Neostability theory
Homework 2 - Due November 28, 2016

November 14, 2016

Observations:

(a) The following is a list of problems includling some standard examples that we will discuss during the course. If you have any questions about the problems, do not hesitate to write me an e-mail.

(b) If you plan to count the hours of the course towards the official assessed taught course requirements for graduate students in mathematics, please hand out the five problems.

Exercise 1. 1. Determine if the theory of the random graph eliminates imaginaries.

2. Show that the theory $T = RCF := Th(\mathbb{R}, +, \cdot, 0, 1, <)$ of real closed fields has elimination of imaginaries.

Exercise 2. Let L be the language having unary predicates $\langle P_s : s \in 2^{<\omega} \rangle$ (where $2^{<\omega}$ denotes the set of all finite sequences of 0's and 1's). Consider the theory T axiomatized by the following sentences:

- $\forall x\, P_\Lambda(x)$ (where Λ denotes the empty string)
- $\{\exists x\, P_s(x) : s \in 2^{<\omega}\}$
- $\{\forall x((P_{s^\frown 0}(x) \vee P_{s^\frown 1}(x)) \leftrightarrow P_s(x)) : s \in 2^{<\omega}\}$
- $\{\forall x\, \neg(P_{s^\frown}0(x) \wedge P_{s^\frown 1}(x)) : s \in 2^{<\omega}\}$

Essentially, this theory says that the predicates $\langle P_s : s \in 2^{<\omega} \rangle$ form a binary decomposition of the universe.

1. Show that T is complete and has quantifier elimination.

2. Prove that T is stable.

> **Definition A.** *Let T be a complete theory and λ an infinite cardinal. We say that T is* λ-stable *if for every $n < \omega$ and every $A \subseteq \mathbb{M}$ of cardinality $|A| \leq \lambda$, $|S_n(A)| \leq \lambda$.*

Exercise 3. Let T be a complete theory. Show that the following are equivalent:

(a) T is stable.

(b) T is λ-stable for all λ such that $\lambda^{|T|} = \lambda$.

(c) T is λ-stable for some λ.

Exercise 4. Consider the language $L = \{E_1, E_2, \ldots\}$ of countably many binary relation symbols, and consider the theory T establishing that each E_i is an equivalence relation with infinite classes, and E_{i+1} refines E_i. Namely, T includes the axioms

- $\{\forall x, y, z(E_i(x,x) \wedge E_i(x,y) \leftrightarrow E_i(y,x) \wedge ((E_i(x,y) \wedge E_i(y,z)) \rightarrow E_i(x,z)) : 1 \leq i < \omega\}$.
- $\{\forall x, y(E_{i+1}(x,y) \rightarrow E_i(x,y)) : 1 \leq i < \omega\}$.

Let $T^2 \supseteq T$ be the theory that states that E_1 has two classes, and each E_i-class is the union of two E_{i+1}-classes, and let $T^\infty \supseteq T$ be the theory establishing that E_1 has infinitely many classes, and each E_i-class is the union of infinitely many E_{i+1}-classes.

1. Show that T^2 is λ-stable, for all $\lambda \geq 2^{\aleph_0}$.
2. Show that T^∞ is λ-stable if and only if $\lambda^{\aleph_0} = \lambda$.

Definition B. *We say that a formula $\phi(\overline{x}, \overline{y})$ has the* independence property *(in T) if for every $k < \omega$ there are sequences $\langle \overline{a}_i : 1 \leq i \leq k \rangle$ and $\langle \overline{b}_W : W \subseteq \{1, \ldots, k\} \rangle$ such that $\models \phi(\overline{a}_i; \overline{b}_W)$ if and only if $i \in W$.*

Exercise 5.

1. Show that if $\phi(\overline{x}, \overline{y})$ has the independence property then $\phi(\overline{x}, \overline{y})$ is unstable.
2. Show that the formula $x < y$ does not have the independence property in $T = DLO$.

Let $(G, \cdot, e, ^{-1})$ be a group, and let $T = Th(G, \cdot, e, ^{-1})$. Suppose $\phi(x, \overline{y})$ is a formula such that for each $b \in \mathbb{M}^{|\overline{y}|}$, the set $H_{\overline{b}} := \phi(\mathbb{M}, \overline{b})$ defines a subgroup of G. We call these sets *ϕ-definable groups.*

3. Suppose that $\phi(x, \overline{y})$ does not have the independence property. Then, there is a number n_ϕ such that any finite intersection of ϕ-definable **groups** is equal to the intersection of at most n_ϕ-members of the intersection.
4. Suppose T is stable. Then G has the *intersection chain condition* for ϕ-definable groups: There is not an infinite chain of groups $G_1 \gneq G_2 \gneq \cdots$ each of which is definable as an intersection of ϕ-definable groups.