Neostability theory
Homework 1 - Due October 14, 2016

October 1, 2016

Observations:

(a) The following is a list of problems inclulding some standard examples that we will discuss during the course. If you have any questions about the problems, do not hesitate to write me an e-mail.

(b) If you plan to count the hours of the course towards the official assessed taught course requirements for graduate students in mathematics, please hand out (4) problems, selecting one problem in each of the following groups: 1-3 ; 4-7; 8-10; 11-13.

(c) To show quantifier elimination, you can look for one of the different criteria in Hodges' book or Marker's book, reference it (i.e., writting it down) and apply it to the problem you are working on.

Exercise 1. Let T be the common theory of all ordinals in the language $L = \{<\}$ (i.e., an L-sentence σ is in T if and only if $(\alpha, <) \models \sigma$ for every ordinal α).

1. For a given set $A \subseteq M$, describe the definable closure and the algebraic closure of A.
2. Show that there is a structure $(M, <) \models T$ which is not a well-order. (*Hint:* consider an expansion of the language by constants $\{c_n : n < \omega\}$ and a expansion of the theory $T' = \{c_n > c_{n+1} : n < \omega\}$. Is T' finitely consistent?)

Exercise 2.

1. Let $L = \{<\}$ and let M be an infinite linear order. Show that there is an elementary extension N of M for which there is an order-preserving embedding $\sigma : \mathbb{Q} \to N$ of the rational numbers into N.

Recall that a *universal sentence* is a sentence of the form $\forall x_1 \forall x_2 \cdots \forall x_n(\phi(x_1, \ldots, x_n))$ where $\phi(x_1, \ldots, x_n)$ is a quantifier free formula, and the *universal part* of a complete theory is simply the collection of all universal sentences in the theory.

2. Show that all the infinite linear orders have the same universal theory. (i.e., given M, N infinite linear orders, the universal parts of $\mathrm{Th}(M)$ and $\mathrm{Th}(N)$ are equal.)

Exercise 3.

1. Let T be the theory of abelian groups. Show that there is no collection of L_{groups}-sentences Γ such that $G \models T \cup \Gamma$ if and only if G is cyclic.

2. Let T be the theory of graphs. Show that there is no collection of L_{graphs}-sentences Γ such that $G \models T \cup \Gamma$ if and only if G is a connected graph.

Exercise 4. Let T be a complete extension of Peano arithmetic. Show that $|S_1(T)| = 2^{\aleph_0}$, where $S_1(T)$ is the collection of 1-types in T with parameters over the empty set.

Exercise 5. Suppose that M is an L-substructure of N.

1. Show that if M is isomorphic to N then $M \equiv N$.
2. Show that if $M \prec N$ then $M \equiv N$.
3. Consider the groups $M_1 = (2\mathbb{Z}, +), M_2 = (\mathbb{Z}, +)$. Show that $M_1 \subseteq M_2$ and $M_1 \cong M_2$, but $M_1 \not\prec M_2$.
4. Consider the groups $M_2 = (\mathbb{Z}, +)$ and $M_3 = (\mathbb{Z} \oplus \mathbb{Z}, +)$. Show that $M_2 \not\equiv M_3$.

Exercise 6. Let $L = \{E\}$ and let T be the L-theory asserting that E is an equivalence relation with infinitely many infinite classes.

1. Show that T has quantifier elimination.
2. Given a model $M \models T$, give a classification of the types $S_1(M)$. (*Hint:* given a single element $m \in M$, think about the cases where $x = m \in p$, or $xEm \in p$ or neither $x = m$ or xEm belongs to p)

Exercise 7.

1. Let T be the theory of the structure $(\mathbb{Z}, S)$ where $s(x) = x + 1$. Determine the types in $S_n(T)$. Which of these types contain algebraic formulas?
2. Do the same for the structure $(\mathbb{Z}, <, S)$.

Exercise 8.

1. Let $M = (X, <)$ be a dense linear order without end points, $A \subseteq M$ and $\bar{b}, \bar{c} \in M^n$, with $b_1 < \cdots < b_n$ and $c_1 < \ldots < c_n$. Show that $\mathrm{tp}(\bar{b}/A) = \mathrm{tp}(\bar{c}/A)$ if and only if for every $i = 1, \ldots, n$ and $a \in A$, we have $b_i > a \Leftrightarrow c_i > a$ and $c_i = a \Leftrightarrow b_i = a$.
2. Let $M = (\mathbb{Q}, <)$ and take $A = \left\{1 - \frac{1}{n} : n = 1, 2, \ldots\right\} \cup \left\{2 + \frac{1}{n} : n = 1, 2, \ldots\right\}$. Show that $b = 1$ and $c = 2$ have the same type over A, but there is no isomorphism of $\mathbb{Q}$ fixing A pointwise and sending 1 to 2

Exercise 9. Consider the theory T_{dLO} of *discrete* linear orders in the language $L = \{<\}$, and let $L' = \{<, S, S^{-1}\}$ where S, S^{-1} are unary function symbols, interpreted in a model of T as follows:

$$M \models y = S(x) \Leftrightarrow M \models x < y \wedge \forall z(x < z \rightarrow y \leq z).$$
$$M \models y = S^{-1}(x) \Leftrightarrow M \models x > y \wedge \forall z(x > z \rightarrow y \geq z).$$

1. Prove that T_{dLO} have quantifier elimination in the language L'. If you wish, you can look for a quantifier elimination criterion in Hodges' book or Marker's book, reference it and apply it to this problem.

2. Let L be any linear order. Show that $M = (L \times \mathbb{Z}, <_{lex})$ is a model of T_{dLO}, where $<_{lex}$ is the lexicographical order.

3. Is the structure $(\mathbb{R} \times \mathbb{Z}, <_{lex})$ an $\aleph_0$-saturated model of T_{dLO}?

Exercise 10. Let $\mathbb{F}$ be a fixed countable field, and consider the theory $T_{\mathbb{F}-vs}$ of $\mathbb{F}$-vector spaces in the language $L = \{+, 0, (f_\alpha(x) : \alpha \in \mathbb{F})\}$.

1. Show that the theory $T_{\mathbb{F}-vs}$ has quantifier elimination.

2. Show that $\mathbb{F}^n$ is not $\aleph_0$-saturated for any $n < \omega$.

3. Show that a model of $T_{\mathbb{F}-vs}$ is saturated if and only if it has infinite dimension.

4. Show that every infinite dimensional model of $T_{\mathbb{F}-vs}$ of cardinality κ is strongly-κ-homogeneous.

Exercise 11. Consider the structure $M = (\mathbb{C}, +, \times, 0, 1)$. In this exercise feel free to use the quantifier elimination result for algebraically closed fields.

1. For a given $A \subseteq M$, show that $\operatorname{acl}(A) = \overline{A}^{alg}$, the algebraic closure of A with respect to the field $\mathbb{C}$.

2. For a given $A \subseteq M$, describe the definable closure of A, $dcl(A)$.

3. Show that if $b, b' \notin \overline{A}^{alg}$, then $tp(b/A) = tp(b'/A)$.

4. Show that if $b, b' \in \overline{A}^{alg}$ and $tp(b/A) = tp(b'/A)$, then b, b' have the same minimal polynomial over $\mathbb{Q}(A)$.

5. Prove that the set $X = \mathbb{R}$ is not definable in the structure $(\mathbb{C}, +, \times, 0, 1)$. (even with parameters!).

Exercise 12. Consider the language of graphs $L = \{R\}$, and the theory T_d of *graphs with fix valency d*, axiomatized by the following sentences:

- $\forall x(\neg(xRx)) \wedge \forall x, y(xRy \rightarrow yRx)$ (i.e., (M, R) is a graph).
- $\forall x\ \exists y_1, \ldots, y_d \left(\bigwedge_{1 \leq i < j \leq d} y_i \neq y_j \wedge \bigwedge_{1 \leq i \leq d} xRy_i \wedge \forall z \left(zRx \rightarrow \bigvee_{1 \leq i \leq d} z = y_i \right) \right)$
  (that is, every vertex has exactlly d neighbhours).
- $\left\{ \forall x_1, \ldots, x_n \left(\neg \left(\bigwedge_{1 \leq i \leq n-1} x_i R x_{i+1} \wedge x_n R x_1 \right) \right) : n < \omega \right\}$ (i.e., there are no cycles)

1. Show that this theory is $\aleph_1$-categorical (thus complete), but it is not ω-categorical.

2. Let $M \models T_d$, and describe $\operatorname{acl}(a)$ for $a \in M$.

3. Given $A \subseteq M$, show that $\operatorname{acl}(A) = \bigcup_{a \in A} \operatorname{acl}(a)$

4. Prove that every uncountable model of T_d is saturated and strongly homogeneous.

Exercise 13.

1. Suppose $b \in \text{acl}(A)$. Show that there are $b_1, \ldots, b_m$ such that if σ is an automorphism of M with $\sigma(a) = a$ for all $a \in A$, then $\sigma(b) = b_i$ for some i. In other words, there are only finitely many conjugates of x under automorphisms of M fixing A pointwise.

2. Show that for any subsets $A, B \subseteq M$, $\text{acl}(\text{acl}(A)) = \text{acl}(A)$ and if $A \subseteq B$, $\text{acl}(A) \subseteq \text{acl}(B)$.

3. Show that $b \in \text{acl}(A)$ if and only if $b \in \text{acl}(A_0)$ for some finite $A_0 \subseteq A$.