1.

Let M be a closed connected smooth¹ manifold of dimension $2n$. Suppose M admits a symplectic structure ω and denote by \mathcal{J}_ω the space of almost complex structures J on M such that ω is J -invariant and tames J . For an almost complex structure $J \in \mathcal{J}_\omega$ by J -curve in M we mean a J -holomorphic map $u : \Sigma \rightarrow M$, where Σ is a closed Riemannian surface; J -holomorphic spheres are also called *rational J -curves*.

Recall that a diffeomorphism $\phi : M \rightarrow M$ is called *symplectomorphism* if it preserves the symplectic structure, $\phi^*\omega = \omega$. Let $Symp(M)$ be a group of symplectomorphisms of M endowed with the compact open topology. By ev_u we denote the evaluation map at a base point $u \in M$ given by

$$Symp(M) \ni \varphi \longmapsto \varphi(u) \in M.$$

We also use the notation $ev_u^{\natural}[\phi]$ for the homotopy class $[ev_u \circ \phi]$, the image under the evaluation map of a homotopy class $[\phi]$ of maps $\Sigma \rightarrow Symp(M)$.

Our principle result shows that the existence of a certain 2-cycle in $Symp(M)$ whose orbits are non-contractible is related to the presence of rational J -curves.

Theorem 1. *Let (M, ω) be a closed symplectic manifold and suppose that either:*

- (i) *there exists a homotopy class $[\phi]$ of 2-spheres in $Symp(M)$ such that the evaluated class $ev_u^{\natural}[\phi]$ is non-trivial;*

*Supported by EPSRC grant GR/S68712/01

¹by this we always mean C^∞ -category unless there is an explicit statement to the contrary

- (ii) the Euler-Poincaré number $\chi(M)$ does not vanish and there exists a homotopy class $[\phi]$ of 2-tori in $\text{Symp}(M)$ such that the evaluated class $\text{ev}_u^{\natural}[\phi]$ is non-trivial.

Then for any almost complex structure $J \in \mathcal{J}_\omega$ there exists a J -holomorphic sphere in M .

The proof is based on the analysis of the Cauchy-Riemann equations perturbed by a term defined by a 2-parameter family of symplectomorphisms. We show that the corresponding moduli space formed by solutions in $\text{ev}_u^{\natural}[\phi]$ is diffeomorphic to M . In the case (i) we use the absence of sphere bubbles to conclude that this moduli space is null-cobordant or the homotopy class $\text{ev}_u^{\natural}[\phi]$ is trivial. Further analysis shows that only the latter occurs. The argument in the case (ii) is similar, but uses the Morse-Bott theory. Theorem 1 is a consequence of a more general statement in the next subsection.

An example with a torus shows that the conclusion of the theorem no longer holds if the condition $\chi(M) \neq 0$ in the part (ii) is dropped. More precisely, the 2-torus in $\text{Symp}(\mathbb{T}^2)$ given by the action

$$\mathbb{R}^2/\mathbb{Z}^2 \ni (s, t) : \mathbb{T}^2 \rightarrow \mathbb{T}^2, \quad (x, y) \mapsto (x + s, y + t)$$

evaluates into the fundamental cycle and, clearly, there are no non-trivial pseudo-holomorphic spheres in \mathbb{T}^2 .

As a direct consequence we arrive at the following vanishing theorem.

Corollary 2. *Suppose that a symplectic manifold (M, ω) does not admit non-trivial J -holomorphic spheres for at least one compatible almost complex structure $J \in \mathcal{J}_\omega$. Then:*

- (i) the evaluation map $\text{ev}_u : \text{Symp}(M) \rightarrow M$ induces the trivial homomorphism on π_2 ;
- (ii) if $\chi(M) \neq 0$, the image $\text{ev}_u^{\natural}[\phi]$ of any homotopy class $[\phi]$ of 2-tori in $\text{Symp}(M)$ is trivial.

This statement is interesting from the point of view of Gottlieb's theory [3, 5]. We discuss this in the next subsection in more detail. Now we end with simple examples of symplectic manifolds without J -spheres. First, we introduce more notation.

Recall that the energy of a map u from a Riemannian surface (Σ, i_Σ) to (M, J) , where $J \in \mathcal{J}_\omega$, is defined as the integral

$$E(u) = \int_{\Sigma} \frac{1}{2} \|du(z)\|^2 d\text{Vol}_{\Sigma}(z),$$

where $\|du(z)\|$ is the Hilbert-Schmidt norm of the operator $du(z) : T_z\Sigma \rightarrow T_{u(z)}M$ with respect to the metric $\omega(\cdot, J\cdot)$ on M and any metric on Σ in the conformal class determined by i_Σ . Any J -holomorphic curve $u : \Sigma \rightarrow M$ minimises the energy in a given homology class and, in particular, enjoys the following identity

$$E(u) = \langle u^*[\omega], \Sigma \rangle, \tag{1.1}$$

where the right-hand side stands for the evaluation of $u^*[\omega]$ on the fundamental cycle. As is known [11, Chapter 4], and can be easily proved, the quantity

$$S_\omega(J) = \inf \{E(u) : u \text{ is a non-constant } J\text{-sphere in } M\}$$

is positive. Here the infimum over the empty set is supposed to be equal to infinity. The latter, for example, occurs in the examples below.

Example 1. A symplectic manifold (M, ω) is called *symplectically aspherical* if $\omega|_{\pi_2} = 0$. As follows from the energy identity for J -curves (relation (1.1) above) such manifolds do not have non-trivial J -spheres for any $J \in \mathcal{J}_\omega$. The existence of symplectically aspherical manifolds with non-trivial π_2 was an open question until the examples due to Kollar and Gompf [6] appeared. There are also examples of the latter with arbitrarily large Euler-Poincaré numbers; it was observed in [2] that one can construct these as the symplectic submanifolds described by Auroux [1].

Example 2. Let (M, ω) be a 4-dimensional symplectic manifold such that its first Chern class is a non-positive multiple of ω ,

$$[c_1]|_{\pi_2} = k [\omega]|_{\pi_2}, \quad \text{where } k \leq 0, \quad k \in \mathbb{R}. \quad (1.2)$$

Then for a generic almost complex structure $J \in \mathcal{J}_\omega$ the manifold M contains no J -spheres. Indeed, first recall that a J -curve is called simple if it is not a (branched) cover of degree greater than one of another J -curve. Clearly, if there exists a J -sphere, then there exists a simple J -sphere. Further, due to the standard Fredholm theory [11, Chapter 3], for a generic J the dimension of unparametrised simple J -spheres representing $A \in H_2(M, \mathbb{Z})$ is equal to $2c_1(A) - 2$. In particular, if such a sphere exists, then $c_1(A) \geq 1$. On the other hand, due to relation (1.2), we have $c_1(A) \leq 0$ for any homology class A that can be represented by a rational J -curve. The latter is a consequence of the energy identity for J -curves, relation (1.1).

1.2.

In this subsection we state a more precise condition on a 2-cycle which guarantees that its image under the evaluation map is homotopically trivial. For this we define a certain energy-type characteristic of its action on M .

First, we suppose that tori and spheres in $Symp(M)$ under consideration are represented by maps ϕ such that $ev_u \circ \phi$ is smooth for any $u \in M$. Due to the following lemma this does not affect the topological conclusions.

Lemma 3. *Let Σ and (M, ω) be an arbitrary closed manifold and a closed symplectic manifold respectively. Then continuous maps $\phi : \Sigma \rightarrow Symp(M)$ such that $ev_u \circ \phi$ is smooth for any $u \in M$ form a dense subset in the space of all continuous maps $\Sigma \rightarrow Symp(M)$ with respect to the compact open topology.*

The proof follows essentially from the Moser isotopy argument and is explained at the end of Section 3.

Now we define the *evaluation energy* of a map $\phi : \Sigma \rightarrow Symp(M)$ as

$$E^{\text{ev}}(\phi, \omega, J) = \int_{\Sigma} \frac{1}{2} \max_{u \in M} \|d(ev_u \circ \phi)\|^2 dVol_{\Sigma},$$

where the Hilbert-Schmidt norm of $d(ev_u \circ \phi)$ is taken with respect to the metric $\omega(\cdot, J\cdot)$ on M and some (and, hence, any) metric in the conformal class of i_{Σ} . Further, a map $\phi : \Sigma \rightarrow Symp(M)$ and a given metric on Σ define a function Λ on

the product $\Sigma \times M$ by the relation $(\text{ev}_u \circ \phi)^* \omega = \Lambda_u d\text{Vol}_\Sigma$. Using this function we construct the second functional

$$\Delta(\phi, \omega) = \int_\Sigma \left(\max_u \Lambda_u - \min_u \Lambda_u \right) d\text{Vol}_\Sigma \geq 0.$$

Finally, the *corrected evaluation energy* is defined as the sum

$$\mathcal{E}(\phi, \omega, J) = E^{\text{ev}}(\phi, \omega, J) + \Delta(\phi, \omega).$$

In general, this quantity depends on the conformal class of metrics or, equivalently, the complex structure i_Σ on the Riemannian surface.

Let \mathcal{M}_g be the Riemannian moduli space of all complex structures on a Riemannian surface Σ of genus g up to the pull-back by an orientation preserving diffeomorphism. Recall that for a sphere and a torus the space \mathcal{M}_g is identified with a single point and the fundamental domain for the action of $\text{PSL}(2, \mathbb{Z})$ on the upper half-plane, respectively. Denote by $\mathcal{E}_\Pi([\phi], \omega, J)$ the infimum of the corrected evaluation energy \mathcal{E} over pairs (ϕ, i_Σ) , where ϕ represents a given homotopy class $[\phi]$ and i_Σ ranges over \mathcal{M}_g .

We are ready to state a quantitative version of Theorem 1.

Theorem 4. *Let (M, ω) be a symplectic manifold and Σ be a Riemannian surface. Suppose that a homotopy class $[\phi]$ of mappings $\Sigma \rightarrow \text{Symp}(M)$ is such that*

$$\sup_{J \in \mathcal{J}_\omega} (S_\omega(J) - \mathcal{E}_\Pi([\phi], \omega, J)) > 0. \quad (1.3)$$

Then:

- (i) if Σ is a sphere, the homotopy class $\text{ev}_u^h[\phi]$ is trivial;
- (ii) if Σ is a torus, the homotopy class $\text{ev}_u^h[\phi]$ is trivial or $\chi(M) = 0$.

The statement of the theorem can be also regarded as an estimate for the energy $\mathcal{E}(\phi, \omega, J)$ from below. That is the “energy” required for a sphere or a torus in $\text{Symp}(M)$ to evaluate into a homotopically non-trivial one is at least $S_\omega(J)$.

The hypothesis on the evaluation energy in Theorem 4 can be relaxed, if we are concerned only with the action of the evaluation map on homology classes. Let A be a class from $H_2(\text{Symp}, \mathbb{Z})$. Denote by $\mathcal{E}_H(A, \omega, J)$ the infimum of the evaluation energy over pairs (ϕ, i_Σ) , where ϕ is a map of a Riemannian surface Σ of a fixed genus g into $\text{Symp}(M)$ such that $\phi_*[\Sigma] = A$ and $i_\Sigma \in \mathcal{M}_g$. We have the following version of Theorem 4.

Theorem 4’. *Let (M, ω) be a symplectic manifold and A be a homology class in $H_2(\text{Symp}, \mathbb{Z})$ that can be represented by an image of a given Riemannian surface Σ . Suppose that*

$$\sup_{J \in \mathcal{J}_\omega} (S_\omega(J) - \mathcal{E}_H(A, \omega, J)) > 0. \quad (1.4)$$

Then:

- (i) if Σ is a sphere, the homology class $(\text{ev}_u)_* A$ is trivial;
- (ii) if Σ is a torus, the homology class $(\text{ev}_u)_* A$ is trivial or $\chi(M) = 0$.

Remark 3. It is a simple exercise to show that the infimums of the corrected evaluation energy \mathcal{E} on the homology classes A and $-A$ coincide. Thus, condition (1.4) is natural with respect to the fact that the map ev_u vanishes or not on these classes simultaneously.

Finally, we mention that our results can be viewed as symplectic versions of Gottlieb's vanishing theorems. To illustrate the relationship more clearly we recall the following assertion, which is due to [4, Theorem 8.9].

Gottlieb's theorem. *Let N be a closed oriented manifold and $\text{Diff}(N)$ be its group of diffeomorphisms. Then the homomorphisms $\chi(N) \text{ev}_u^*$ and $c \cdot \sigma(N) \text{ev}_u^*$ of the cohomology groups $H^k(N, R) \rightarrow H^k(\text{Diff}, R)$ vanish for any $k > 0$ and any unitary ring R ; here c is an appropriate non-zero integer which depends only on the dimension of N and $\chi(N)$ and $\sigma(N)$ stand for the Euler-Poincaré number and the signature respectively.*

As the example below shows the Euler-Poincaré number $\chi(N)$ in the theorem is essential and, in general, the homomorphism induced by ev_u is not expected to be trivial on cohomology or homology.

Example 4. Let S^2 be a unit sphere in \mathbb{R}^3 and $SO(3)$ be its group of orientation preserving isometries. The evaluation map $\text{ev}_u : SO(3) \rightarrow S^2$ defines a bundle with fibre $SO(2)$. Note that this map induces the trivial homomorphisms on the reduced homology. However, the homomorphism on the cohomology

$$\text{ev}_u^* : H^2(S^2, \mathbb{Z}_2) \rightarrow H^2(SO(3), \mathbb{Z}_2)$$

is not trivial. Indeed, the fundamental class $[\omega] = \text{PD}[pt]$ maps to

$$\text{ev}_u^*[\omega] = \text{ev}_u^* \text{PD}[pt] = \text{PD}[\text{ev}_u^{-1}(pt)].$$

Since the fiber $\text{ev}_u^{-1}(pt)$ is not homologous to zero and the Poincaré Duality PD is an isomorphism, we conclude that $\text{ev}_u^*[\omega] \neq 0$. This illustrates Gottlieb's theorem – the presence of the Euler-Poincaré number is essential. In particular, we see that the evaluation map is not contractible on the 2-skeleton of $SO(3)$. In fact, there are 2-tori in $SO(3)$ which evaluate into homotopically non-trivial ones and, hence, condition (1.3) in Theorem 4 is necessary. As such a torus one can take, for example, a subset in $SO(3)$ generated by rotations around two different axes in \mathbb{R}^3 ; since $SO(3)$ is not commutative one needs to specify which rotation applies first.

Acknowledgements. I am much obliged to Elmer Rees for a number of discussions on the subject, which encouraged me to write this note.

2. Preliminaries

2.1. Perturbed Cauchy-Riemann equations

Let Σ be an oriented closed Riemannian surface and (M, ω) be a closed symplectic manifold of dimension $2n$ endowed with an almost complex structure $J \in \mathcal{J}_\omega$. For mappings $u : \Sigma \rightarrow M$ we consider the non-linear Cauchy-Riemann operator

$$\bar{\partial}u = \frac{1}{2}(du + J \circ du \circ i_\Sigma),$$

the J -complex anti-linear part of the differential du . Denote by $\Omega^{0,1}$ the vector bundle with base $\Sigma \times M$ whose fibre over (z, u) is formed by J -anti-linear operators $T_z \Sigma \rightarrow T_u M$. In this notation the differential operator $\bar{\partial}$ sends

$$\text{Maps}(\Sigma, M) \ni u \longmapsto \bar{\partial}u \in \text{Sections}(\tilde{u}^* \Omega^{0,1}),$$

where $\tilde{u} : \Sigma \rightarrow \Sigma \times M$ is the graph of u , given by $z \mapsto (z, u(z))$. More generally, let f be a section of the bundle $\Omega^{0,1}$. Consider the *perturbed Cauchy-Riemann equations*

$$\bar{\partial}u(z) = f(z, u(z)), \quad z \in \Sigma; \quad (2.1)$$

its solutions are called *perturbed J -curves*. Below we suppose that the right-hand side f is $W^{p,\ell+1}$ -smooth in the Sobolev sense, where $p > 2(n+1)$ and $\ell > 3$, and a solution u is $W^{2,2}$ -smooth. Due to elliptic regularity theory, these suppositions imply that solutions of equation (2.1) are, in fact, $C^{\ell+1}$ -smooth.

For a given homotopy class $[v]$ of mappings $\Sigma \rightarrow M$ denote by $\mathfrak{M}([v], J)$ the *universal moduli space* formed by pairs (u, f) of such maps $u \in [v]$ and sections f of $\Omega^{0,1}$ which satisfy equation (2.1). We consider $\mathfrak{M}([v], J)$ as a subspace in the product $C^{\ell+1}(\Sigma, M) \times \{C^\ell\text{-smooth sections } f\}$ and endow it with the induced topology. The symbol π denotes the natural projection

$$\mathfrak{M}([v], J) \ni (u, f) \mapsto f \in \{W^{p,\ell+1}\text{-smooth sections}\}.$$

Thus, each fiber $\pi^{-1}(f)$ is simply the moduli space of solutions (homotopic to v) of equation (2.1) with a given section f .

It is a simple exercise to show that a solution of equation (2.1) satisfies the following energy estimate:

$$E(u) \leq \int_{\Sigma} \max_u \|f(\cdot, u)\|^2 d\text{Vol}_{\Sigma} + \langle u^*[\omega], \Sigma \rangle.$$

Using this and the standard rescaling technique we arrive at the following statement.

Compactness theorem. *Let (M, ω) be a closed symplectic manifold endowed with an almost complex structure $J \in \mathcal{J}_{\omega}$. Denote by \mathcal{C} the set formed by homotopy classes $[v]$ of mappings $\Sigma \rightarrow M$ such that*

$$V_{\mathcal{C}} = \sup \{ \langle v^*[\omega], \Sigma \rangle : [v] \in \mathcal{C} \} < S_{\omega}(J).$$

Then the natural projection

$$\pi : \bigcup_{[v] \in \mathcal{C}} \mathfrak{M}([v], J) \rightarrow \{W^{p,\ell+1}\text{-smooth sections } f\}, \quad (u, f) \mapsto f,$$

restricted on the domain $\pi^{-1}(\mathcal{U}_{\ell})$ is proper, where

$$\mathcal{U}_{\ell} = \left\{ W^{p,\ell+1}\text{-smooth } f : \int_{\Sigma} \max_u \|f(\cdot, u)\|^2 < S_{\omega}(J) - V_{\mathcal{C}} \right\}. \quad (2.2)$$

In particular, the space of solutions of equation (2.1) within the homotopy classes $[v]$ such that $\langle v^[\omega], \Sigma \rangle \leq 0$ is always compact provided the section f satisfies $\int \max_u \|f(\cdot, u)\|^2 < S_{\omega}(J)$.*

In applications below the set \mathcal{C} is a single homotopy class or the set of homotopy classes representing a given homology class of mappings. In both cases the constant $V_{\mathcal{C}}$ is equal to $\langle v^*[\omega], \Sigma \rangle$.

Now we linearise equation (2.1) with respect to a linear connection ∇^Ω on the vector bundle $\Omega^{0,1}$. By definition the corresponding linearised at a (C^3 -smooth) map u Cauchy-Riemann operator sends a section v of the pull-back bundle u^*TM to a section of $\tilde{u}^*\Omega^{0,1}$,

$$v \longmapsto (\bar{\partial}u)_*v = \nabla_{\partial/\partial t}^\Omega \Big|_{t=0} (\bar{\partial}u_t);$$

here u_t is a family of mappings $\Sigma \rightarrow M$ such that

$$u_t|_{t=0} = u \quad \text{and} \quad (\partial/\partial t)|_{t=0} u_t = v.$$

Such a connection ∇^Ω on the vector bundle $\Omega^{0,1}$ can be, for example, built up from a canonical J -linear connection on M and any Levi-Civita connection (of a metric compatible with the complex structure) on Σ . More precisely, let ∇ be a Levi-Civita connection of the metric $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$. Then the connection $\tilde{\nabla}$ given by

$$\tilde{\nabla}_Y X = \nabla_Y X - \frac{1}{2}J(\nabla_Y J)X,$$

where X and Y are vector fields on M , is J -linear. The corresponding linearised Cauchy-Riemann operator is given by the formula

$$(\bar{\partial}u)_*v = (\nabla v)^{0,1} - \frac{1}{2}J(u)(\nabla_v J)\partial u$$

and, in particular, does not depend on a connection on Σ . Here v is a vector field along u , the symbol $(\nabla v)^{0,1}$ stands for the (J -)complex anti-linear part of the form ∇v , and ∂u is the J -linear part of du . For more details we refer to [11, Chapter 3].

Analogously, the linearisation of equation (2.1) at a map u defines the differential operator $(\bar{\partial}u)_* - f_*(\cdot, u)$. This operator differs from the linearised Cauchy-Riemann operator by zero-order terms depending on derivatives of f . Moreover, the corresponding operator linearised at a solution of equation (2.1) does not depend on the choice of a connection ∇^Ω used and can be defined as

$$v \longmapsto \frac{\partial}{\partial t} \Big|_{t=0} [\bar{\partial}u_t - f(\cdot, u_t(\cdot))],$$

where u_t is a family of mappings as above. Recall that a section f in the perturbed Cauchy-Riemann equations is called *regular*, if the cokernel of this differential operator is trivial for any solution u of equation (2.1). In particular, so is any section f for which equation (2.1) does not have solutions, i.e. $\pi^{-1}(f) = \emptyset$.

The following statement is folkloric and its analogues are proved by many authors in different frameworks. Our closest references are [11, Chapter 3] and [8, 9].

Theorem 5 (Folklore). *Let (M, ω) be a symplectic manifold of dimension $2n$ endowed with an almost complex structure $J \in \mathcal{J}_\omega$ and Σ be a closed oriented Riemannian surface (with a fixed complex structure). Suppose that a given homotopy class $[v]$ of mappings $\Sigma \rightarrow M$ is such that $\langle v^*[\omega], \Sigma \rangle < S_\omega(J)$ and let \mathcal{U}_ℓ be the domain given by (2.2) with the integer ℓ such that*

$$\ell > n\chi(\Sigma) + 2\langle v^*[c_1], \Sigma \rangle + 3.$$

Then for any regular section $f \in \mathcal{U}_\ell$ the space of solutions $\pi^{-1}(f)$ within $[v]$ is either empty or a closed $C^{\ell-2}$ -smooth manifold whose dimension is equal to $n\chi(\Sigma) + 2\langle v^*[c_1], \Sigma \rangle$; besides, $\pi^{-1}(f)$ carries a natural orientation. Further, two regular sections f_0 and $f_1 \in \mathcal{U}_\ell$ can be joined by a path $f_t \in \mathcal{U}_\ell$ such that the set $\cup_t \pi^{-1}(f_t)$ is a smooth oriented manifold with boundary $\pi^{-1}(f_0) \cup \pi^{-1}(f_1)$. The boundary orientation agrees with the orientation of $\pi^{-1}(f_1)$ and is opposite to the orientation of $\pi^{-1}(f_0)$.

We end with a few comments on the proof. First, one shows that the universal moduli space $\mathfrak{M}([v], J)$ is a $C^{\ell-2}$ -smooth Banach manifold and the projection π is a $C^{\ell-2}$ -smooth Fredholm map. Its index coincides with the index of the linearised Cauchy-Riemann operator $(\bar{\partial}u)_*$ and by Riemann-Roch theorem is given by the formula

$$\text{ind } \pi = n\chi(\Sigma) + 2\langle v^*[c_1], \Sigma \rangle.$$

The regular values of π are identified with regular sections f and, hence, the preimage $\pi^{-1}(f)$ is a $C^{\ell-2}$ -smooth manifold whose dimension is equal to $\text{ind } \pi$. The proof that two regular fibers are cobordant uses the transversality argument which requires that the order of smoothness of π is greater than $(\text{ind } \pi + 1)$; see also [12, Section 3] for a similar argument. This explains the formula for ℓ in the theorem.

For the sequel we point out that the cobordism manifold $N = \cup_t \pi^{-1}(f_t)$ is a $C^{\ell-2}$ -smooth submanifold of the universal moduli space $\mathfrak{M}([v], J)$; the latter is a submanifold in the product $W^{2,2}(\Sigma, M) \times \{W^{p,\ell+1}\text{-smooth sections } f\}$. For given a reference point $z_* \in \Sigma$ consider the map

$$N \ni (u, f) \longmapsto u(z_*) \in M. \quad (2.3)$$

The latter factors as the composition of the projection onto $W^{2,2}(\Sigma, M)$ and the evaluation at the point z_* , and is clearly $C^{\ell-2}$ -smooth.

The case when the dimension of the space of solutions $\pi^{-1}(f)$ is equal to zero is of particular interest and have been studied in [9] in a slightly different framework. We discuss this below in more detail.

2.2. Elements of Morse-Bott theory

For the rest of the section we suppose that the genus of a Riemannian surface Σ is equal to one and a given homotopy class $[v]$ is such that $\langle v^*[c_1], \Sigma \rangle = 0$. Then, under the conditions of Theorem 5, the space of solutions in $[v]$ of equation (2.1) with a regular $f \in \mathcal{U}_\ell$ is finite and its oriented cobordism class defines an integer $\text{deg } \pi$ – the algebraic number of solutions. Note also that in this case the linearised Cauchy-Riemann operator sends sections of u^*TM into themselves (the bundles u^*TM and $\tilde{u}^*\Omega^{0,1}$ are naturally isomorphic) and, hence, one can speak about its resolvent set.

Let \mathfrak{S} be a space, regarded as a subspace of $W^{2,2}(\Sigma, M)$, formed by solutions of the equation

$$\bar{\partial}u(z) = g(z, u(z)), \quad z \in \Sigma, \quad (2.4)$$

within a fixed homotopy class. Suppose that g above is a smooth section of $\Omega^{0,1}$ and, hence, due to elliptic regularity, \mathfrak{S} is formed by smooth mappings. In sequel we use the notation $\bar{\mathcal{D}}(u)$ for the linearised operator $(\bar{\partial}u)_* - g_*(\cdot, u)$.

By the implicit function theorem any $u \in \mathfrak{S}$ has a neighbourhood in the space \mathfrak{S} which can be identified with a subset of a ball in the space $\text{Ker } \bar{\mathcal{D}}(u)$; see [8,

Proposition 4.1]. In particular, if there exists a neighbourhood which can be identified with a ball in $\text{Ker } \widehat{\mathcal{D}}(u)$, then the space of solutions \mathfrak{S} is called *non-degenerate at a point u* . We call the space \mathfrak{S} , or its connected component, *non-degenerate (in the sense of Morse-Bott)* if it is non-degenerate at any point. Alternatively, one can say that \mathfrak{S} is non-degenerate if each of its connected components \mathfrak{S}^α is a smooth submanifold of $W^{2,2}(\Sigma, M)$ whose dimension is equal to the dimension of $\text{Ker } \widehat{\mathcal{D}}(u)$, where $u \in \mathfrak{S}^\alpha$.

Definition. The space of solutions \mathfrak{S} (or its connected component) is called *strongly non-degenerate* if it is non-degenerate in the sense of Morse-Bott and for any $u \in \mathfrak{S}$ the linearised operator $\widehat{\mathcal{D}}(u)$ does not have adjoint vectors corresponding to the zero eigenvalue; i.e. the algebraic multiplicity of the zero eigenvalue is equal to the dimension of $\text{Ker } \widehat{\mathcal{D}}(u)$.

Example 5. Suppose that a pull-back bundle u^*TM , where $u \in \mathfrak{S}$, is endowed with a Riemannian metric. This together with a volume form on Σ gives rise to a natural L_2 -scalar product on the vector fields along u . Recall that a linear differential operator is called *formally normal* if it commutes with its formally adjoint operator. Formally normal operators do not have adjoint vectors corresponding to the zero eigenvalue [7, Chapter 5]; see also [9, Section 6.1]. Thus, if the space of solutions \mathfrak{S} is non-degenerate and the operator $\widehat{\mathcal{D}}(u)$ is formally normal for any $u \in \mathfrak{S}$, then \mathfrak{S} is strongly non-degenerate.

The following theorem is proved in [9]; see [9, Theorem 3] and also the discussion in [9, Section 10].

Theorem 6. Let (M, ω) be a symplectic manifold endowed with an almost complex structure $J \in \mathcal{J}_\omega$ and $[v]$ be a homotopy class of mappings $\Sigma = \mathbb{T}^2 \rightarrow M$ such that

$$\langle v^*[c_1], \Sigma \rangle = 0 \quad \text{and} \quad \langle v^*[\omega], \Sigma \rangle < S_\omega(J).$$

Suppose that there exists a smooth section $\mathfrak{g} \in \mathcal{U}_\ell$, $\ell > 3$, such that the space \mathfrak{S} formed by solutions in $[v]$ of equation (2.4) is strongly non-degenerate in the sense of Morse-Bott and the evaluation map

$$\Sigma \times \mathfrak{S} \ni (z, u) \xrightarrow{\text{ev}} (z, u(z)) \in \Sigma \times M \tag{2.5}$$

is an embedding. Then the algebraic number $\deg \pi$ of solutions in $[v]$ of equation (2.1) for a regular section $\mathfrak{f} \in \mathcal{U}_\ell$, $\ell > 3$, is given by the formula

$$\deg \pi = \sum_{\alpha} \pm \chi(\mathfrak{S}^\alpha),$$

where \mathfrak{S}^α is a connected component of \mathfrak{S} and $\chi(\mathfrak{S}^\alpha)$ stands for its Euler-Poincaré number.

Corollary 7 (Theorem 6 in [9]). Let (M, ω) be a symplectic manifold endowed with an almost complex structure $J \in \mathcal{J}_\omega$. Then the algebraic number $\deg \pi$ of null-homotopic perturbed J -tori for a regular section \mathfrak{f} such that $\int \max_u \|\mathfrak{f}(\cdot, u)\|^2 < S_\omega(J)$ is equal to the Euler-Poincaré number $\chi(M)$. For a non-trivial homotopy class $[v]$ such that

$$\langle v^*[c_1], \Sigma \rangle = 0 \quad \text{and} \quad \langle v^*[\omega], \Sigma \rangle \leq 0$$

the degree $\deg \pi$ is equal to zero.

Proof. The proof follows directly from Theorem 6 by setting $g \equiv 0$. Indeed, the space of null-homotopic J -tori consists of all constant mappings only. The corresponding linearised operator $\tilde{\mathcal{D}}(u)$ is the Cauchy-Riemann operator on vector-functions $\Sigma \rightarrow T_u M \simeq \mathbb{C}^n$. Due to the Liouville principle $\text{Ker } \tilde{\mathcal{D}}(u)$ consists of constant vector-functions only and, hence, the space of null-homotopic solutions $\mathfrak{S} \simeq M$ is non-degenerate in the sense of Morse-Bott. Moreover, the operator $\tilde{\mathcal{D}}(u)$ is formally normal and, due to Example 5, we see that \mathfrak{S} is strongly non-degenerate. The other hypotheses of the theorem in this case are obvious. The statement about the non-trivial homotopy class $[v]$ simply follows from the definition of the degree, since the suppositions of the theorem imply that $[v]$ does not contain J -tori, i.e. $\pi^{-1}(0) = \emptyset$. \square

Note that, since the compactness theorem holds for homology classes of mappings, Theorem 6 also has a version concerned with the algebraic number of perturbed J -tori within homology classes. In particular, Corollary 7 implies that for a regular section f in equation (2.1) such that $\int \max_u \|f(\cdot, u)\|^2 < S_\omega(J)$ the algebraic number of null-homologous perturbed J -tori is also equal to $\chi(M)$. The condition in Theorem 6 that the map given by (2.5) is an embedding can be, in fact, relaxed. In [8, Appendix 4.B] it is shown how to deal with the case when the latter map is only an immersion.

3. The proofs

Let $\phi : \Sigma \rightarrow \text{Diff}(M)$ be a fixed map from a Riemannian surface Σ such that the map $(\text{ev}_u \circ \phi)(z) = \phi_z(u)$ is smooth with respect to $z \in \Sigma$ for any $u \in M$. Define a section g of the bundle $\Omega^{0,1}$ by the following formula:

$$g(z, u) = \bar{\partial}(\phi_z(\bar{u}))|_{\bar{u}=\phi_z^{-1}(u)} \in \Omega_{(z,u)}^{0,1}, \quad z \in \Sigma, \quad u \in M. \quad (3.1)$$

Clearly, for any $u \in M$ the map $\text{ev}_u \circ \phi$ is a solution of the equation

$$\bar{\partial}u(z) = g(z, u(z)), \quad z \in \Sigma. \quad (3.2)$$

Thus, within the homotopy class $\text{ev}_u^{\sharp}[\phi]$ we have the family of solutions $\{\text{ev}_u \circ \phi\}$ parameterised by $u \in M$. Our observation is that the Morse-Bott theory applies to equation (3.2). To implement this we need the following lemmas.

Lemma 8. *For any map $\phi : \Sigma \rightarrow \text{Diff}(M)$ such that $\text{ev}_u \circ \phi$ is smooth for any $u \in M$ the following inequality holds:*

$$\int_{\Sigma} \max_u \|\bar{\partial}(\text{ev}_u \circ \phi)\|^2 d\text{Vol}_{\Sigma} \leq \mathcal{E}(\phi, \omega, J) - \langle (\text{ev}_u \circ \phi)^*[\omega], \Sigma \rangle. \quad (3.3)$$

Proof. Fix a Riemannian metric g_{Σ} within the given conformal class on Σ . Denote by Λ_u the function defined by the relation $(\text{ev}_u \circ \phi)^*\omega = \Lambda_u d\text{Vol}_{\Sigma}$. Then direct calculations yield the following identity:

$$\|\bar{\partial}(\text{ev}_u \circ \phi)\|^2 + \Lambda_u = \frac{1}{2} \|d(\text{ev}_u \circ \phi)\|^2.$$

This implies the inequality

$$\max_u \|\bar{\partial}(\text{ev}_u \circ \phi)\|^2 + \min_u \Lambda_u \leq \frac{1}{2} \max_u \|d(\text{ev}_u \circ \phi)\|^2.$$

Integrating the latter over Σ with respect to the volume form $dVol_\Sigma$ and using the definition of the functional \mathcal{E} we arrive at the following inequality

$$\int_\Sigma \max_u \|\bar{\partial}(\text{ev}_u \circ \phi)\|^2 dVol_\Sigma \leq \mathcal{E}(\phi, \omega, J) - \int_\Sigma \max_u \Lambda_u dVol_\Sigma.$$

This immediately implies the claim since the last term in the right-hand side is not greater than $(-\langle (\text{ev}_u \circ \phi)^*[\omega], \Sigma \rangle)$. \square

The following lemma is the only place where the hypothesis that ϕ takes values in $\text{Symp}(M)$ is used.

Lemma 9. *Suppose that the map ϕ takes values in $\text{Symp}(M)$. Then the maps $\text{ev}_u \circ \phi$, where $u \in M$, are the only solutions of equation (3.2) within their homology class, i.e. the space of mappings w such that $w_*[\Sigma] = (\text{ev}_u \circ \phi)_*[\Sigma]$. In particular, there are no other solutions within the homotopy class $\text{ev}_u^*[\phi]$.*

Proof. Let $w(z)$ be a solution of equation (3.2) which is homologous to the map $(\text{ev}_u \circ \phi)(z) = \phi_z(u)$, where $z \in \Sigma$, and $u \in M$. Due to the definition of the section \mathfrak{g} , see formula (3.1), this means that

$$\bar{\partial}w(z) = \bar{\partial}(\phi_z(\bar{u}))\big|_{\bar{u}=\phi_z^{-1} \circ w(z)}, \quad z \in \Sigma. \quad (3.4)$$

Represent the map $w(z)$ as the composition $(\phi_z \circ v)(z)$, where the map $v : \Sigma \rightarrow M$ is defined as $(\phi_z^{-1} \circ w)(z)$. In particular, the latter map v is null-homologous, i.e. $v_*[\Sigma] = 0$. We need a formula for the value of the Cauchy-Riemann operator $\bar{\partial}$ on the composition $(\phi_z \circ v)(z)$. First, we have

$$d(\phi_z \circ v)(z) = d_z \phi(\bar{u})\big|_{\bar{u}=v(z)} + (d\phi_z \circ dv)(z), \quad z \in \Sigma;$$

here the left-hand side stands for the differential of the map $z \mapsto (\phi_z \circ v)(z)$ and by $d_z \phi$ we mean the differential of the map $\phi_z(u)$ with respect to $z \in \Sigma$. Taking the J -anti-linear parts of these differentials we arrive at the following identity

$$\bar{\partial}(\phi_z \circ v)(z) = \bar{\partial}(\phi_z(\bar{u}))\big|_{\bar{u}=v(z)} + d\phi_z \circ \bar{\partial}v(z), \quad z \in \Sigma. \quad (3.5)$$

The symbol $\bar{\partial}v$ above denotes the Cauchy-Riemann operator for the z -dependent almost complex structure

$$\tilde{J}(z, u) = d\phi_z^{-1}(u) \circ J(\phi_z(u)) \circ d\phi_z(u), \quad z \in \Sigma, \quad u \in M.$$

Combining identities (3.4) and (3.5) and the fact that ϕ_z is a diffeomorphism for any $z \in \Sigma$, we see that the map v has to satisfy the equation $\bar{\partial}v = 0$.

For a proof of the lemma we have to show that this map v is constant; the latter would imply that $w(z)$ is equal to $(\text{ev}_u \circ \phi)(z)$ for some $u \in M$. The assertion about v follows from an energy-type identity for solutions of equation $\bar{\partial}v = 0$. First, it is straightforward to see that for any $z \in \Sigma$ the symplectic structure $\omega = \phi_z^* \omega$ tames \tilde{J}_z and is \tilde{J}_z -invariant. Thus, the bilinear form $\omega(\cdot, \tilde{J}_z \cdot)$ defines a scalar product on the tangent space $T_u M$ for any $z \in \Sigma$ and, hence, a scalar product $\langle \cdot, \cdot \rangle$ on the space of linear operators $T_z \Sigma \rightarrow T_u M$. We claim that for any solution v of the equation $\bar{\partial}v = 0$ the following energy identity holds

$$\frac{1}{2} \int_\Sigma \langle dv, dv \rangle dVol_\Sigma = \langle v^*[\omega], \Sigma \rangle.$$

This relation immediately implies that a null-homologous solution has to be constant. Its proof follows the same argument as a proof of the standart energy identity; see [11, Chapter 2]. \square

Lemma 10. *The family of solutions $\{\text{ev}_u \circ \phi\}$, where $u \in M$, of equation (3.2) is non-degenerate in the sense of Morse-Bott. Moreover, if the genus of Σ is equal to one, then this family of solutions is strongly non-degenerate.*

Proof. First, we show that the space of solutions $\{\text{ev}_u \circ \phi\}$, where $u \in M$, is non-degenerate in the sense of Morse-Bott. For this we have to prove that a vector field \mathbf{v} from the kernel of the operator $\bar{\mathcal{D}}(\text{ev}_u \circ \phi)$ has the form $d\phi_z \cdot \mathbf{v}_0$, where \mathbf{v}_0 is a vector from $T_u M$. (Clearly, any vector field of this form belongs to the kernel of $\bar{\mathcal{D}}(\text{ev}_u \circ \phi)$.) Recall that the operator $\bar{\mathcal{D}}(\text{ev}_u \circ \phi)$ can be defined by the following relation

$$\bar{\mathcal{D}}(\text{ev}_u \circ \phi)\mathbf{v} = \left. \frac{\partial}{\partial t} \right|_{t=0} [\bar{\partial} w_t - \mathbf{g}(\cdot, w_t(\cdot))],$$

where w_t is a family of mappings $\Sigma \rightarrow M$ such that

$$w_t|_{t=0} = \text{ev}_u \circ \phi \quad \text{and} \quad (\partial/\partial t)|_{t=0} w_t = \mathbf{v}.$$

As in the proof of Lemma 9 we represent $w_t(z)$ as the composition $(\phi_z \circ v_t)(z)$, where the family of contractible mappings $v_t(z)$ is defined as $(\phi_z^{-1} \circ w_t)(z)$. In particular, the map $v_0(z) \equiv u$ is constant. Using identity (3.5) we obtain the following relations

$$\begin{aligned} \bar{\mathcal{D}}(\text{ev}_u \circ \phi)\mathbf{v}(z) &= \left. \frac{\partial}{\partial t} \right|_{t=0} [d\phi_z \circ \bar{\partial} v_t(z)] \\ &= d\phi_z \circ [\bar{\partial} v_0]_* \left. \frac{\partial}{\partial t} \right|_{t=0} v_t = (d\phi_z \circ [\bar{\partial} v_0]_* \circ d\phi_z^{-1}) \mathbf{v}(z), \quad z \in \Sigma. \end{aligned} \quad (3.6)$$

This implies that a vector field $\mathbf{v}(z)$ belongs to the kernel of the operator $\bar{\mathcal{D}}(\text{ev}_u \circ \phi)$ if and only if the composition $d\phi_z^{-1} \cdot \mathbf{v}(z)$, where $z \in \Sigma$, belongs to the kernel of $[\bar{\partial} v_0]_*$. The latter operator acts in accordance with the formula

$$[\bar{\partial} v_0]_* \tilde{\mathbf{v}} = \frac{1}{2} \left(\nabla \tilde{\mathbf{v}} + \tilde{J}(\cdot, v_0) \circ \nabla \tilde{\mathbf{v}} \circ i_\Sigma \right)$$

on sections of the trivial bundle $\mathbb{R}^{2n} \times \Sigma$ endowed with the almost complex structure $\tilde{J}(z, v_0)$ in the fibre over $z \in \Sigma$. In particular, up to an isomorphism (for example given by $d\phi_z$) the operator $[\bar{\partial} v_0]_*$ can be regarded as the usual Cauchy-Riemann operator on vector-functions $\Sigma \rightarrow \mathbb{C}^n$. Due to the Liouville principle, any vector-function from its kernel has to be constant. Thus, the vector field $d\phi_z^{-1} \cdot \mathbf{v}(z)$ is constant and we obtain that any \mathbf{v} from the kernel of $\bar{\mathcal{D}}(\text{ev}_u \circ \phi)$ has the form $d\phi_z \cdot \mathbf{v}_0$ for some vector $\mathbf{v}_0 \in T_u M$. This demonstrates that the space of solutions $\{\text{ev}_u \circ \phi\}$ is non-degenerate in the sense of Morse-Bott.

Now suppose that Σ is a torus. Then the Cauchy-Riemann operator $[\bar{\partial} v_0]_*$ is formally normal and, in particular, does not have adjoint vectors corresponding to the zero eigenvalue; see Example 5. Due to relation (3.6) so does the operator $\bar{\mathcal{D}}(\text{ev}_u \circ \phi)$. This ends the proof of the lemma. \square

Proof of Theorem 4. First, note that the quantities $\mathcal{E}(\phi, \omega, J)$ and $S_\omega(J)$ are invariant under the simultaneous changes $\omega \mapsto -\omega$ and $J \mapsto -J$, where the map ϕ is arbitrary and an almost complex structure J belongs to \mathcal{J}_ω . Besides, the groups of diffeomorphisms preserving the forms ω and $(-\omega)$ coincide. Thus, we can suppose that for a given homotopy class $[\phi]$ the symplectic structure on M is such that

$$\langle (\text{ev}_u \circ \phi)^* \omega, \Sigma \rangle \leq 0. \quad (3.7)$$

Under the conditions of the theorem there exist a complex structure i_Σ on Σ , an almost complex structure $J \in \mathcal{J}_\omega$, and a map $\phi : \Sigma \rightarrow \text{Symp}(M)$, representing a given homotopy class, such that

$$\mathcal{E}(\phi, \omega, J) < S_\omega(J). \quad (3.8)$$

Define a section \mathfrak{g} of the bundle $\Omega^{0,1}$ according to formula (3.1). Clearly, we have the identity

$$\max_u \|\mathfrak{g}(\cdot, u)\|^2 = \max_u \|\bar{\partial}(\text{ev}_u \circ \phi)\|^2.$$

Combining this with Lemma 8 and inequality (3.8) we see that

$$\int_\Sigma \max_u \|\mathfrak{g}(\cdot, u)\|^2 dVol_\Sigma < S_\omega(J) - \langle (\text{ev}_u \circ \phi)^* \omega, \Sigma \rangle.$$

Thus, the section \mathfrak{g} belongs to the domain \mathcal{U}_ℓ (from the Compactness theorem) given by relation (2.2) with the constant $V_\mathcal{E}$ equalled to $\langle (\text{ev}_u \circ \phi)^* \omega, \Sigma \rangle$. Note that the first Chern class $[c_1](M)$ also vanishes on the image of $\text{ev}_u \circ \phi$. Indeed, the vector bundle $(\text{ev}_u \circ \phi)^* TM$ is trivial and, hence, all its characteristic classes vanish; it is isomorphic to $T_u M \times \Sigma$ under the morphism which equals $d\phi_z^{-1}(u)$ on the fiber over $z \in \Sigma$. Hence, due to the Riemann-Roch theorem, the index of the linearised operator $\bar{\mathcal{D}}(\text{ev}_u \circ \phi)$ is equal to $n\chi(\Sigma)$.

Case (i). Suppose Σ is a sphere. Then the index of $\bar{\mathcal{D}}(\text{ev}_u \circ \phi)$ is equal to $2n$, the dimension of M . On the other hand, due to Lemmas 9 and 10, the space of solutions $\pi^{-1}(\mathfrak{g})$ is formed by the mappings $\{\text{ev}_u \circ \phi\}$, where $u \in M$, and is non-degenerate in the sense of Morse-Bott. In particular, we see that $\pi^{-1}(\mathfrak{g})$ is diffeomorphic to M and the dimension of the kernel of $\bar{\mathcal{D}}(\text{ev}_u \circ \phi)$ is equal to $2n$, the dimension of M . Thus, the index of the operator $\bar{\mathcal{D}}(\text{ev}_u \circ \phi)$ is equal to its kernel and, hence, the section \mathfrak{g} is regular. Suppose the homotopy class $\text{ev}_u^{\natural}[\phi]$ is not trivial. Then the energy identity (1.1) and the hypothesis (3.7) imply that this homotopy class does not contain J -spheres. Hence, the space of solutions $\pi^{-1}(0)$ is empty and the zero section of $\Omega^{0,1}$ is also regular as a right-hand side of equation (2.1). Now Theorem 5 applies and we see that there is a deformation $\mathfrak{g}_t \in \mathcal{U}_\ell$ of the section \mathfrak{g} to zero such that the preimage $N = \cup_t \pi^{-1}(\mathfrak{g}_t)$ is a compact oriented manifold with boundary $\pi^{-1}(\mathfrak{g}) \simeq M$. Choose a reference point $z_* \in \Sigma$ and consider the map $N \rightarrow M$ given by formula (2.3). Its restriction to the boundary $\partial N \simeq M$ acts by the rule

$$M \ni u \mapsto \phi_{z_*}(u) \in M$$

and, in particular, is a diffeomorphism. Thus, we have a continuous map $N \rightarrow \partial N$ whose restriction to the boundary induces an isomorphism on the top homology. Since the fundamental class of a closed oriented manifold is non-trivial, the latter is impossible and, hence, the homotopy class $\text{ev}_u^{\natural}[\phi]$ has to be trivial.

Case (ii). Suppose Σ is a torus. Then the index of $\bar{\mathcal{D}}(ev_u \circ \phi)$ vanishes and, due to the discussion in the preceding section, the invariant $\deg \pi$ (the algebraic number of solutions for a regular section $f \in \mathcal{U}_\ell$) is well-defined. Due to Lemmas 9 and 10 the Morse-Bott theory applies to equation (3.2): the space of solutions $\pi^{-1}(g) \simeq M$ is strongly non-degenerate in the sense of Morse-Bott and Theorem 6 implies that the degree $\deg \pi$ is equal to $\pm \chi(M)$. Suppose that the homotopy class $ev_u^1[\phi]$ is not trivial. Then the energy identity (1.1) and the hypothesis (3.7) imply that this homotopy class does not contain J -tori. Hence, the space of solutions $\pi^{-1}(0)$ is empty and the zero section of $\Omega^{0,1}$ is regular. This implies that the degree $\deg \pi$ vanishes. Thus, we obtain that the Euler-Poincaré number $\chi(M)$ is equal to zero and the theorem is demonstrated. \square

The proof of Theorem 4' follows along similar lines, since the compactness theorem and Theorems 5 and 6 have analogues concerning the moduli space of solutions within homology classes of mappings.

We end the paper with explaining the proof of Lemma 3.

Proof of Lemma 3. First, without loss of generality we can consider only maps that take values in the connected component of the identity $Symp_0(M)$. Second, a given continuous map $\phi : \Sigma \rightarrow Symp_0(M)$ can be C^1 -approximated by a map $\hat{\phi}$ with values in $Diff_0(M)$ such that $ev_u \circ \hat{\phi}$ is smooth for any $u \in M$. Indeed, one can regard any map of a surface Σ into the diffeomorphism group as a map from the product $\Sigma \times M$ to M and approximate it by a smooth map with respect to the first variable. Finally, to obtain an approximation with values in the symplectomorphism group we apply to $\hat{\phi}$ the canonical retraction \mathcal{R} of a C^1 -neighbourhood of $Symp_0(M)$ in the diffeomorphism group to $Symp_0(M)$. We describe an explicit construction for the latter in terms of the so-called Moser isotopy now.

Let $\hat{\phi}$ be a diffeomorphism from $Diff_0(M)$ which is C^1 -close to $Symp(M)$ such that all forms

$$\omega_t = \omega + t(\hat{\phi}^* \omega - \omega), \quad t \in [0, 1],$$

are non-degenerate. Since $\hat{\phi}$ is homotopic to the identity, the forms ω_t 's are co-homologous. By Moser's isotopy theorem [10, Section 3.2] there exists a canonical family of diffeomorphisms such that $\psi_t^* \omega_t = \omega_0$ and $\psi_0 = \text{Id}$. Clearly, the diffeomorphism $\psi_1 \circ \hat{\phi}$ preserves the symplectic form ω and we define the retraction \mathcal{R} by the rule $\hat{\phi} \mapsto \psi_1 \circ \hat{\phi}$.

Now suppose that a diffeomorphism $\hat{\phi}$ depends smoothly on a parameter $z \in \Sigma$ (in the sense that the map $\Sigma \times M \rightarrow M$ is smooth), then so do the forms $\hat{\phi}^* \omega_t$'s. By the construction of the Moser isotopy, the diffeomorphisms ψ_t are defined as solutions of certain differential equations and also depend smoothly on the parameter z . Hence, so does the diffeomorphism ψ_1 as well as the composition $\psi_1 \circ \hat{\phi}$. \square

Added in proof. As was pointed out to us by Dieter Kotschick, since the fundamental group of $Symp(M)$ is Abelian, any 2-cycle in $Symp(M)$ is generated by tori and spheres. As a consequence Theorem 1 implies that under the absence of rational curves (for at least one compatible almost complex structure) the evaluation map ev_u is homotopically trivial on the 2-skeleton of $Symp(M)$ provided $\chi(M) \neq 0$.

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