Model theory of pseudofinite structures

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Introduction

First order model theory on infinite structures is a well–developed field of abstract mathematics with powerful methods and results an whose subject is to study the interaction between mathematical structures such as groups, fields, graphs, and the formal languages used to describe them.

Throughout the history of model theory the subject have been fueled and redefined by its applications, especially in other areas of mathematics. Early examples of these applications are Tarski's elimination of quantifiers for real closed fields and Ax-Kochen/Ershov mode theory of henselian fields. More recently, connections with algebraic and diophantine geometry have been found and used to prove the Mordell-Lang conjecture for function fields (Hrushovski) and to advance towards the proof of the André-Oort conjecture for $\mathbb{C}^n$ (due to Pila) by using tools from o-minimality to find estimates of rational points in definable sets.

Most of the applications of model theory to other areas in mathematics come in two stages: first by identify-ing abstract (often combinatorial) properties of first-order theories that make them more tractable or "tame" (such as stability, simplicity, NIP, and more recently rosiness and NTP_2), and second when we realize that theories of mathematically meaningful structures satisfy those properties. The leading idea behind the most recent applications from model theory to other areas has been the slogan proposed by Hrushovski: "model theory is the geography of tame mathematics" (see [12], page 38), where model-theorists use informally the terms "tame" or "wild" to distinguish between having desirable or undesirable model-theoretic behavior.

In contrast, Finite Model Theory - the specialization of model theory to the study finite structures - has very different methods, and usually refers to a field of mathematics which has more to do with computer science than to classical mathematical structures.

The fundamental theorem of ultraproducts is due to Jerzy Łoś, and provides a transference principle between the finite structures and their limits. Roughly speaking, Łoś' Theorem states that a formula is true in the ultraproduct M of the structures $\langle M_n : n \in \mathbb{N}\rangle$ if and only if it is true for "almost every" M_n.

When applied to ultraproducts of finite structures, Łoś' theorem presents an interesting duality between the finite structures and the infinite structures. We start with a family of finite structures and produce infinite first-order structure with the same properties. This kind of finite/infinite connection can sometimes be used to prove qualitative properties of large finite structures using the powerful known methods and results coming from infinite model theory, and in the other direction, quantitative properties in the finite structures often induced desirable qualitative properties in their ultraproducts.

The idea is that the counting measure on a class of finite structures can be lifted using Łoś' theorem to give notions of dimension and measure on their ultraproduct. This allows ideas from geometric model theory to be used in the context of pseudofinite theories, and potentially we can prove results in finite combinatorics (of graphs, groups, fields, etc) by studying the corresponding properties in the ultraproducts. This approach was used by Hrushovski in [14], but was better explored in his striking papers [15] and [16], where he applies ideas from geometric model theory to additive combinatorics, locally compact groups and linear approximate subgroups.

Goldbring and Towsner developed in [10] the Approximate Measure Logic, a logical framework that serves as a formalization of connections between finitary combinatorics and diagonalization arguments in measure theory or ergodic theory that have appeared in various places throughout the literature. Using AML- structures, Goldbring and Towsner gave proofs of the Furstenberg's correspondence principle, Szemerédi's Regularity Lemma, the triangle removal lemma, and Szemerédi's Theorem: every subset of the integers with positive density contains arbitrarily long arithmetic progressions.

There is a variety of possible questions about what is the relationship between the different concepts in model theory (stability, NIP, simplicity, geometries coming from independence relations, etc) once the assumption of pseudofiniteness is added, and how these classical model-theoretic properties on the ultraproducts of a class of finite structures reflect on quantitative properties for the definable sets along the class.

Outline

The purpose of this series of lectures is to present the main topics about the model theory of pseudofinite structures, together with some recent developments and applications to combinatorics.

On the model-theoretic perspective, we will review the so-called "pseudofinite dimension" (cf. [16]) and its relationship with the forking and model-theoretic dividing lines in pseudofinite structures as presented in [8]. We will also study folklore results appearing in [24] where some counting arguments on pseudofinite structures and the existence of *Zilber polynomials* are shown to imply that every strongly minimal pseudofinite structure is locally modular.

On the applications to combinatorics we will see a proof of Szemerédi's Regularity Lemma using ultraproducts of finite structures, (cf. [10]) as well as some improvements of this result under the assumption of stability due to Malliaris and Shelah. This last approach was used by Chernikov and Starchenko in [6] to obtain some progress in the study of the Erdös-Hajnal conjecture.

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Lecture 1

Basic concepts, examples and results.

In this chapter we will study the main definitions and examples in the study of L-structures M which are *pseudofinite*, that is, L-structures which satisfy every first order sentence which hold for all finite L-structures. As we will see in Proposition 1.2.3, being pseudofinite is equivalent to being elementary equivalent to an ultraproduct of finite structures, and also equivalent to having the *finite model property*: every first order L-sentence σ true in M has a finite model.

We start by recalling some definitions and basic results regarding ultraproducts of finite structures.

1.1 Ultraproducts of finite structures

Definition 1.1.1. If I is a set of indices and $\mathcal{U}$ is an ultrafilter on I, we say that $I' \subseteq I$ is *$\mathcal{U}$-large* simply if $I' \in \mathcal{U}$. Suppose now that $\{M_i : i < \omega\}$ is a collection of finite structures and $\mathcal{U}$ is an ultrafilter on ω.

1. We say that a property P *holds for $\mathcal{U}$-almost all* M_i if $\{i \in I : M_i \models P\} \in \mathcal{U}$.
2. We write $|M_i| \to_{\mathcal{U}} \infty$ if for every $N \in \mathbb{N}$, the set $\{i \in I : |M_i| \geq N\} \in \mathcal{U}$.
3. Given a sequence $\langle r_i : i \in I\rangle$ of real numbers and $s \in \mathbb{R}$, we write $\lim\limits_{n \to \mathcal{U}} r_n = s$ to mean: for all $\epsilon > 0$, $\{n \in \omega : |r_n - s| < \epsilon\} \in \mathcal{U}$.
4. We denote by $\prod_{\mathcal{U}} M_i$ the ultraproduct of the structures $\{M_i : i \in I\}$ with respect to the ultrafilter $\mathcal{U}$.

Remark 1.1.2. A trivial (but possibly clarifying) case occurs when $\mathcal{U}$ is a *principal* ultrafilter of the form $\mathcal{U} = \{X \subseteq I : k \in X\}$ for some fix $k \in I$. Then we have: $\prod_{\mathcal{U}} M_i \cong M_k$, P holds for $\mathcal{U}$-almost all M_n if and only if $M_k \models P$, $|M_n| \not\to_{\mathcal{U}} +\infty$ because M_k was assumed to be finite, and for any sequence $\langle r_n : n < \omega\rangle$, $\lim_{n\to\infty} r_n = r_k$.

The fundamental theorem of ultraproducts is due to Jerzy Łoś, and provides a powerfu transfer principle between the factor structures and their ultraproduct.

Theorem 1.1.3 (Łoś, 1955). *Let $M = \prod_{\mathcal{U}} M_i$ be n ultraproduct of $\{M_i : i \in I\}$ with respect to an ultrafilter $\mathcal{U}$ on I. Then, for every first-order formula $\varphi(\overline{x}) = \varphi(x_1, \ldots, x_n)$ and every tuple $\overline{c} = ([c_1^i]_{\mathcal{U}}, \ldots, [c_n]_{\mathcal{U}}^i)$ of elements in M, we have*

$$M \models \varphi(\overline{c}) \text{ if and only if } \{i \in I : M_i \models \varphi(c_1^i, \ldots, c_n^i)\} \in \mathcal{U}.$$

A very important property of the ultraproducts of first-order structures is the fact that they are $\aleph_1$-*saturated* (also referred as *countably saturated*): for any countable $A \subseteq M$ and every (partial) type $p(\overline{x})$ over M that is finitely satisfiable in M, there is a tuple $\overline{c}$ from M such that $\overline{c}$ realizes $p(\overline{x})$.

Proposition 1.1.4. *Let $M = \prod_{\mathcal{U}} M_i$ be an ultraproduct with respect to a non-principal ultrafilter $\mathcal{U}$ on $I = \omega$. Then, M is $\aleph_1$-saturated.*

Proof. Suppose $p(\overline{x}) = \{\varphi_m(\overline{x}) : m < \omega\}$ is an enumeration of the formulas in $p(\overline{x})$. Since $p(\overline{x})$ is finitely satisfiable in M, we have that for every $k < \omega$ the set $\varphi_1(M) \cap \cdots \varphi_k(M)$ is non-empty.

By Łośtheorem, this implies that the set

$$S_k' := \{i \in \omega : M_i \models \exists \overline{x}(\varphi_1(\overline{x}) \wedge \cdots \wedge \varphi_k(\overline{x}))\}$$

belongs to $\mathcal{U}$. Let $S_k = S_k' \cap [k, +\infty)$. Note that these sets are $\mathcal{U}$-large, $S_k \supseteq S_{k+1}$ for every k, and $\bigcap_{k<\omega} S_k = \emptyset$.

Given $i \in S_1$, let k_i denote the largest natural number k such that $i \in S_k$, and let $\overline{a}_i \in \varphi_1(M_i) \cap \cdots \varphi_{k_i}(M_i)$. For each $m < \omega$, we have by construction that

$$\{i \in \omega : \overline{a}_i \in \varphi_m(M_i)\} \supseteq \{i \in \omega : m \leq k_i\} \supseteq S_1 \cap S_m = S_m \in \mathcal{U}.$$

Thus, by Łoś' theorem, if $\overline{a} = [\overline{a}_i]_{\mathcal{U}}$ then $M \models \varphi_m(\overline{a})$ for all $m < \omega$, and so $\overline{a} \in M$ realizes $p(\overline{x})$. □

It is sometimes useful to know the cardinality of certain ultraproducts, in order to use some results of categoricity. When the index set is arbitrary, computing the exact size of ultraproducts is a very difficult problem even for ultraproducts of finite structures. The main references in this subject are [21] and [25]. In the latter paper, it was shown that if an ultraproduct of finite sets is infinite of size κ, then $\kappa^{\aleph_0} = \kappa$.

However, when we consider ultraproducts of finite structures over a countable set of indices, we can obtain the much simpler but still powerful result below.

Proposition 1.1.5. *If $M = \prod_{\mathcal{U}} M_i$ is an ultraproduct with $I = \omega$ and $|M_i| \to_{\mathcal{U}} \infty$, then $|M| = 2^{\aleph_0}$.*

Proof. Note first that $\left|\prod_{\mathcal{U}} M_i\right| \leq \left|\prod_{i\in\omega} M_i\right| \leq \left|\mathbb{N}^{\mathbb{N}}\right| = |\mathbb{R}|$, so $|M| \leq 2^{\aleph_0}$. For the other inequality, given a set $A \subseteq \mathbb{N}$, we can consider the function $f_A : \mathbb{N} \to \mathbb{N}$ defined by

$$f_A(n) = \sum_{k<n} \chi_A(k) \cdot 2^k$$

where χ_A is the characteristic function of A. Consider the family $\mathcal{F} = \{f_A : A \subseteq \mathbb{N}\}$.

Claim: *For every $n \in \mathbb{N}$, $f_A(n) < 2^n$ and for different subsets A, B of $\mathbb{N}$, $\{n \in \mathbb{N} : f_A(n) = f_B(n)\}$ is finite.*

Proof of the Claim: It is clear that $f_A(n) < 2^n$ for every $f_A \in \mathcal{F}$. Also, given two different subsets A, B of $\mathbb{N}$, we can even show that $\{n \in \mathbb{N} : f_A(n) = f_B(n)\} = \{n \in \mathbb{N} : n \leq \min(A \triangle B)\}$.

Let $t = \min(A \triangle B)$, and assume without loss of generality that $t \in A \setminus B$. Note that $f_A(t+1) = \sum_{k \leq t} \chi_A(k) \cdot 2^k = 2^t + \sum_{k<t} \chi_A(k) \cdot 2^k = f_B(t+1) + 2^t$, so $f_A(t+1) > f_B(t+1)$. Suppose now that $f_A(n) = f_B(n)$ for some $n > t+1$. Then we have:

$$f_A(n) = \sum_{k<t} \chi_A(k) \cdot 2^k + 2^t + \sum_{t<k<n} \chi_A(k) \cdot 2^k = \sum_{k<t} \underbrace{\chi_A(k)}_{=\chi_B(k)} \cdot 2^k + \sum_{t<k<n} \chi_B(k) \cdot 2^k = f_B(n)$$

and substracting the first summand in both sides we obtain

$$2^t + \sum_{t<k<n} \chi_A(k) \cdot 2^k = \sum_{t<k<n} \chi_B(k) \cdot 2^k$$

$$2^t \left(1 + \sum_{t<k<n} \chi_A(k) \cdot 2^{k-t}\right) = 2^t \cdot \sum_{t<k<n} \chi_B(k) \cdot 2^{k-t}$$

$$1 + \sum_{t<k<n} \chi_A(k) \cdot 2^{k-t} = \sum_{t<k<n} \chi_B(k) \cdot 2^{k-t}$$

a contradiction, because the left hand side is an odd number while the right hand side is even. $\square_{\text{Claim}}$.

Consider now the set $I_n = \{i \in I : 2^n \leq |M_i| \leq 2^{n+1}\}$. The sets I_n are not in $\mathcal{U}$, but they form a partition of I. For each $i \in I_n$, let $\{a_{i,j} : j < 2^n\}$ be a list of 2^n different elements from M_i. For every subset $A \subseteq \mathbb{N}$, consider the element

$$a_A = [a_{i, f_A(n)}]_{\mathcal{U}}$$

where n is the only natural number such that $i \in I_n$.

Note that if A, B are different subsets of $\mathbb{N}$, then

$$\begin{aligned} \{i \in I : a_A^i = a_B^i\} &= \{i \in I : a_{i,f_A(n)} = a_{i,f_B(n)}\} \\ &= \{i \in I : i \in I_n \text{ and } f_A(n) = f_B(n)\} \subseteq \bigcup \{I_n : f(n) = g(n)\} \end{aligned}$$

which is a finite union of sets not in $\mathcal{U}$. Thus, $\{i \in I : a_A^i = a_B^i\} \notin \mathcal{U}$, and we conclude that $a_A \neq a_B$ by Łoś' theorem. Therefore, $\left|\prod_{\mathcal{U}} M_i\right| \geq 2^{\aleph_0}$. $\square$

Exercise 1.1.6.

1. If $\mathcal{U}$ is an ultrafilter on an index set I and for every $i \in I$, $\mathcal{F}_i$ is an ultrafilter on an index set J_i, consider the index set $K = \{(i,j) : i \in I, j \in J_i\}$. Define the collection $\mathcal{W}$ of subsets of K determined by

$$X \in \mathcal{W} \text{ if and only if } \{i \in I : \{j \in J_i : (i,j) \in X\} \in \mathcal{F}_i\} \in \mathcal{U}$$

   Show that $\mathcal{W}$ is an ultrafilter on K.

2. Show that every ultraproduct of ultraproducts of finite structures is isomorphic to an ultraproduct of finite structures.

Ultraproducts of finite models have been classically used to prove results about non-definability of certain properties in classes of finite structures. Here we mention some of them:

Lemma 1.1.7. *[Classification of ultraproducts of finite linear orders] Every infinite ultraproduct of finite linear orders have order type of the form $\omega + I \times \mathbb{Z} + \omega^*$, where I is an $\aleph_1$-saturated dense linear order without end points.*

Proof. Let $M = \prod_{\mathcal{U}} L_i$ be an infinite ultraproduct of finite linear orders. First, let us consider the formulas $\varphi_{\min}(x) := \forall y(y = x \vee x < y)$ and $\varphi_{\max}(x) := \forall y(y = x \vee y < x)$ and notice that the sentences $\sigma_{\min} = \exists x(\varphi_{\min}(x))$ and $\sigma_{\max} := \exists x(\varphi_{\max}(x))$ (asserting the existence of minimal and maximal elements, respectively) are true in every L_i. So, by Łoś' Theorem, L has minimal and maximal elements c_0, c_1, respectively.

Also, notice that for every $i < \omega$,

$$L_i \models \forall x\big[\neg\varphi_{\max}(x)) \to \exists y(x < y \wedge \forall w(x < w \to (y \leq w)))\big]$$

That is, every element in L_i different from the maximum has an immediate succesor, and similarly, every element in L_i different from the minimum has an immediate predecessor. This property transfers to L by Łoś' theorem and we can consider the equivalence relation on L given by:

$$a \sim b \Leftrightarrow \text{there is } n \in \mathbb{Z} \text{ such that } S^n(a) = b.$$

This equivalence relation is not definable, but we still have that a description of the classes $\overline{a} := a/\sim$ for $a \in L$ given by:

$$\overline{a} := a/\sim = \begin{cases} \{c_0, S(c_0), S^2(c_0), \ldots\} & \text{if } a \sim c_0 \\ \{c_1, S^{-1}(c_1), S^{-2}(c_1), \ldots, \} & \text{if } a \sim c_1 \\ \{S^n(a) : n \in \mathbb{Z}\} & \text{if } a \not\sim c_0, c_1 \end{cases}$$

Notice that the original order in L induces a linear order in $L/\sim$, given by $\overline{a} < \overline{b}$ if and only if $a < b$ (for $a \not\sim b$), which has maximum elements $\overline{c}_0 = c_0/\sim$ and $\overline{c}_1 = c_1/\sim$. Let $I = L/\sim \setminus\{\overline{c}_0, \overline{c}_1\}$. By construction, we already know that L is isomorphic to $\overline{c}_0 \cup I \times \mathbb{Z} \cup \overline{c}_1 \cong \omega \cup I \times \mathbb{Z} \cup \omega^*$, so it only remains to show that I is an $\aleph_1$-saturated dense linear order without end points, which also will have size $2^{\aleph}_0$ because by Proposition 1.1.5 we have

$$2^{\aleph_0} = |L| = |\omega + I \times \omega + \omega| = |I|.$$

Suppose $\overline{a}, \overline{b}$ are two elements in I with $\overline{a} < \overline{b}$, and consider the type

$$\begin{aligned} p_{a,b}(x) &= \{S^n(a) < x < S^{-n}(b) : n < \omega\} \\ &= \{\exists x_1, \dots, x_n, y_1, \dots, y_n (a < x_1 < \cdots < x_n < x < y_n < \cdots < y_1 < b) : n < \omega\} \end{aligned}$$

Since L is $\aleph_1$-saturated, there is an element $c \in L$ realizing $p_{a,b}(x)$, and we will have that $\overline{a} < \overline{c} < \overline{b}$, so I is dense. A similar argument can be used to show that I has no end points, and this completes the proof. □

Proposition 1.1.8 (Gurevich, 1988). *"Having even cardinality" is not expressible in first-order logic of finite linear orders, that is, there is no a first-order sentence σ such that for every finite structure, $M \models \sigma$ if and only if M has even cardinality.*

Proof. Let $L = \{<\}$ and let $\mathcal{K}$ be the class of finite L-structures which are also linear orders. Suppose there is an L-sentence σ such that for all $M \in \mathcal{K}$, $M \models \sigma$ if and only if $|M|$ is even.

Let A_n be the linear order $(\{1, 2, \dots, n\}, <)$, and consider the structures $M = \prod_{\mathcal{U}} A_{2n}$ and $M' = \prod_{\mathcal{U}} A_{2n+1}$ where $\mathcal{U}$ is a non-principal ultrafilter on ω.

By the Lemma 1.1.7, and using a back & forth argument, we can show easily that $A \cong \omega + I \times \mathbb{Z} + \omega^* \equiv \omega + I' \times \mathbb{Z} + \omega^* \cong B$, and so since each A_{2n} is even, $M \models \sigma$ which implies $M' \models \sigma$, and so, $A_{2n+1} \models \sigma$ for $\mathcal{U}$-almost all n. A contradiction. □

Proposition 1.1.9 (0-1 law for the finite linear orders). *For every sentence σ in the language $L = \{<\}$, there is $n < \omega$ such that either $L \models \sigma$ for every finite linear order of size $|L| \geq n$, or $L \models \sigma$ for every finite linear order of size $|L| \geq m$.*

Proof. Suppose otherwise. Then for some sentence σ there are finite linear orders $\langle L_n, L'_n : n < \omega \rangle$ such that $|L_n|, |L'_n| \geq n$ and $L_n \models \sigma$, $L'_n \models \neg\sigma$. However, this contradicts the fact that $\prod_{\mathcal{U}} L_n \equiv \prod_{\mathcal{U}} L'_n$. □

Hajek proved in [11] that "connectedness" is not first-order expressible in the class of finite graphs. Turán observed in [28] that this can be proved using ultraproducts of finite structures.

Lemma 1.1.10. *Suppose that $\mathcal{U}$ is a non-principal ultrafilter on ω, and $\langle G_n : n < \omega \in \mathbb{N} \rangle$ is a collection of graphs such that every $x \in G_n$ has degree 2 and G_n does not contain cycles of length less than n. Then, $G = \prod_{\mathcal{U}} G_n$ is a union of $2^{\aleph_0}$ $\mathbb{Z}$-chains.*

Proof. First, consider the $\{R\}$-sentences

$$\varphi := \forall x \exists y, z (y \neq z \wedge R(y,x) \wedge R(z,x) \wedge \forall w (R(w,x) \rightarrow w = y \vee w = z))$$
$$\sigma_k := \neg \exists x_1, \dots, x_n \left(\bigwedge_{1 \leq i < j \leq k} x_i \neq x_j \wedge R(x_1, x_k) \wedge \bigwedge_{1 \leq i < k} R(x_i, x_{i+1}) \right)$$

asserting that the every vertex in the graph has degree exactly two, and there are no cycles of size k for any $k \geq 3$. Every G_n realizes φ, and so G is a graph on which every vertex has degree 2. On the other hand, for a fixed $k \geq 3$, we have that

$$\{n < \omega : G_n \models \sigma_k\} \supseteq \{n < \omega : n \geq k\} \in \mathcal{U}$$

and by Łoś' theorem, $G \models \sigma_k$ for every $k \geq 3$. That is, G does not contain cycles.

Using the same idea of in the proof of Lemma 1.1.7, we consider a (non-definable) equivalence relation $\sim$ on G given by

$$a \sim b \Leftrightarrow a \text{ and } b \text{ are connected.}$$

Claim: *Every class $C_a := a/\sim$ is isomorphic to a $\mathbb{Z}$-chain.*

From the claim we can conclude that $|C_a| = \aleph_0$ and given $a \not\sim b$, there are no edges between C_a and C_b. So, since G has cardinality $2^{\aleph_0}$, we conclude that G is isomorphic to the disjoint union of the $2^{\aleph_0}$ $\mathbb{Z}$-chains given by C_a.

Proof of the Claim: Let $C_a := a/\sim$ be the class of an element $a \in G$, and let b_1, c_1 be the unique two elements in G that are adjacent to a. Put $b_0 = a = c_0$, and define inductively the element b_{n+1} (resp. c_{n+1}) to be the unique element in G such that $G \models b_{n+1} R b_n \wedge b_{n+1} \neq b_{n-1}$ (similarly for c_{n+1}).

Consider the function

$$f : \mathbb{Z} \longrightarrow C_a$$
$$n \longmapsto f(n) := \begin{cases} b_n & \text{if } n > 0 \\ a & \text{if } n = 0 \\ c_n & \text{if } n < 0 \end{cases}$$

Clearly f is well-defined, because each vertex in G has degree 2. In fact, $f(n)Rf(m)$ if and only if $|n - m| = 1$. To show that f is injective, let n be the integer with minimal absolute value such that $f(m) = f(n)$ for some $m \neq n$. Without loss of generality, we may assume that $n > 0$ since the case $n < 0$ can be proved similarly. Thus, $f(n) = b_n$, and we have two cases:

- If $m > 0$, then $f(m) = b_m$ and $m > n$, so we would obtain the cycle

  $$b_n - b_{n+1} - \cdots - b_m = b_n,$$

  a contradiction since G does not contain cycles.

- If $m < 0$, then $f(m) = c_m$ and we would obtain the cycle

  $$c_m - c_{m-1} - \cdots - c_1 - c_0 = a = b_0 - b_1 - \cdots - b_n = c_m,$$

  again, a contradiction because G does not contain cycles.

To show that f is surjective, if $d \in C_a$ then there is some $n < \omega$ such that $\text{dist}(a, d) = n$. We will show by induction the following:

- *If* $\text{dist}(a, d) = n$ *then either* $d = b_n$ *or* $d = c_n$.

It is clear for $n = 0, 1$. Suppose now that $\operatorname{dist}(a, d) = n + 1$. Then, there are elements $d_0, \ldots, d_n, d_{n+1}$ all diferent such that $a = d_0 - d_1 - \cdots - d_n - d_{n+1} = d$. By induction hypothesis, either $d_n = b_n$ or $d_n = c_n$, and also either $d_{n-1} = b_{n-1}$ or $d_{n-1} = c_{n-1}$.

If $d_n = b_n$ and $d_{n-1} = c_{n-1}$, then we obtain the cycle given by $d_{n-1} = c_{n-1} - \cdots - c_1 - a - b_1 - \cdots - b_n = d_n - d_{n-1}$, a contradiction. Similarly, if $d_n = c_n$ and $d_{n-1} = b_{n-1}$. Thus, the only remaining case are $d_{n-1} = b_{n-1}$ and $d_n = b_n$, which implies by construction $d = b_{n+1}$ or$d_{n-1} = c_{n-1}$ and $d_n = c_n$ which implies $d = c_{n+1}$.

Thus, f is an isomorphism of graphs, and we conclude that C_a is a $\mathbb{Z}$-chain. $\square_{\text{Claim}}$ $\square$

Proposition 1.1.11. *"Being connected" is not first-order expressible in the class of finite graphs.*

Proof. Suppose σ is a sentence such that for every finite graph G, $G \models \sigma$ if and only if G is connected. For every $n < \omega$, let A_n be a cycle of length $2n$ and let B_n be be the union of two disjoint cycles of length n. Notice that each A_n is connected while each B_n is not, so $A_n \models \sigma$ and $B_n \models \neg\sigma_n$. However, by the Lemma 1.1.10 $M = \prod_{\mathcal{U}} A_n$ and $M' = \prod_{\mathcal{U}} B_n$ are both unions of $2^{\aleph_0}$ $\mathbb{Z}$-chains, so $M \cong M'$, but by Łoś' theorem we have $M \models \sigma$ and $M' \models \sigma$. A contradiction. $\square$

Koponen and Luosto showed in [22] using ultraproducts of finite structures that neither simplicity nor nilpotency are first-order definable in the class of finite groups.

Let $\mathcal{U}$ be a non-principal ultrafilter over the set of primes $\mathbb{P}$. We can show that simplicity is not definable in finite groups by showing that the groups $G = \prod_{\mathcal{U}} \mathbb{Z}/p\mathbb{Z}$ and $G' = \prod_{\mathcal{U}} (\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z})$.

Notice that both G and G' are infinite torsion-free divisible abelian groups of cardinality $2^{\aleph_0}$, so $G \cong G'$ follows from the following statement:

Proposition 1.1.12. *The theory of divisible abelian groups is κ-categorical for all $\kappa \geq \aleph_1$.*

Proof. First, notice that if G is a divisible abelian group, we can see G as a $\mathbb{Q}$-vector space by defining $\dfrac{p}{q} \cdot x := p \cdot z$, where z is the only element satisfying the equation $q \cdot z = x$, for $q > 0$.

This is well-defined because if $q \cdot y_1 = x = q \cdot y_2$, then $q \cdot (y_1 - y_2) = 0$ which implies $y_1 = y_2$ because G is torsion-free. So, let G, H be two torsion-free divisible abelian groups of the same cardinality $\kappa \geq \aleph_1$.

Any two basis of G and H as $\mathbb{Q}$-vector spaces (respectively) have the same cardinality κ, and any bijection between these basis will induce an isomorphisms $G \cong H$. $\square$

1.2 Pseudofinite structures and theories

Definition 1.2.1. An L-structure M is *pseudofinite* if for every L-sentence σ such that $M \models \sigma$ there is a finite L-structure $M_0 \models \sigma$. That is, M is pseudofinite if every sentence true in M has a finite model.

Definition 1.2.2. if L is a first-order language, we denote by $\Gamma_{f,L}$ the common theory of all finite L-structures.

That is, $\sigma \in \Gamma_{f,L}$ if and only if σ is true in every finite L-structure.

The following result describes several equivalent definitions for a *structure* to be pseudofinite.

Proposition 1.2.3. *Fix a first-order language L, and let M be an L-structure. Then the following are equivalent:*

1. *M is pseudofinite.*
2. *M is elementarily equivalent to an ultraproduct of finite structures.*
3. *$M \models \Gamma_{f,L}$.*

Proof. (2) $\Rightarrow$ (3): Suppose $M \equiv \prod_{\mathcal{U}} M_i$ where $\{M_i : i \in I\}$ is a collection of finite structures and $\mathcal{U}$ is an ultrafilter on I. Then, for every $\sigma \in \Gamma_{f,L}$ we have $M_i \models \sigma$. Thus, $\{i \in I : M_i \models \sigma\} = I \in \mathcal{U}$, and by Łoś' theorem, $\prod_{\mathcal{U}} M_i \models \sigma$ which implies $M \models \sigma$. Therefore, $M \models \Gamma_{f,L}$.

(3) $\Rightarrow$ (1): Let σ be an L-sentence such that $M \models \sigma$. If σ has no finite models, then for every finite L-structure M_0 we would have $M_0 \models \neg\sigma$. So, $\neg\sigma \in \Gamma_{f,L}$, and we would obtain $M \models \neg\sigma$, a contradiction.

(1) $\Rightarrow$ (2): Suppose M is pseudofinite and let $\mathrm{Th}(M)$ be the collection of all L-sentences that are true in M. Let I be the collection of all finite subsets of $\mathrm{Th}(M)$. For every $i = \{\phi_1, \ldots, \phi_m\} \in I$, let M_i be a finite L-structure such that $M_i \models \phi_1 \wedge \cdots \wedge \phi_m$.

Let $\mathcal{F}_0$ be the collection of the sets of the form $X_j = \{j \in I : M_j \models \phi \text{ for all } \phi \in i\}$. We will show that $\mathcal{F}_0$ has the *finite intersection property*: note that

$$\begin{aligned} X_i \cap X_j &= \{k \in I : M_k \models \phi \text{ for all } \phi \in i\} \cap \{k \in I : M_k \models \phi \text{ for all } \phi \in j\} \\ &= \{k \in I : M_k \models \phi \text{ for all } \phi \in i \cup j\} = X_{i \cup j} \neq \emptyset. \end{aligned}$$

So, $\mathcal{F}_0$ can be extended first to a filter $\mathcal{F}$, and then to an ultrafiter (a maximal filter) $\mathcal{U}$.

Now we show that $M \equiv \prod_{\mathcal{U}} M_i$. If $M \models \sigma$, then the set $\{i \in I : M_i \models \sigma\} \supseteq X_{\{\sigma\}} \in \mathcal{U}$, and so, by Łoś' theorem, $\prod_{\mathcal{U}} M_i \models \sigma$. $\square$

Remark 1.2.4. The main difference between pseudofinite structures and ultraproducts of finite structures is that the form may omit types, while the latter are always $\aleph_1$-saturated (if infinite). For instance, in Example 1.3.7 it is shown that the structures $(\mathbb{Q}, +)$ is pseudofinite, but since it is countable, it cannot be isomorphic to an ultraproduct of finite structures.

Recall the following known result:

Theorem 1.2.5 (Keisler-Shelah isomorphism theorem)**.** *Let M, N be two L-structures. Then $M \equiv N$ if and only if there exists an ultrafilter $\mathcal{U}$ (over some set of indices) such that*

$$M^{\mathcal{U}} := \prod_{\mathcal{U}} M \cong \prod_{\mathcal{U}} N := N^{\mathcal{U}}.$$

Thus, a structure M is pseudofinite if and only if it has an ultrapower which is *isomorphic* to an ultraproduct of finite structures. This gives a purely algebraic definition of pseudofiniteness, where no sentences are mentioned.

There is a deep result in model theory due to Cherlin, Harrington and Lachlan (see [4]) that can be used to find pseudofinite models:

Theorem 1.2.6 (Cherlin, Harrington, Lachlan, 1985). *All models of totally categorical (κ-categorical for each $\kappa \geq \aleph_0$) are pseudofinite.*

For theories, defining pseudofiniteness might be a little bit more complicated, and there are two versions.

Definition 1.2.7. Let T be a consistent theory, not necessarily complete.

1. T is *weakly pseudofinite* if whenever $T \models \sigma$ then σ is true in some finite L-structure (not necessarily a model of T).

2. T is *strongly pseudofinite* if whenever σ is consistent with T there is a finite structure $M_0 \models \sigma$.

For instance, if $T = T_{LO}$ is the theory of linear orders, then T is weakly pseudofinite but not strongly pseudofinite: the sentence $\forall x, y(x < y \rightarrow \exists z(x < z < y))$ is consistent with T_{LO} but does not have finite models. However, we have the following facts (left as exercises to the reader)

Exercise 1.2.8. Show that T is weakly pseudofinite if and only if $T \cup \Gamma_{f,L}$ has some pseudofinite model.

Exercise 1.2.9. T is strongly pseudofinite if and only if $T \models \Gamma_{f,L}$ if and only if every model of T is pseudofinite.

Exercise 1.2.10. If T is complete, then T is weakly pseudofinite if and only if T is strongly pseudofinite.

In the light of Exercise 1.2.10, whenever T is a complete theory, we say T is pseudofinite to mean either weakly or strongly pseudofinite.

1.3 Examples of pseudofinite structures and theories

We now will show some examples of pseudofinite structures and theories.

Example 1.3.1. *For a language L, the common theory of all L-structures is weakly pseudofinite*

Let T be the common theory of all L-structures. Note that every finite L-structure M is a pseudofinite model of $T \cup \Gamma_{f,L}$. So, by Exercise 1.2.8, T is weakly pseudofinite.

Example 1.3.2. *Let T_0 be any L-theory. Let $T_{0,f}$ be the common first-order theory of all finite models of T_0. Then $T_{0,f}$ is strongly pseudofinite.*
Note that $T_{0,f}$ may not be the weakest strongly pseudofinite theory extending T_0.

Of course, all these makes sense only when T_0 has finite models. First, note that $T_0 \subseteq T_{0,f}$ by definition. Suppose now that $T_{0,f} \cup \{\sigma\}$ is consistent, say $M \models T_{0,f} \cup \{\sigma\}$. Notice that if σ does not have finite models, then $M_0 \models \neg\sigma$ for every finite model of T_0. Thus, $\neg\sigma \in T_{0,f}$, but this contradicts the consistency of $T_{0,f} \cup \{\sigma\}$.

Example 1.3.3. *The theory DLO of dense linear orders is* not *pseudofinite.*

The sentence $\sigma := \forall x, y \exists z(x < z < y) \wedge$"$<$ is a linear order" does not have finite models.

Example 1.3.4. *ACF :="Theory of algebraically closed fields" is not weakly pseudofinite.*

Suppose $ACF \cup \Gamma_{f,L_{rings}}$ has pseudofinite models, say $M = \prod_{\mathcal{U}} M_i \models ACF$ for a collection $\{M_i : i \in I\}$ of finite rings and an ultrafilter $\mathcal{U}$ on I. We have two cases:

If $\mathrm{char}(M) \neq 2$, consider the sentence $\sigma_1 = \forall x \exists y(y^2 = x) \in ACF$. Then, for $\mathcal{U}$-almost all finite fields M_i we have $M_i \models \sigma \wedge 1 + 1 \neq 0$. So, the function $f : M_i \to M_i$ given by $f(x) = x^2$ is surjective, and since M_i is finite, it is also injective.

Thus, $M_i \models \forall x, y(x^2 = y^2 \to x = y)$, and in particular, $1 = -1$, which contradicts that $\mathrm{char}(M_i) \neq 2$.

Now, if $\mathrm{char}(M) = 2$, we consider the sentence $\sigma_2 = \forall x \exists y(y^3 = x) \in ACF$. Again, for $\mathcal{U}$-almost all M_i we have $M_i \models \sigma_2 \wedge 1 + 1 = 0$, and so the map $g : M_i \to M_i$ given by $g(x) = x^3$ is surjective, thus injective since M_i is finite.

Therefore, $M_i \models \forall x, y(x^3 = y^3 \to x = y)$, and since $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$ we have that all roots of $x^2+xy+y^2 = 0$ satisfy $x = y$. In particular, if $M_i \models \exists x(x^2+x+1 = 0)$ then the only root is $x = 1$, and we have $M_i \models 1^2+1+1 = 1+1+1 = 1 = 0$, a contradiction since M_i is a field.

Example 1.3.5. *Fix a finite field $\mathbb{F}$, and let T be the theory of $\mathbb{F}$-vector spaces in the language of additive groups. Then T is strongly pseudofinite.*

The language for T is $L = \{+, 0, (f_\alpha)_{\alpha\in\mathbb{F}}\}$ and T is axiomatized by the axioms of $\mathbb{F}$-vector spaces. For instance,

(a) $(V, +, 0)$ is an abelian group.

(b) $\forall x(f_\alpha(f_\beta(x)) = f_{\alpha\cdot\beta}(x))$ for $\alpha, \beta \in \mathbb{F}$.

(c) $\forall x, y(f_\alpha(x + y) = f_\alpha(x) + f_\alpha(y))$ for $\alpha \in \mathbb{F}$.

(d) $\forall x(f_{\alpha+\beta}(x) = f_\alpha(x) + f_\beta(x))$ for $\alpha, \beta \in \mathbb{F}$.

Let T' be the theory of infinite models of T, which is axiomatized by the axioms above together with the collection of sentences

$$\left\{\sigma_n := \exists x_1, \dots, x_n \left(\bigwedge_{\alpha\in\mathbb{F}, i<j} \neg(f_\alpha(x_i) = x_j)\right) : n < \omega\right\}$$

Clearly T' is ω-categorical (if $M_1, M_2 \models T'$ are countable models, any bijection between bases induces an isomorphism $M_1 \cong M_2$)

So, T' is complete, and we have two cases for a sentence σ such that $T \cup \{\sigma\}$ is consistent: either $T' \not\models \sigma$, and so $T' \models \neg\sigma$ which implies that the original model of $T \cup \{\sigma\}$ was finite, or $T' \models \sigma$, and so by compactness there is some $n < \omega$ such that (a)-(d) $\cup\, \sigma_n \models \sigma$. Thus, $M_n = (\mathbb{F}^{n+1}, +, \overrightarrow{0}) \models \sigma$. We conclude that T is strongly pseudofinite

Example 1.3.6. *The theory of $\mathbb{Q}$-vector spaces in the language of groups with a function symbol for scalar multiplication is pseudofinite and complete.*

In this case, the theory is axiomatized by the axioms of $\mathbb{Q}$-vector space, together with the axioms

$$\left\{ \sigma_{\frac{r}{s}} := \forall x \left(\underbrace{f_{\frac{r}{s}}(x) + \cdots + f_{\frac{r}{s}}}_{s\text{-times}} = f_r(x) \right) : r, s \in \mathbb{N}, s \neq 0. \right\}$$

This theory is $\aleph_1$-categorical, so it is complete. To show that it is pseudofinite it is enough to find a finite model for every finite subset of the axioms. Assume that $T_0 = \{\varphi_1, \dots, \varphi_m\} \cup \{\sigma_{\frac{r_1}{s_1}}, \dots, \sigma_{\frac{r_n}{s_n}}\}$ is a finite subset of T, where the formulas ϕ_i are axioms for the theory of $\mathbb{Q}$-vector spaces containing finitely many function symbols f_α.

Let $N = \max\{|r_i|, |s_i|, \text{height}(\alpha) : i \leq n,\ f_\alpha \text{ mentioned in a formula } \varphi_j\}$, and pick a prime $p > N$. Note that if $\alpha = \dfrac{r}{s}$, we can assign $f_{\frac{r}{s}}(x) = r \cdot \left(\frac{x}{s}\right)$ for every $x \in \mathbb{Z}/p\mathbb{Z}$ because $\gcd(p, s_j) = 1$. Similarly with $\frac{r_i}{s_i}$.

Thus, by interpreting putting $f_\alpha(x) = 0$ whenever α is not mentioned in T_0, we conclude that $(\mathbb{Z}/p\mathbb{Z}, +, 0, f_\alpha)$ is a finite model of T_0.

Example 1.3.7. *The theory* $\text{Th}(\mathbb{Q}, +)$ *is pseudofinite.*

Notice that $\text{Th}(\mathbb{Q}, +)$ is the theory of abelian divisible groups, which is $\aleph_1$-categorical. Let $\mathbb{P}$ be the collection of prime numbers, and $\mathcal{U}$ a non-principal ultrafilter on $\mathbb{P}$. Then, $\prod_{\mathcal{U}} \mathbb{Z}/p\mathbb{Z}$ is a divisible abelian group (as at the end of Example 1.3.6). So $\text{Th}(\mathbb{Q}, +)$ has a pseudofinite model, and therefore it is pseudofinite (because it is complete).

Example 1.3.8. *The theory of the random graph is pseudofinite.*

Recall that the random graph is the generic Fraïssé limit of the class of finite graphs. Its theory can be axiomatized in the language $L = \{R\}$ by the sentences

$$P_{k,\ell} = \forall x_1, \dots, x_k \forall y_1, \dots, y_\ell \left(\bigwedge_{i,j} x_i \neq y_j \to \exists z \left(\bigwedge_{i,j} zRx_i \wedge \neg zRy_j \right) \right)$$

We will show that each of this sentences have finite models using a probabilistic argument. Fix $n \in \mathbb{N}$ and take $V = \{1, 2, \dots, n\}$. For every possible edge $e \in [V]^2$, consider the probability space $\Omega_e = \{0_e, 1_e\}$ with $P_e(\{0_e\}) = 1-p$, $P_e(\{1_e\}) = p$ for some fixed $p \in [0,1]$. Let $\mathcal{G}(n,p)$ be the probability space $\Omega := \prod_{e\in[V]^2} \Omega_e$ with the product measure.

Note that every element in Ω is in correspondence with a unique graphs G with set of vertices V. So, the events in $\mathcal{G}(n,p)$ are simply collections of graphs on V. For example, for $e \in [V]^2$, the set $A_e = \{\overline{x} \in \Omega : x_e = 1_e\} = \{G : e \in E(G)\}$ is the event of having e as an edge, and its probability is $Pr(A_e) = Pr_e(\{1_e\} \times \prod_{e'\neq e} 1 = p$.

***Claim 1**: The events A_e are independent and occur with probability p* By definition, is $S = \{e_1, \ldots, e_k\} \subseteq [V]^2$, then

$$Pr(A_{e_1} \cap A_{e_k}) = Pr\left(\prod_{e'\notin S} \Omega_{e'} \times \prod_{i=1}^{k} \{1_{e_i}\}\right) = 1 \cdot p^k = P(A_1) \cdots P(A_{e_k}). \qquad \square$$

Consider now for $k, \ell \geq 1$, the event defined by $\mathcal{P}_{k,\ell} = \{G \in \mathcal{G}(n,p) : G \models P_{k,\ell}\}$ =the collection of graphs G such that for any disjoint $U, W \subseteq G$ with $|U| \leq k$, $|W| \leq \ell$ there is $v \notin U \cup W$ such that uRv and $\neg(uRw)$ for all $u \in U, w \in W$.

So, to show that the theory of the random graph is pseudofinite, it is enough to show that given $k, \ell \geq 1$, the set $\mathcal{P}_{k,\ell}(\mathcal{G}(n,p)) \neq \emptyset$ for some $n \in \mathbb{N}$ and some $p \in [0,1]$.

In fact, we have something stronger:

Theorem 1.3.9. *For any $k, \ell \geq 1$ and every constant $p \in (0,1)$, almost every graph in $G \in \mathcal{G}(n,p)$ satisfies the property $P_{k,\ell}$. That is,*

$$\lim_{n\to\infty} Pr(\mathcal{P}_{k,\ell}(\mathcal{G}(n,p)) = 1.$$

Proof. For fixed n, disjoint subsets of vertices U, W and $v \in [n] \setminus (U \cup W)$, we have $Pr(\forall u \in U, \forall w \in W(uRv \wedge \neg(uRw)) = p^{|U|} \underbrace{(1-p)}_{q}^{|W|} \geq p^k q^\ell$. Hence,

$$\begin{aligned} &Pr(\text{There is no suitable } v \text{ for the pair } (U,W)) \\ &\qquad = (1 - p^{|U|} q^{|W|})^{|[n]\setminus(U\cup W)|} = (1 - p^{|U|} q^{|W|})^{n-|U|-|W|} \\ &\qquad \leq (1 - p^k q^\ell)^{n-k-\ell}. \end{aligned}$$

Notice that there are no more than $n^{k+\ell}$ of pairs (U,W) with $U \cap W = \emptyset$ and $|U| \leq k, |W| \leq \ell$, since every such pair can be encoded with a function $f : \{a_1, \ldots, a_k\} \cup \{b_1, \ldots, b_\ell\} \to \{1, \ldots, n\}$ (if $|U| < k$ or $|W| < \ell$, the pair (U,W) would be encoded with a non-injective function). Thus, the probability that some pair U, W has no suitable element v is at most $(1-p^k q^\ell)^{n-k-\ell} \cdot n^{k+\ell}$. So, since $k+\ell$ is constant and $p^k q^\ell = p^k(1-p)^\ell \leq p(1-p) = p - p^2 < 1$, we conclude that

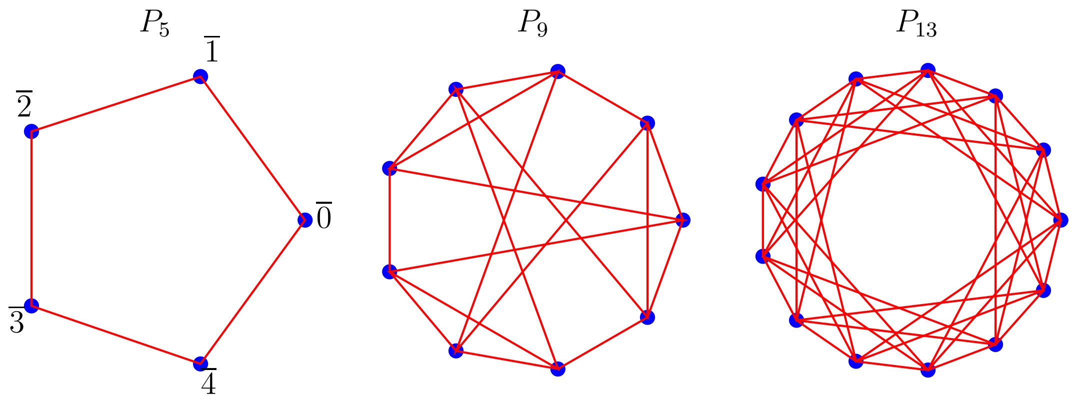
$$\lim_{n\to\infty} Pr((\mathcal{P}_{k,\ell}(\mathcal{G}(n,p)))^c) \leq \lim_{n\to\infty} n^{k+\ell} \cdot (1-p^k q^\ell)^{n-k-\ell} = \lim_{n\to\infty} n^{k+\ell} \cdot r^{n-k-\ell} = 0$$

because the exponential decay dominates the polynomial growth. $\square$

In particular, given $k, \ell \geq 1$ there is some $n \in \mathbb{N}$ and some graph G with n-vertices such that $G \models P_{k,\ell}$. This shows that the theory of the random graph is pseudofinite.

The following is a curious note regarding the random graph:

Definition 1.3.10 (Paley graphs)**.** Let $q = p^n$ be a prime power with $q \equiv 1 \pmod 4$. We define the *Paley graph* P_q to be the graph with set of vertices $V = \mathbb{F}_q$ and the edge relation defined by xRy if and only if $x \neq y$ and $(x - y)$ is a square.



The hypothesis $q \equiv 1 \pmod 4$ allows us to ensure that (-1) is a square in $\mathbb{F}_q$, and thus R is symmetric relation. A rather technical theorem of Bollobás and Thomason (see Theorem 10 in Ch. XIII.2 of [3]) states the following:

Theorem 1.3.11 (Bollobás - Thomason, 1981)**.** *Let U, W be disjoint sets of vertices of P_q with $|U \cup W| = m$, and define*

$$v(U, W) = \{v \in P_q \setminus (U \cup W) : vRu \wedge \neg vRw \textit{ for all } u \in U, w \in W\}.$$

Then,

$$|v(U, W) - 2^{-m}q| \leq \frac{1}{2}(m - 2 + 2^{m+1})q^{1/2} + \frac{m}{2}.$$

Using this result as a black box, we can conclude the following:

Corollary 1.3.12. *Let $\mathcal{U}$ be an ultrafilter on the set $I = \{q : q$ is a prime power and $q \equiv 1 \pmod 4\}$. Then, $P = \prod_{\mathcal{U}} P_q$ is a model of the theory of the random graph.*

Proof. Let $U = \{u_1, \ldots, u_k\}$, $W = \{w_1, \ldots, w_\ell\}$ be disjoint subsets of P. Then, the sets $U^q = \{u_1^q, \ldots, u_k^q\}$, $W^q = \{w_1^q, \ldots, w_k^q\}$ are disjoint subsets of P_q for $\mathcal{U}$-almost all q. By Bollobás-Thomason Theorem, we have $|v(U^q, W^q)| \geq \frac{1}{2^{k+\ell}}q - C_{k+\ell}q^{1/2}$ for some fixed constant $C_{k+\ell} > 0$. So, for sufficiently large q ($q \geq 2^{k+\ell+1} \cdot C_{k+\ell}$), we have

$$|v(U^q, W^q)| \geq \frac{1}{2}\left(\frac{1}{2^{k+\ell}}\right) q > 0,$$

and we have

$$\{q \in I : v(U^q, w^q) \neq \emptyset\} = \left\{q \in I : P_q \models \exists z \left(\bigwedge_{i,j} zRu_i^q \wedge \neg(zRw_j^q)\right)\right\} \in \mathcal{U}$$

and by Łoś' theorem, $P \models \exists z \left(\bigwedge_{i,j} zRu_i \wedge \neg(zRw_j)\right)$. Since this was shown for arbitrary disjoint U, W of sizes k, ℓ respectively, we conclude that $P \models P_{k,\ell}$ for all $k, \ell \geq 1$, and so P is a model of the theory RG of the random graph. $\square$

Example 1.3.13. *Almost sure theories:*

Let L be a countable language and suppose $\mathcal{K}$ is a class of finite structures which is closed under isomorphisms, that has only finitely many non-isomorphic models of size n for every $n \in \mathbb{N}$. Let μ_n be a uniform probability measure on the set $\mathcal{K}_n(L) = \{M : M \text{ is an } L\text{-structure with universe } \{1, \ldots, n\}\}$ and define, for any L-sentence σ,

$$\mu(\sigma) = \lim_{n\to\infty} \mu_n \left(\{A \in \mathcal{K}_n(L) : A \models \sigma\}\right).$$

Definition 1.3.14. Given $\mu, \mu_n, \mathcal{K}$ and $\mathcal{K}_n(L)$ as above, we define the *almost sure theory of* $\mathcal{K}$ as

$$T_{as}(\mathcal{K}) = \{\sigma : \sigma \text{ is a first-order } L\text{-sentence and } \mu(\sigma) = 1\}.$$

Proposition 1.3.15. *If $M \models T_{as}(\mathcal{K})$ then M is pseudofinite. Moreover, $\sigma \in T_{as}$ if and only if σ is true in* almost all *finite L-structures.*

Proof. By definition, $\sigma \in T_{as}(\mathcal{K})$ if and only if $\lim_{n\to\infty} \mu_n(\{M \in \mathcal{K}_n(L) : M \models \sigma\}) = 1$, which is means by definition that σ is true in almost all finite models of $\mathcal{K}$.

Now, suppose that $M \models T_{as}(\mathcal{K})$, and let σ be an L-sentence such that $M \models \sigma$. If σ does not have finite models, then for all $n < \omega$ and $M \in \mathcal{K}_n(L)$ we have $M \models \neg\sigma$. So, $\mu(\neg\sigma) = 1$, which implies $\neg\sigma \in T_{as}$. This contradicts that $M \models T_{as} \cup \{\sigma\}$. □

Lecture 2

Pseudofinite dimensions, forking and simplicity.

Dimension theory is one of the most important concepts in model theory and it can be used to give a combinatorial description of the definable sets of first order structures, allowing also the use of inductive arguments to prove properties about definable sets.

One of the recurrent themes around the notions of rank is their relationship with forking-independence. It is often desired that any instance of forking (on types or formulas) can be detected by a decrease of some dimension in the same way that any instance of linear dependence is witnessed by a decrease in the linear dimension, or any algebraic dependence can be detected by analyzing the transcendence degree.

In [14], Hrushovski and Wagner defined the notion of quasidimension on a structure M as a way to generalize the concept of dimension allowing values in an ordered group instead of allowing only integer values. The main example is what I call "logarithmic pseudofinite dimension" which is defined on ultraproducts of finite structures by taking the logarithm of the cardinality of nonstandard finite sets and factor them out by the convex hull of the reals.

The purpose of this chapter is to study the connection between forking in a pseudofinite structure and the logarithmic pseudofinite dimension, and use this connections to characterize desirable model-theoretic properties of the ultraproduct via conditions on the pseudofinite dimension.

2.1 Measures and dimension in ultraproducts of finite structures

Throughout this section, we will assume that L is a first order language, and $\mathcal{C}$ is a class of finite L-structure index by some set I, and let $\mathcal{U}$ be a non-principal ultrafilter on I. Suppose also that $|M_i| \to_{\mathcal{U}} \infty$.

We can enrich the language L to a language L^+ with 2-sorts: a sort $\mathbb{D}$ carrying the language L and another sort $\mathbb{OF}$, carrying the language of ordered rings. Also, for every L-formula $\phi(\overline{x}, \overline{y})$, add a new function symbol $f_\phi : \mathbb{D}^{|y|} \to \mathbb{OF}$.

Given a finite structure $M_i \in \mathcal{C}$, there is a natural way to expand M_i to an L^+-structure K_i by doing:

- $\mathbb{D}(K_i) = M_i$, with its original L-structure.
- $\mathbb{OF}(K_i) = (\mathbb{R}, +, \cdot, 0, 1, <)$.
- $f_\phi : M_i^{|y|} \to \mathbb{R}$ is a function defined by $f_\phi(\overline{b}) = |\phi(M_i^{|x|}; \overline{b})|$, the cardinality of the set defined by $\phi(\overline{x}, \overline{b})$ in the structure M_i.

We consider now the ultraproduct of the structures K_i with respect to $\mathcal{U}$,

$$K := \prod_{\mathcal{U}} K_i = \left(\prod_{\mathcal{U}} M_i, \mathbb{R}^* \right).$$

This structure will look like a two-sorted structure, having a pseudofinite structure M in the first sort, the non-standard reals in the second sort, and for every definable subset $X = \phi(M^r; \overline{b})$ of M a definable non-standard cardinality given by $f_\phi(\overline{b}) = |X|$.

Note that since we are taking an ordered field in the second sort, we will be allowed to take sums, products, and quotients of cardinalities of definable sets, and even compare them with rational numbers, all definably in L^+.

One of the most useful features of pseudofinite structures is the fact that we can use *counting measures* on the algebra of definable sets in the ultraproducts.

For a non-empty definable subsets D of M, there is a finitely-additive real valued probability measure μ_D defined as:

$$\mu_D(X) := \text{st}\left(\frac{|X|}{|D|} \right) = \lim_{i \to \mathcal{U}} \frac{|X(M_i) \cap D(M_i)|}{|D(M_i)|}.$$

Note that the language L^+ is able to encode significant information about these measures. For instance, $\mu_D(\phi(\overline{x}, \overline{b})) \leq \frac{p}{q}$ if and only if $M \models q \cdot f_\phi(\overline{b}) \leq p \cdot f_D$. The counting measures in pseudofinite structures have been used to obtain model-theoretic proofs of classical results in extremal combinatorics, such as the Szemerédi's regularity lemma, the correspondence Furstenberg principle, and the Szemerédi's theorem in number theory: Every subset of the integers with positive density contains arbitrarily long arithmetic progressions.

It is routine to show that these counting measures are finiteliy-additive real valued probability measures on the boolean algebra of definable subsets of M (or M^n), and by Carathéodory's Extension Theorem it extends uniquely to a countably-additive probability measure on the σ-algebra generated by the definable sets of M.

The following result shows that, at least for definable sets, the measure μ_D is definable "up to infinitesimals" in the new structure K.

Proposition 2.1.1. *1. If $K \models f_\phi(\overline{b}) \leq r \cdot f_D$ then $\mu(\phi(x, b)) \geq r$.*

2. If $\mu(\phi(x, b)) < r$ then $K \models f_\phi(b) \leq r \cdot f_D$.

3. If $K \models f_\phi(\overline{b}) \geq r \cdot f_D$ then $\mu_D(\phi(\overline{x}, \overline{b}) \geq r$.

Proof. 1. Recall that $\mu_D(\phi(\overline{x},\overline{b})) = st\left(\frac{|\phi(M^{|x|};\overline{b})|}{|D|}\right) = \inf\left\{q \in \mathbb{Q} : K \models \frac{f_\phi(b)}{f_D} \leq q\right\}$. So (1) is clear by definition.

2. If $\mu(\phi(x,b)) < r$, then there is $q \in \mathbb{Q}$ such that $K \models f_\phi(\overline{b}) \leq q \cdot f_D$, which clearly implies $K \models f_\phi(\overline{b}) \leq r \cdot f_D$.

3. If $K \models f_\phi(\overline{b}) \geq r \cdot f_D$, then for every $r' < r$ we have $K \models \neg(f_\phi(\overline{b}) \leq r' \cdot f_D)$. That is, for every $r' < r$, $r' \notin \{q \in \mathbb{Q} : K \models f_\phi(\overline{b}) \leq q \cdot f_D\}$, and so $r' \leq \mu_D(\phi(\overline{x},\overline{b}))$ for every $r' < r$. This implies that $\mu(\phi(\overline{x};\overline{b}) \geq r$.

□

Example 2.1.2. *Notice however that there can be some "infinitesimal disagreement" between $\mu_D(\phi(\overline{x};\overline{b})$ and $f_\phi(\overline{b})$ in K:* let $r = \frac{p}{q} \in \mathbb{Q} \cap (0,1)$. We can define the structures M_n in the language $L = \{P\}$ where P is an unary predicate, given by $(M_n, P(M_n)) = (\{1,2,\ldots,q \cdot n\}, \{1,\ldots,pn+1\})$. Then we have that $f_P^{M_n} = \frac{pn+1}{qn} = \frac{p}{q} + \frac{1}{n}$.

So, $K_n \models \neg f_P^{M_n} \leq r$ for every $n < \omega$, and by Łoś' theorem, $K \models \neg(f_P \leq r)$. However,

$$\mu(P(M)) = \inf\{r' \in \mathbb{Q} : K \models f_P \leq r'\} = \inf\left\{r + \frac{2}{n} : n \in \mathbb{N}\right\} = r.$$

Exercise 2.1.3. Show that this "infinitesimal disagreement" cannot occur with an irrational number, that is, if $\mu(\phi(x,\overline{b})) = \alpha \notin \mathbb{Q}$, then

$$\{r \in \mathbb{Q} : \mu(\phi(x,\overline{b})) \leq r\} = \{r \in \mathbb{Q} : K \models f_\phi(\overline{b}) \leq r\}$$

Goldbring and Towsner developed an extension of the first order logic known as *approximate measure logic* where they take this "infinitesimal variation" into consideration.

2.2 The pseudofinite dimensions

In this section we show that the logarithm of a non-standard finite set behaves like a dimension theory, as soon as one factors out a non-trivial convex subgroup of the non-standard reals. This dimension operator will be called the *pseudofinite dimension* and we show here its main features.

Definition 2.2.1. A non-empty subset $S \subseteq \mathbb{R}^*$ is said to be *convex* if whenever $s_1, s_2 \in S$ and $s_1 < r < s_2$, then we also have $r \in S$.

Example 2.2.2. 1. Any non-empty interval $(a,b) := \{x \in \mathbb{R}^* : a < x < b\}$ with $a, b \in \mathbb{R}^* \cup \{+\infty, -\infty\}$ is a convex subset of $\mathbb{R}^*$

2. Given $r \in \mathbb{R}^*$, the *monad of* r defined as $S_r := \{x \in \mathbb{R}^* : x \in (r - \frac{1}{n}, r + \frac{1}{n})$ for all $n \in \mathbb{N}\}$ is a convex subset of $\mathbb{R}^*$, but it is not an interval.

Example 2.2.3. The following are examples of convex subgroups of $(\mathbb{R}^*, +)$:

1. The trivial subgroup $C = \{0\}$.

2. The group of *infinitesimals*, namely, the monad of 0 in $\mathbb{R}^*$. This is the only monad which is also a subgroup of $(\mathbb{R}^*, +)$.

3. Given a non-empty subset A of $\mathbb{R}^*$, we can consider the *convex hull of* A to be

$$\mathrm{Conv}(A) = \bigcap \{C \leq \mathbb{R}^* : C \text{ is a convex subgroup and } A \subseteq C.\}.$$

   It is clear that this is the smallest convex subgroup of $\mathbb{R}^*$ that contains A. The main example of this kind is the subgroup $\mathrm{Conv}(\mathbb{Z}) = \mathrm{Conv}(\mathbb{R})$.

Proposition 2.2.4. *Let $\alpha \in \mathbb{R}^*$, $\alpha > 0$.*

1. *There exists a convex subgroup C_α which is the smallest convex subgroup of $\mathbb{R}^*$ containing α.*

2. *There exists a convex subgroup $C_{<\alpha}$ which is the largest convex subgroup of $\mathbb{R}^*$ which does not contain α.*

3. *There is a unique isomorphism of rings $\phi : C_\alpha / C_{<\alpha} \to \mathbb{R}$ such that $\phi(\overline{\alpha}) = 1$.*

Proof. 1. Consider the group

$$C_\alpha := \mathrm{Conv}(\{\alpha\}) = \bigcap \{S \leq \mathbb{R}^* : S \text{ is a convex subgroup of } \mathbb{R}^* \text{ and } \alpha \in S\}.$$

   It is clear that C_α is the smallest convex subgroup of $\mathbb{R}^*$ containing α, but we write the proof here for the sake of completeness.

   Clearly $\alpha \in C_\alpha$ since α belongs to every member in the intersection. Also, if $x, y \in C_\alpha$, then $x, y \in S$ for every member in the intersection, and we have that $x + y, -x \in S$ for every S, which prove that S is a subgroup of $(\mathbb{R}^*, +)$. Finally, if $s_1 < r < s_2$ and s_1, s_2 are in the intersection, then $s_1, s_2 \in S$ for each S, and since each of the sets in the intersection is convex, $r \in S$ for every S. So, C_α is also convex.

2. Define $C_{<\alpha} = \{x \in \mathbb{R}^* : n \cdot |x| < \alpha \text{ for every } n \in \mathbb{N}\}$. We have that $\alpha \notin C_{<\alpha}$ since $1 \cdot \alpha \not< \alpha$. Also, if $s_1 < r < s_2$ with $s_1, s_2 \in C_\alpha$, we have for every $n \in \mathbb{N}$ that:

$$n \cdot |r|, n \cdot |s_1 + s_2|, n \cdot |-s_1| \leq 2n \cdot \max\{|s_1|, |s_2|\} < \alpha,$$

   which shows that $C_{<\alpha}$ is a convex subgroup of $\mathbb{R}^*$.

   Now, suppose that $C_{<\alpha} \subsetneq C$ where C is a convex subgroup of $\mathbb{R}^*$. Then, there is some positive $x \in C$ such that $n \cdot x \geq \alpha$, but in this case we have $0 < \alpha < (n+1) \cdot x$ and since both 0 and $(n+1) \cdot x$ belong to C, we conclude that $\alpha \in C$. Thus, $C_{<\alpha}$ is the largest convex subgroup of $\mathbb{R}^*$ not containing α.

3. Consider the map $\varphi : C_\alpha \to \mathbb{R}$ given by $\varphi(\beta) = \sup\{q \in \mathbb{Q} : q \leq \frac{\beta}{\alpha}\} \in \mathbb{R}$.

   First, we leave as an exercise to show that φ is a ring homomorphism. Second, notice that φ is surjective: if $r \in \mathbb{R}$ then $n < r < n+1$ for some $n \in \mathbb{Z}$, and so $n \cdot \alpha < r \cdot \alpha < (n+1) \cdot \alpha$. This shows that $r \cdot \alpha \in C_\alpha$ since it is a convex subgroup containing α and all its integer multiples. Now, we have $\varphi(r \cdot \alpha) = \sup\{q \in \mathbb{Q} : q \leq \frac{r \cdot \alpha}{\alpha} = r\} = r$.

   The kernel of the homomorphism φ is given by

$$\begin{aligned} \ker \varphi = \{x \in C_\alpha : \varphi(x) = 0\} &= \left\{x \in C_\alpha : -\frac{1}{n} < \frac{x}{\alpha} < \frac{1}{n} \text{ for all } n \in \mathbb{N}\right\} \\ &= \{x \in C_\alpha : n \cdot |x| < \alpha\} = C_{<\alpha}. \end{aligned}$$

   Thus, by the isomorphism theorem for rings, we have there is an isomorphism $\phi = \overline{\varphi} : C_\alpha / C_{<\alpha} \to \mathbb{R}$ with $\phi(\overline{\alpha}) = \varphi(\alpha) = 1$.

□

In some sense, the different convex subgroups of $\mathbb{R}^*$ correspond to different "orders of magnitude": if $C_1 \subsetneq C_2$ and $\alpha \in C_1, \beta \in C_2 \setminus C_1$, then α is infinitesimally smaller than β. This idea will play a key role in the next section when we consider the *pseudofinite dimension*

Note also that if C is a convex proper subgroup of $\mathbb{R}^*$, then the quotient $\mathbb{R}^*/C$ is an abelian ordered group, with the order given by $\overline{x} < \overline{y}$ in $\mathbb{R}^*/C$ if and only if $x < y$ in $\mathbb{R}^*$.

Definition 2.2.5. Let $M = \prod_{\mathcal{U}} M_i$ be an ultraproduct of finite structures, and let C be a convex subgroup of $\mathbb{R}^*$ containing $\mathbb{Z}$.. For a given $A \subseteq M$ non-empty definable subset, we define the *pseudofinite dimension of* A (with respect to C) as:

$$\delta_C(A) = \log |A| + C,$$

that is, the image of $\log |A|$ under the canonical projection of $\mathbb{R}^*$ onto $\mathbb{R}^*/C$.

Remark 2.2.6. The hypothesis that C contains $\mathbb{Z}$ ensures that finite sets have dimension zero, allowing δ to be a non-trivial dimension operator: if C does not contain $\mathbb{Z}$, then C would be contained in the convex subgroup of the infinitesimals, and for instance, that $\delta(M \setminus \{a\}) < \delta(M)$ for any $a \in M$, which would be absurd.

Notice that this dimension operator does not take integer values, but rather in the group $\mathbb{R}^*/C$. The following proposition provided some evidence to consider the non-integer valued function δ_C as a dimension operator.

Proposition 2.2.7. *Let $M = \prod_{\mathcal{U}} M_i$ be an ultraproduct of finite structures, and let A, B definable subsets Then:*

1. *If $A \subseteq B$, then $\delta_C(A) \leq \delta_C(B)$.*
2. *For any non-empty finite set X, $\delta(X) = 0$.* [1]

[1]We can extend the dimension to the empty set by using the notation $\delta_C(\emptyset) = -\infty$

3. $\delta_C(A \times B) = \delta_C(A) + \delta_C(B)$.

4. $\delta_C(A \cup B) = \max\{\delta_C(A), \delta_C(B)\}$

5. *If $f : X \to Y$ is a definable function, then $\delta(f(X)) \le \delta(X)$. In particular, if X is a definable bijection, $\delta(X) = \delta(Y)$.*

6. Subadditivity: *Let X, Y be definable subsets and $f : X \to Y$ be a definable surjective function such that for all $\overline{b} \in Y$, $\delta_C(f^{-1}(\overline{b})) \le \beta$ for some $\beta \in \mathbb{R}^*/C$. Then, $\delta(X) \le \delta(Y) + \beta$.*

Proof. Assertion (1) follows directly because log is an increasing function. For (2), notice that if $X = \{\overline{a}_1, \dots, \overline{a}_m\} \subseteq M^n$, then $\delta_C(X) = \log\ |X| + C = C = 0$ since $\log|X| = \log m \le m \in C$. For (3), assume $A = \prod_{\mathcal{U}} A_i$ and $B = \prod_{\mathcal{U}} B_i$. and notice that for every index i we have $\log(|A_i \times B_i|) = \log|A_i| + \log|B_i|$, obtaining

$$\delta_C(A \times B) = \log|A \times B| + C = \log|A| + \log|B| + C = \delta_C(A) + \delta_C(B).$$

For (4), suppose without loss of generality that for an $\mathcal{U}$-large set of indices we have $|A_i| \ge |B_i|$. Then we have $|A_i \cup B_i| \le 2|A_i|$ for $\mathcal{U}$-almost all indices i, obtaining:

$$\begin{aligned}
\log|A_i| \le \log|A_i \cup B_i| &\le \log(2 \cdot |A_i|) = \log 2 + \log|A_i| \\
\log|A| \le \log|A \cup B| &\le \log 2 + \log|A| \\
\log|A| + C \le \log|A \cup B| + C &\le \log 2 + \log|A| + C = \log|A| + C \\
\delta_C(A) \le \delta_C(A \cup B) &\le \delta_C(A)
\end{aligned}$$

because $\log(2 \cdot |A|) - \log|A| = \log 2 \in C$.

For (5), let $X = \prod_{\mathcal{U}} X_i$ and $Y = \prod_{\mathcal{U}}(Y_i)$. By counting in the finite structures, we have that for $\mathcal{U}$-almost all i, $|f(X_i)| \le |X_i|$, and so $|f(X)| \le |X|$ which implies

$$\delta_C(f(X)) = \log|f(X)| + C \le \log|X| + C = \delta_C(X).$$

Finally, for (6), suppose that X, Y are definable subsets and $f : X \to Y$ is a definable surjective function such that for all $\overline{b} \in Y$, $\delta_C(f^{-1}(\overline{b})) \le \beta$ for some constant $\beta \in \mathbb{R}^*/C$. Let r denote the element in $\mathbb{R}^*$ such that $\beta = r + C$. Suppose $X = \prod_{\mathcal{U}} X_i$ and $Y = \prod_{\mathcal{U}} Y_i$. Then for $\mathcal{U}$-almost all i there is a definable surjective function $f_i : X_i \to Y_i$, and we can choose an element $\overline{b}_i^* \in Y_i$ such that $|f_i^{-1}(\overline{b}_i)|$ is maximal. Then, by counting in the finite structures, we have:

$$|X_i| = \left| \bigcup_{\overline{b}_i \in Y_i} f_i^{-1}(\overline{b}_i) \right| = \sum_{\overline{b}_i \in Y_i} |f^{-1}(\overline{b}_i) \le |f^{-1}(\overline{b}_i^*)| \cdot |Y_i|.$$

Let $\overline{b}^* = [\overline{b}_i^*]_{\mathcal{U}} \in Y$. By hypothesis, since $\delta_C(f^{-1}(\overline{b}^*)) \le \beta$, there is $c \in C$ such that $\log|f^{-1}(\overline{b}^*)| \le r + c$. Then we have the following:

$$\begin{aligned}
|X| &\le |f^{-1}(\overline{b}^*)| \cdot |Y| \\
\log|X| &\le \log|f^{-1}(\overline{b}^*)| + \log|Y| \\
\log|X| &\le (r + \log|Y| + c) \\
\delta_C(X) = \log|X| + C &\le (r + C) + (\log|Y| + C) = \beta + \delta_C(Y).
\end{aligned}$$

∎

In principle, for every different convex subgroup C of $\mathbb{R}^*$ there would be a different notion of pseudofinite dimension, and this will allow us to distinguish between various degrees of graininess. However, there are two main examples that will appear later in this notes which correspond to different special convex subgroups of $\mathbb{R}^*$.

The first one is when we consider C to be the minimal interesting example: the convex hull of the standard reals, denoted C_{fin}. We thus denote the corresponding pseudofinite dimension by δ_{fin}.

Remark 2.2.8. In general, the map $\overline{a} \mapsto \delta_{\text{fin}}(\phi(\overline{x};\overline{a}))$ is not definable even in the language L^+, since $C = \text{Conv}(\mathbb{Z})$ and hence $\mathbb{R}^*/C$ are not definable.

The characteristic feature of δ_{fin} is that every possible value α for the dimension comes with a real-valued measure μ_α, defined up to a scalar multiple. Such measure is characterized by putting $\mu_\alpha(X) = 0$ iff $\delta_{\text{fin}}(X) < \alpha$, $\mu_\alpha(X) = \infty$ iff $\delta(X) > \alpha$, and when $\delta_{\text{fin}}(X) = \delta_{\text{fin}}(Y) = \alpha$, $\mu_\alpha(X) = \text{st}\left(\frac{|X|}{|Y|}\right) \cdot \mu_\alpha(Y)$.

In particular, we have the following:

Proposition 2.2.9. *Suppose X, Y are definable sets. Then $\delta_{\text{fin}}(X) = \delta_{\text{fin}}(Y)$ if and only if there is a natural number n such that $\frac{1}{n} < \frac{|X|}{|Y|} < n$.*

Proof. We have $\delta_{\text{fin}}(X) = \delta_{\text{fin}}(Y)$ if and only if $\log|X|/\,\text{Conv}(\mathbb{Z}) = \log|Y|/\,\text{Conv}(\mathbb{Z})$, if and only if $\log\left(\frac{|X|}{|Y|}\right) \log|X| - \log|Y| \in \text{Conv}(\mathbb{Z})$. So, there is a positive integer k such that:

$$-k \leq \log\left(\frac{|X|}{|Y|}\right) \leq k$$
$$e^{-k} \leq \frac{|X|}{|Y|} \leq e^k$$

and it suffices to pick n bigger than e^k to have the desired inequality. □

Corollary 2.2.10. *In the language of Section 2.1, given $X \subseteq Y$, we have $\delta_{\text{fin}}(X) < \delta_{\text{fin}}(Y)$ if and only if $\mu_Y(X) = 0$.*

On the other hand, if we have in mind some definable set X, with $\alpha = \log\,|X|$, let $C_{<\alpha}$ be the maximal convex subgroup of $\mathbb{R}^*$ not containing α, and C_α be the smallest convex subgroup containing α. If we restrict our attention to definable sets Y with $\log|Y| \in C_\alpha$, then the corresponding dimension theory can be viewed as real valued, using the natural isomorphisms $C_\alpha/C_{<\alpha} \to \mathbb{R}$ mapping α to 1. It can be defined directly:

$$\delta_\alpha(Y) = \text{st}\left(\frac{\log|Y|}{\alpha}\right).$$

A particular but important example of δ_α is when we consider α to be $\log|M|$:

Definition 2.2.11. Let $M = \prod_{\mathcal{U}} M_i$ be a pseudofinite structure, and $A \subseteq M$ be a non-empty definable subset. We defined the *normalized pseudofinite dimension* of A $\delta_M(A)$, (or $\delta_{\mathcal{C}_0}(A)$ following notation from [16]) to be

$$\mathrm{st}\left(\frac{\log|A|}{\log|M|}\right) \in [0,1].$$

Alternitavely, if $A = \prod_{\mathcal{U}}(A_i)$ and we put $\ell_i = \dfrac{\log|A_i|}{\log|M_i|}$, (so that $|A_i| = |M_i|^{\ell_i}$), then $\delta_M(A)$ can be also defined as $\delta_M(A) = \lim_{i\to\mathcal{U}} \ell_i$.

Remark 2.2.12. For the normalized pseudofinite dimension, we have some more standard properties of the dimension. For instance, $\delta_\alpha(M^n) = n$, and more generally if $|X| \approx |M|^r$ for some $r \in \mathbb{R}$, then $\delta_M(X) \approx r$.

The map δ is extended to infinitely definable sets in [15] and [16]. For $\epsilon \in \mathbb{R}^*$, chosen sufficiently large and with $\epsilon > C$, we define $V_0 = V_0(\epsilon)$, to be the smallest convex subgroup of $\mathbb{R}^*/C$ containing ϵ.

Lemma 2.2.13. *Let $V = V(\epsilon)$ be the set of cuts in V_0, i.e., non-empty subsets bounded above a closed downwards. Then $V(\epsilon)$ is a semigroup, under set addition, linearly ordered by inclusion.*

Proof. $(V, +)$ is clearly a semigroup, because addition in $\mathbb{R}^*/C$ is associative. Now, let r, s be cuts in V. If $s \neq r$, we may assume without loss of generality that there is $a \in s \setminus r$ (recall that $r, s \subseteq V_0$).

Since r is closed downwards, it follows that $x < a$ for all $x \in r$. Now, since S is closed downwards, $r \subseteq \{x \in V_0 : x < a\} \subseteq S$, and we conclude that $r \leq s$. □

The set V_0 embeds into V, by the map $a \mapsto \{v : v \leq a\}$. We can identify V_0 with its image in V under this map, and we can conclude now that any subsets of V that is bounded below has an infimum in V:

Proof. Let $S \subseteq V$, bounded below by a (which means that for all $s \in S$, $a \subseteq s$). Let $\alpha = \bigcap S = \bigcap_{s\in S} s$. Note that $\alpha \neq \emptyset$, because since a is a lower bound of S, $a \subseteq s$ for all $s \in S$, and since S is closed downwards, $\{v : v \leq a\} \subseteq \alpha$.

- *α is a cut:* if $b \in \mathbb{R}^*/C$ is an upper bound for $s \in S$, it is also an upper bound for $\alpha \subseteq s$ So, α is bounded above.

  Now, if $x < y$ and $y \in \alpha$, then $y \in s$ for all $s \in S$, which implies $x \in s$ for all $s \in S$, because all elements in S are closed downwards. Thus, $x \in \bigcap_{s\in S} s = \alpha$.

- $\alpha = \inf\{s : s \in S\}$: Assume β is a lower bound for S. Then, $\beta \leq s$ for all $s \in S$ (with the order in V), so $\beta \subseteq s$ for all $s \in S$, and we have that $\beta \subseteq \bigcap_{s\in S} s = \alpha$, which proves that $\beta \leq \alpha$ with the order given in V.

□

Definition 2.2.14. For a $\bigwedge$-definable set X, define

$$\delta(X) := \inf\{\delta(D) : D \supseteq X, D \text{ definable}\},$$

the infimum evaluated in $V(\epsilon)$ for sufficiently large ϵ. Given $B \subseteq M$ and a tuple $\overline{a}$ from M, $\delta(\overline{a}/B)$ denotes $\delta(\mathrm{tp}(\overline{a}/B))$, and $\delta^{\phi}(\overline{a}/B)$ denotes $\delta(\mathrm{tp}^{\phi}(\overline{a}/B))$, that is, the dimension of the corresponding partial positive ϕ-type.

Lemma 2.2.15. *Properties for the dimension, even with $\bigwedge$-definable sets.*

1. *$\delta(\emptyset) = -\infty$, and $\delta(X) = 0$ for any finite definable set X.*

2. *If X_1, X_2 are $\bigwedge$-definable, then $\delta(X_1 \cup X_2) = \max\{\delta(X_1), \delta(X_2)\}$.*

3. *If X_1, X_2 are $\bigwedge$-definable, then $\delta(X_1 \times X_2) = \delta(X_1) + \delta(X_2)$.*

4. *If $(\alpha_n), (\beta_n)$ are decreasing sequences of cuts in V_0, then*

$$\inf_n(\alpha_n + \beta_n) = \inf_n \alpha_n + \inf_n \beta_n.$$

5. *If $\alpha, \alpha', \beta, \beta' \in V$ with $\alpha < \alpha'$ and $\beta < \beta'$, then $\alpha + \beta < \alpha' + \beta'$.*

6. *If $X = \bigcap_n X_n$ with $X_1 \supseteq X_2 \supseteq \cdots$ all $\bigwedge$-definable, then $\delta(X) = \inf_n \delta(X_n)$.*

7. *If X is $\bigwedge$-definable, f is a definable map and $\delta(f^{-1}(a) \cap X) \leq \gamma$ for some $\gamma \in V_0$ and all a, then $\delta(X) \leq \delta(f(X)) + \gamma$.*

Proof. 1. By convention, $\log 0 = \log|\emptyset| = -\infty$. Now, if X is finite, there is $k \in \mathbb{Z}$ such that $|X| \leq k$, and so $\log|X| \leq \log k \leq n$ for some $n \in \mathbb{Z}$, so $\log|X| \in \mathrm{Conv}(\mathbb{Z})$ which implies $\delta(X) = 0$.

2. We have already proved this for definable sets, so we show it here for $\bigwedge$-definable sets. Since $X_1 \cup X_2$ contains both X_1, X_2, we have that $\delta(X_1 \cup X_2) \geq \max\{\delta(X_1), \delta(X_2)\}$. Suppose now without loss of generality that $\delta(X_1) > \delta(X_2)$. Then, there is a definable set $D \supseteq X_2$ such that $\delta(D) < \delta(X_1)$.

   So, $X_1 \cup X_2 \subseteq X_1 \cup D$, and we have

$$\begin{aligned}
\delta(X_1 \cup X_2) &\leq \delta(X_1 \cup D) \\
&= \inf\ \{\delta(E) : E \supseteq X_1 \cup D,\ E \text{ definable}\} \\
&= \inf\{\delta(E' \cup D) : E' \supseteq X_1,\ E' \text{ definable}\} && (E' = E \cup D) \\
&= \inf\{\max\{\delta(E'), \delta(D)\} : E' \supseteq X_1,\ E' \text{ definable}\} \\
&= \inf\{\delta(E') : E' \supseteq X_1, E' \text{ definable}\} && (\text{because } \delta(D) < \delta(X_1) \leq \delta(E')) \\
&= \delta(X_1).
\end{aligned}$$

3. We have already showed that $\delta(X_1 \times X_2) = \delta(X_1) + \delta(X_2)$ for definable sets X_1, X_2. Now, if we assume $X_1 = \bigwedge_i X_1^i$ and $X_2 = \bigwedge_i X_2^i$ then we have $X_1 \times X_2 = \bigwedge_{i,j}(X_1^i \times X_2^j)$, and so

$$\begin{aligned}
\delta(X_1 \times X_2) &= \inf_{i,j}\{\delta(X_1^i \times \delta(X_2^j)\} \\
&= \inf_{i,j}\{\delta(X_1^i) + \delta(X_2^j)\} \\
&= \inf_i\{\delta(X_1^i\} + \inf_j\{\delta(X_2^j)\} && \text{(by (4) below)} \\
&= \delta(X_1) + \delta(X_2).
\end{aligned}$$

4. Since $\inf_n(\alpha_n) + \inf_n(\beta_n)$ is a lower bound for $\{\alpha_b + \beta_n : n < \omega\}$, we have that

$$\inf_n(\alpha_n) + \inf_n(\beta_n) \leq \inf_n(\alpha_n + \beta_n).$$

Suppose now the inequality is strict. Then if $\alpha = \inf_n \alpha_n$ and $\beta = \inf_n \beta_n$ we have $\alpha + \beta < \inf_n(\alpha_n + \beta_n)$, that is, $\inf_n \beta_n = \beta < \inf_n(\alpha_n + \beta_n) - \alpha$, and so there is β_m such that $\beta_m < \inf(\alpha_n + \beta_n) - \alpha$.

We would have then $\alpha < \inf(\alpha_n + \beta_n) - \beta_m$, and so there is some α_k such that $\alpha_k < \inf(\alpha_n + \beta_n) - \beta_m$. Since $(\alpha_n), (\beta_n)$ are both decreasing, then if $m \geq k$ we conclude that $\alpha_m + \beta_m < \inf(\alpha_n + \beta_n)$, a contradiction.

So, $\inf_n(\alpha_n + \beta_n) = \inf_n(\alpha_n) + \inf_n(\beta_n)$.

□

2.3 Fine pseudofinite dimension and forking

Throughout the rest of this chapter, we will fork with the pseudofinite dimension $\delta = \delta_{\text{fin}}$. During this section, we will show that as well as several examples of dimension in different model-theoretic contexts, the pseudofinite dimension is able to detect instances of dividing in the ultraproducts of pseudofinite structures.

This will lead to establish conditions under which the natural notion of dimension provided by the pseudofinite dimension operator is equivalent to non-forking independence, or conditions that will ensure the ultraproducts are simple or supersimple.

2.3.1 Dividing and drop of pseudofinite dimension

In showing that dividing can be detected by the fine pseudofinite dimension, the following lemma will be very important. This result roughly states that in a probability space, every infinite collection of sets with a fix positive measure have to start accumulating with positive measure.

Proposition 2.3.1. *Let Ω be a measure space with $\mu(\Omega) = 1$, and fix $0 < \epsilon \leq \frac{1}{2}$. Let $(A_i : i < \omega)$ be a sequence of measurable subsets of Ω such that $\mu(A_i) \geq \epsilon$ for every $i < \omega$.*

Then, for every $k < \omega$, there are indices $i_1 < \ldots < i_{2^k}$ such that

$$\mu\left(\bigcap_{j=1}^{2^k} A_{i_j}\right) \geq \epsilon^{3^k}.$$

Proof. By induction on k. For $k = 1$, we have to find indices $i_1 < i_2$ such that $\mu(A_{i_1} \cap A_{i_2}) \geq \epsilon^3$.

Suppose not, then for all $i \neq j$ we have $\mu(A_i \cap A_j) < \epsilon^3$. By the truncated inclusion-exclusion, we know that for every $N \in \mathbb{N}$,

$$\begin{aligned}
\mu\left(\bigcup_{i=1}^{N} A_i\right) &\geq \sum_{i=1}^{N} \mu(A_i) - \sum_{1\leq i<j\leq N} \mu(A_i \cap A_j) \\
&\geq \epsilon \cdot N - \binom{N}{2}\epsilon^3 \\
&= -\frac{N^2}{2}\epsilon^3 + N\left(\epsilon + \frac{\epsilon^3}{2}\right).
\end{aligned}$$

The function $f(x) = -\frac{x^2}{2}\epsilon^3 + x\left(\epsilon + \frac{\epsilon^3}{2}\right)$ achieve its maximum at $x_0 = \frac{1}{\epsilon^2} + \frac{1}{2} > 0$, and if $N \in [x_0 - 1, x_0]$, we obtain

$$\begin{aligned}
\mu\left(\bigcup_{i=1}^{N} A_i\right) &\geq f(N) \leq f(x_0 - 1) \\
&= \frac{1}{2\epsilon} + \frac{\epsilon}{2} - \frac{3}{8}\epsilon^3 \\
&\geq 1 + \epsilon\left(\frac{1}{2} - \frac{3}{8}\epsilon^3\right) \qquad \text{(because } \epsilon \leq 1/2\text{)} \\
&> 1,
\end{aligned}$$

a contradiction. So, there are $i_1 < i_2$ such that $\mu(A_{i_1} \cap A_{i_2}) \geq \epsilon^1$.

Now, by induction hypothesis, there is a tuple $(i_1, \ldots, i_k)$ with $i_1 < \ldots < i_{2^k}$ and

$$\mu\left(\bigcap_{j=1}^{2^k} A_{i_j}\right) \geq \epsilon^{3^k}.$$

In fact, there are infinitey many of such tuples: otherwise, by taking ℓ to be the maximum of all indices appearing in the tuples $(i_1, \ldots, i_k)$, then $(A_j : j \leq \ell + 1)$ would contradict the induction hypothesis.

Let $(\alpha_j : j < \omega)$ be an enumeration of all tuples satisfying (*) and put $B_j = \bigcap_{i\in\alpha_j} A_i$. By construction, $\mu(B_j) \geq \epsilon^{3^k}$ for all $j < \omega$. By the case $k = 2$, there are $j_1 \neq j_2$ with $\mu(B_{j_1} \cap B_{j_2}) \geq (\epsilon^{3^k})^3 = \epsilon^{3^k \cdot 3} = \epsilon^{3^{k+1}}$. In particular, there are at least $k + 1$ indices $i_1 < \cdots < i_k < i_{k+1}$ such that

$$\mu\left(\bigcap_{j=1}^{k+1} a_{i_j}\right) \geq \mu(B_{j_1} \cap B_{j_2}) \geq \epsilon^{3^{k+1}}.$$

□

Theorem 2.3.2. *Let $\psi(\overline{x}, \overline{a})$ be a formula over A, and $\phi(\overline{x}, \overline{b})$ a formula implying $\psi(\overline{x}, \overline{a})$ that divides A. Then, there is an element $\overline{b}' \equiv_A \overline{b}$ such that $\delta(\phi(\overline{x}, \overline{b}')) < \delta(\psi(\overline{x}, \overline{a})$.*

Proof. Let D be the set defined by $\psi(\overline{x}, \overline{a})$, and suppose the result does not hold. Then for every $\overline{b}'$ with the same type of $\overline{b}$ over A we have $\delta(\phi(\overline{x}, \overline{b}')) = \delta(D)$, and so there is a natural number $n_{\overline{b}'}$ such that $|\phi(\overline{x}, \overline{b}')| \geq \frac{1}{n_{\overline{b}'}}|D|$.

By compactness, there is a uniform $n \in \mathbb{N}$ such that $|\phi(\overline{x}, \overline{b}')| \geq \frac{1}{n}|D|$, since otherwise, the L^+-type given by $\Gamma(\overline{y}) = \mathrm{tp}(\overline{b}/A) \cup \{|\phi(\overline{x}, \overline{y})| \cdot n < |D| : n < \omega\}$ would be realized in M, and the realization $\overline{b}'$ will satisfy $\delta(\phi(\overline{x}, \overline{b}')) < \delta(D)$.

Now, since $\phi(\overline{x}, \overline{b})$ divides over A, there is an A-indiscernible sequence $(\overline{b}_i : i < \omega)$ in $\mathrm{tp}(\overline{b}/A)$ such that the set $\{\phi(\overline{x}, \overline{b}_i) : i < \omega\}$ is k-inconsistent for some $k < \omega$.

Consider now the probability measure given by μ_D, and let $A_i := \phi(\overline{x}, \overline{b}_i)$ for $i < \omega$. By the previous consideration, $\mu_D(\phi(\overline{x}, \overline{b}_i) \geq \frac{1}{n}$ for every $i < \omega$, and by Proposition 2.3.1 we have that there are indices $i_1 < \ldots < i_{2^k}$ such that $\mu\left(\bigcap_{j=1}^{2^k} A_i\right) \geq \left(\frac{1}{n}\right)^{3^k} > 0$. In particular, $\{\phi(\overline{x}, \overline{b}_{i_1}), \ldots, \phi(\overline{x}, \overline{b}_{i_k})\}$ is consistent, which is a contradiction. □

The theorem above allows us to conclude that the number of possible different values for pseudofinite dimensions of definable sets is a bound for the length of dividing chains, providing also a bound for the U-rank in types. We will explore this idea in Section 2.4.

We might think about two possible generalizations of Theorem 2.3.2: either changing dividing by forking or showing that the original formula (instead of replacing the parameters by a conjugate) has lower pseudofinite dimension. The following two examples who limitations for these attempts:

Example 2.3.3. Consider the class of finite structures $M_n = ([1, 2^n], E_n)$ where E_n is an equivalence relation with classes

$$[1, 2^{n-1}, 1], [2^{n-1}, 2^{n-1} + 2^{n-2} - 1], [2^{n-1} + 2^{n-2}, 2^{n-1} + 2^{n-2} + 2^{n-3} - 1], \\ \ldots, [2^{n-1} + 2^{n-2} + \cdots + 2^2, 2^n]$$

The idea here is that M_n is a set with a equivalence relations $E_1, E_2, \ldots, E_n$ with sizes

$$|E_1| = \frac{1}{2}|M_n|, |E_2| = \frac{1}{4}|M_n|, \ldots, |E_n| = \frac{1}{2^{n-1}}|M_n| \geq 1.$$

Let $M = \prod_{\mathcal{U}} M_n$ and $b = [(1, 1, \ldots)]_{\mathcal{U}}$. In the ultraproduct M the relation symbol is interpreted as an equivalence relation with infinitely many infinite classes, and so the formula xEb divides over the empty set. Theorem 2.3.2 shows that there is a conjugate b' of b over $\emptyset$ such that the formula xEb' witnesses the drop of pseudofinite dimension. However, this drip is not witnessed by the formula xEb because

$$\log|M_n| - \log|xE_n 1| = \log(2^n) - \log(2^{n-1}) = \log 2 < 1$$

which implies that $\delta(M) = \delta(xEb)$.

Example 2.3.4. This examples is an adaptation of the classical example of the circular order that shows that the formula $x = x$ may fork over the empty set. Consider the structure $M_n = (\mathbb{Z}/(3n)\mathbb{Z}, R)$ where R is a ternary relation interpreted in M_n as follows: $M_n \models R(b, a, c)$ if and only if there are integers a', b', c' congruent to $a, b, c \pmod{3n}$ respectively, such that $a' < b' < c'$ and $|c' - a'| < n$. [2] Take $M = \prod_{\mathcal{U}} M_n$, and the elements $a := [a_n = 0]_{\mathcal{U}}, b := [b_n = n]_{\mathcal{U}} \in M$.

Claim: *The formula $R(x; a, b)$ divides over $\emptyset$.*

Proof of the Claim: On each M_n consider the sequence given by

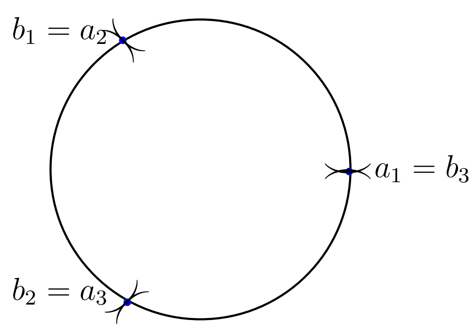
$$\left\langle (a_i^n, b_i^n) = (n + k \cdot [\![\log n]\!], n + (k+1) \cdot [\![\log n]\!]) : k \leq \frac{n}{\log n} \right\rangle,$$

and consider in the ultraproduct the sequence given by $\langle (a_i, b_i) = ([a_i^n]_{n\in\mathcal{U}}, [b_i^n]_{n\in\mathcal{U}}) : i < \omega \rangle$. This is a sequence in $tp(a, b/\emptyset)$ which is indiscernible over the empty set, and by construction we have that the set of formulas $\{R(x; a_i, b_i) : i < \omega\}$ is 2-inconsistent. $\square_{\text{Claim}}$

Consider the elements in the ultraproduct M given by $a_1 := [a_1^n = 0]_{\mathcal{U}}$, $a_2 := [a_2^n = n]_{\mathcal{U}} = b_1$ and $a_3 := [a_3^n = 2n]_{\mathcal{U}} = b_2$ and $b_3 = a_1$. Note that the formula $x = x$ forks over $\emptyset$, because it implies the disjunction

$$\bigvee_{i=1}^{3} R(a_i, x, b_i) \vee \bigvee_{i=1}^{3} x = a_i$$

of formulas that divide over $\emptyset$.



However, the set of realizations of the formula of $x = x$ is M and it does not witness any drop of pseudofinite dimension ($\delta(M)$ is the maximal value of the pseudofinite dimension among subsets of M).

Even if Theorem 2.3.2 only applies for dividing formulas, there are natural settings where dividing is equivalent to forking. For example forking and dividing over arbitrary sets are equivalent in simple theories , and they are also equivalent over models in theories with NTP_2 [5].

[2]These structures are intended to realize the circular order in the ultraproduct.

2.3.2 Conditions on the fine pseudofinite dimension

Definition 2.3.5. The following are conditions on the fine pseudofinite dimension.

1. *Attainability* (A_ϕ). There is no sequence $(p_i : i \in \omega)$ of finite partial positive ϕ-types such that $p_i \subseteq p_{i+1}$ (as sets of formulas) and $\delta(p_i) > \delta(p_{i+1})$ for each $i \in \omega$. We denote by (A^*_ϕ) the corresponding (stronger) condition where the above is assumed for all increasing sequences of finite conjunctions of (possibly negated) ϕ-instances.

2. *Strong attainability* (SA). For each partial type $p(\overline{x})$ over a parameter set B, there is a finite subtype p_0 of p such that $\delta(p(\overline{x})) = \delta(p_0(\overline{x}))$.

3. *Weak order definability* (WOD_ϕ). There is $n = n_\phi \in \mathbb{N}$ such that for all $\overline{a}, \overline{b} \in M^s$,

$$\delta(\phi(\overline{x}, \overline{a})) = \delta(\phi(\overline{x}; \overline{b})) \Leftrightarrow \frac{1}{n} < \frac{|\phi(\overline{x}; \overline{a})|}{|\phi(\overline{x}, \overline{a})|} < n.$$

4. *Dimension Comparison in L^+* (DC_{L^+}). For all formulas $\phi(\overline{x}, \overline{y})$ and $\psi(\overline{x}, \overline{z})$ (with $t = l(\overline{z})$, there is an L^+ formula $\chi_{\phi,\psi}(\overline{y}, \overline{z})$ such that for all $\overline{a} \in M^s$ and $\overline{b} \in M^t$,

$$\chi_{\phi,\psi}(\overline{a}, \overline{b}) \Leftrightarrow \delta_{\overline{x}}(\phi(\overline{x}; \overline{a})) \le \delta_{\overline{x}}(\psi(\overline{x}, \overline{b})).$$

5. *Dimension Comparison in L* (DC_L) This is as for (DC_{L^+}), except that the formula $\chi_{\phi,\psi}$ can be chosen in L.

6. *Finitely many values* (FMV_ϕ) There is a finite set $\{\delta_1, \ldots, \delta_k\}$ such that for each $\overline{b} \in M^s$ there is $i \in \{1, \ldots, k\}$ with $\delta(\phi(M^r, \overline{b})) = \delta_i$.

The conditions (A_ϕ),(A^*_ϕ),(WOD_ϕ) and (FMV_ϕ) have global versions (A),(A^*),(WOD) and (FMV), where they are assumed to hold for all ϕ (with k and δ_i in (FMV), and n in (WOD), dependent on ϕ).

We now proceed to present some easy observations about these conditions. Note that

$$(\mathrm{SA}) \Rightarrow (A^*_\phi) \text{ for all } \phi \Rightarrow (A)$$

We also have the following:

Lemma 2.3.6. *1. For every formula $\phi(\overline{x}, \overline{y})$, the conditions $((A_\phi) \wedge (A_{\neg\phi}) \wedge (A_{\phi(\overline{x},\overline{y}_1) \wedge \neg\phi(\overline{x},\overline{y}_2)}))$ and (A^*_ϕ) are equivalent.*

2. The conditions (A) *and* (A^*) *are equivalent.*

Proof. This follows directly from the definitions. □

Lemma 2.3.7. *The conditions* (WOD) *and* (DC_{L^+}) *are equivalent.*

Proof. (DC_{L^+})⇒(WOD): This is by ω_1-saturation of the ultraproduct. Namely, assume (DC_{L^+} and that there is a formula $\phi(\overline{x},\overline{y})$ such that for all $n<\omega$ there are $\overline{a}_n,\overline{b}_n \in M^{|y|}$ with $\delta(\phi(\overline{x};\overline{a}_n)) = \delta(\phi(\overline{x};\overline{b}_n)$ but $\frac{|\phi(\overline{x};\overline{a}_n)|}{|\phi(\overline{x},\overline{b}_n)|} < \frac{1}{n}$.

Consider the set of L^+-formulas given by

$$\Psi(\overline{y}_1,\overline{y}_2) := \{\chi_{\phi,\phi}(\overline{y}_1,\overline{y}_2), \chi_{\phi,\phi}(\overline{y}_2,\overline{y}_1)\} \cup \left\{\frac{|\phi(\overline{x},\overline{y}_1)|}{|\phi(\overline{x},\overline{y}_2)|} < \frac{1}{n}\right\}$$

This set is consistent and countable, so there are $\overline{a},\overline{b} \in M$ realizing Ψ. So, $\models \chi_{\phi,\phi}(\overline{a},\overline{b}) \wedge \chi_{\phi,\phi}(\overline{b},\overline{a})$ and $\delta(\phi(\overline{x},\overline{a})) < \delta(\phi(\overline{x},\overline{b}))$, a contradiction.

(WOD)⇒(DC_{L^+}): Fix formulas $\phi(\overline{x},\overline{y}), \psi(\overline{x},\overline{z})$, and consider the formula $\rho(\overline{x},\overline{yz}ww_1w_2)$ given by

$$\begin{aligned}&[(w=w_1 \wedge w_1 \neq w_2) \to \phi(\overline{x},\overline{y})] \wedge [(w=w_2 \wedge w_1\neq w_2) \to \psi(\overline{x},\overline{z})]\\ &\wedge [(w_1=w_2 \vee (w\neq w_1,w_2)) \to (\neg\phi(\overline{x},\overline{z}) \wedge \neg\psi(\overline{x},\overline{z}))]\end{aligned}$$

By (WOD), there is a number n_ρ associated with ρ. Note that $\phi(\overline{x},\overline{a}) \equiv \rho(\overline{x};\overline{a}\overline{b}w_1w_1w_2)$ and $\psi(\overline{x},\overline{b}) \equiv \rho(\overline{x},\overline{a}\overline{b}w_2w_1w_2)$. Thus,

$$\begin{aligned}&\delta(\phi(\overline{x},\overline{a})) \leq \delta(\psi(\overline{x},\overline{b}))\\ &\Leftrightarrow \text{There are } w_1\neq w_2 \text{ with } \delta(\rho(\overline{x},\overline{a}\overline{b}w_1w_1w_2)) \leq \delta(\rho(\overline{x},\overline{a}\overline{b}w_2w_1w_2))\\ &\Leftrightarrow \text{There are } w_1\neq w_2 \text{ with } \frac{\rho(\overline{x},\overline{a}\overline{b}w_1w_1w_2)}{\rho(\overline{x},\overline{a}\overline{b}w_2w_1w_2)} \leq n_\rho\\ &\Leftrightarrow M \models \underbrace{\exists w_1,w_2\left(w_1\neq w_2 \wedge |\rho(\overline{x},\overline{a}\overline{b}w_1w_1w_2)| \leq n_\rho\cdot|\rho(\overline{x},\overline{a}\overline{b}w_2w_1w_2\right)|}_{\chi_{\phi,\psi}}\end{aligned}$$

So, we conclude that (DC_{L^+}) holds. □

Lemma 2.3.8.

1. *Let $f: X\to Y$ be a definable map in M between definable sets X and Y. Suppose there is a natural number n such that for all $\overline{a},\overline{b}\in Y$, $\frac{1}{n}|f^{-1}(\overline{b})| \leq |f^{-1}(\overline{a})| \leq n|f^{-1}(\overline{b})|$ (non standard cardinalities). Then, for all $\overline{a}\in Y$ we have $\delta(X)=\delta(Y)+\delta(f^{-1}(\overline{a}))$.*

2. *Assume M satisfies (WOD). Let X be a definable set in M, f a definable map and suppose that $\delta(f^{-1}(\overline{a}) = \gamma$ for all $\overline{a}\in f(X)$. Then, $\delta(X)=\delta(f(X))+\gamma$.*

Proof.

1. By counting in the finite structures, note that for every $i\in I$ there are elements $a_i^1, a_i^2 \in Y_i$ such that $|f^{-1}(a_i^1)| \leq |f^{-1}(b)| \leq |f^{-1}(a_i^2)|$ for all $b\in Y_i$. On the other hand, $|X(M_i)| = \sum_{y\in Y(M_i)}|f^{-1}(y)|$ and so we have

$$\begin{gathered}|Y(M_i)|\cdot|f^{-1}(a_i^1)| \leq |X(M_i)| \leq |Y(M_i)|\cdot|f^{-1}(a_i^2)|\\ \log|Y(M_i)| + \log|f^{-1}(a_i^1)| \leq \log|X(M_i)| \leq \log|Y(M_i)| + \log|f^{-1}(a_i^2)|\end{gathered}$$

and in the limit we obtain

$$\log|Y| + \log|f^{-1}(a^1)| \leq \log|X| \leq \log|Y| + \log|f^{-1}(a^2)|$$

Now, given $a \in Y$, we have $|f^{-1}(a^2)| \leq n \cdot |f^{-1}(a)|$ and $|f^{-1}(a^1)| \geq \dfrac{|f^{-1}(a)|}{n}$. Thus,

$$\begin{aligned} &\log|Y| + \log|f^{-1}(a)| - \log n \leq \log|X| \leq \log|Y| + \log|f^{-1}(a)| + \log n \\ \Rightarrow &\delta(Y) + \delta(f^{-1}(a)) \leq \delta(X) \leq \delta(Y) + \delta(f^{-1}(a)). \end{aligned}$$

2. Since $\delta(f^{-1}(a)) = \gamma$ for all γ and (WOD) holds, there is $n = n_{f^{-1}(y)}$ such that for all $a, b \in Y$,

$$\frac{1}{n_\phi}|f^{-1}(a)| \leq |f^{-1}(b)| \leq n_\phi \cdot |f^{-1}(a)|.$$

The result now follows from (1).

□

Lemma 2.3.9. *Assume (A_ϕ) holds. Then there is $m = m_\phi \in \mathbb{N}$ such that there do not exists $\overline{a}_1, \ldots, \overline{a}_m$ so that if $p_i = \{\phi(\overline{x}, \overline{a}_j) : j \leq i\}$ then p_i is consistent and $\delta(p_1) > \delta(p_2) > \cdots > \delta(p_m)$.*

Proof. This follows again by compactness and ω_1-saturation of M. If the result does not hold, then for every $N < \omega$ there are $\overline{a}_1, \ldots, \overline{a}_N$ such that

$$\left|\bigwedge_{k=1}^{i} \phi(\overline{x}, \overline{a}_k)\right| > N \cdot \left|\bigwedge_{k=1}^{i+1} \phi(\overline{x}, \overline{a}_k)\right|$$

for each $i = 1, \ldots, N$.

Thus, the L^+-type given by $p(\overline{y}_i : i < \omega)$ given by

$$\left\{\left|\bigwedge_{k=1}^{i} \phi(\overline{x}, \overline{a}_k)\right| > N \cdot \left|\bigwedge_{k=1}^{i+1} \phi(\overline{x}, \overline{a}_k)\right| : i \leq N, N < \omega\right\}$$

is consistent, and by ω_1-saturation of M there is a sequence $\langle \overline{a}_i : i < \omega \rangle$ realizing the type p. If we define p_i to be $\bigwedge_{k \leq i} \phi(\overline{x}, \overline{a}_k)$, then we would have $\delta(p_1) > \delta(p_2) > \cdots$, contradicting ($\mathrm{A}_\phi$). □

Lemma 2.3.10. *Assume (SA) holds. Then there is no sequence of definable sets $\langle S_n : n < \omega \rangle$ such that $S_{n+1} \subseteq S_n$ and $\delta(S_{n+1}) < \delta(S_n)$ for each $n < \omega$.*

Proof. Otherwise, we may consider the type $p(\overline{x}) := \{S_n(\overline{x}) : n < \omega\}$. By (SA), there is a finite subtype $p_0 := \{S_{n_1}(\overline{x}), \ldots, S_{n_k}(\overline{x})\}$ with $n_1 < \cdots < n_k$ such that $\delta(p) = \delta(p_0) = \delta(S_{n_k}) > \delta(S_{n_k+1}) \geq \delta(p)$. A contradiction. □

Lemma 2.3.11. *Assume (DC_L). If (FMV_ϕ) fails for some formula $\phi(x, y)$, then T has the strict order property. So, in particular, T is not simple.*

Proof. Since (FMV$_\phi$) fails, there are $\overline{a}_1, \overline{a}_2, \ldots$ such that either $\delta(\phi(\overline{x};\overline{a}_i)) > \delta(\phi(\overline{x};\overline{a}_i))$ for all $i < \omega$, or $\delta(\phi(\overline{x};\overline{a}_i)) < \delta(\phi(\overline{x};\overline{a}_i))$ for all $i < \omega$. Then, there is a definable pre-order with infinite chains given by

$$\overline{y} \leq \overline{y}' \Leftrightarrow \delta(\phi(\overline{x};\overline{y})) \leq \delta(\phi(\overline{x},\overline{y}')) \Leftrightarrow \chi_{\phi,\phi}(\overline{y},\overline{y}').$$

□

Lemma 2.3.12.

1. *Assume* Th(M) *has the SOP, then (FMV) fails.*
2. *If M has (DC$_L$) and* Th(M) *has elimination of imaginaries, then (FMV) holds and so* Th(M) *does not have the SOP.*

Proof. The idea here is that from an order with infinite chains we can get a dense linear order. From there, we can use ω_1-saturation: from $a_0 < a_\epsilon < a_1$ with $\delta([a_0, a_\epsilon]) < \delta([a_0, a_1])$, we can iterate this argument to obtain $\delta([a_0, a_1]) > \delta([a_0, a_{\epsilon_1}]) > \delta([a_0, a_{\epsilon_2}]) > \cdots$, obtaining an infinite drop of dimensions.

For a more detailed proof: For (1), let $\psi(\overline{u},\overline{v})$ be a formula defining a preorder $\prec$ in M^t with an infinite chain. By Erdös-Rado and ω_1-saturation there is an L^+-indiscernible sequence in M^t with $\overline{a}_i \prec \overline{a}_j$ if and only if $i < j$.

Let $\chi(\overline{x}, \overline{uv})$ be express $\overline{u} \prec x \prec \overline{v}$. Hence, since $\chi(\overline{a}_0, \overline{a}_n) \supseteq \bigcup_{i=0}^{n} \chi(\overline{a}_i, \overline{a}_{i+1})$ and $f_\chi(\overline{a}_0, \overline{a}_1) = f_\chi(\overline{a}_i, \overline{a}_{i+1})$, we have $f_\chi(\overline{a}_0\overline{a}_n) \geq n \cdot f_\chi(\overline{a}_0\overline{a}_1)$, for every $n > 0$.

So, by L^+-indiscernibility, $f_\chi(\overline{a}_0\overline{a}_2) \geq n \cdot f_\chi(\overline{a}_0\overline{a}_1)$ for each n, and so, $\delta(\chi(M^t, \overline{a}_0\overline{a}_2)) > \delta(\chi(M^t, \overline{a}_0\overline{a}_1))$. It follows by compactness that the set $\{\delta(\chi(M^t; \overline{a}_0\overline{a}_i)) : i > 0\}$ is infinite.

For (2), let us assume (DC$_L$) and (EI), and suppose for a contradiction that $\{\delta(\phi(M^r; \overline{a})) : \overline{a} \in M^s\}$ is infinite. Let $\psi(\overline{u},\overline{v})$ express $\delta(\phi(M^r, \overline{u})) \leq \delta(\phi(M^r, \overline{v}))$. Then, ψ defines a preorder on M^s.

Let E be the equivalence relation defined by putting $E(u, v) \Leftrightarrow \psi(\overline{u},\overline{v}) \wedge \psi(\overline{v},\overline{u})$. Then ψ induce on M^s/E an empty-definable total order $\prec$. By EI, for some t there is an empty-definable function $g : M^s \to M^t$ with $E(\overline{u},\overline{v}) \Leftrightarrow g(\overline{u}) = g(\overline{v})$.

There is then an empty-definable total order $\prec$ on $I = g(M^s)$ given by $\overline{a} \prec \overline{b} \Leftrightarrow g^{-1}(\overline{a}) < g^{-1}(\overline{b})$.

Since I is pseudofinite, we may find (by ω_1-saturation) a sequence of subintervals $I \supseteq J_0 \supseteq J_1 \supseteq J_2 \supseteq \cdots$ with $|J_i| = 2|J_{i+1}|$ (non-standard cardinalities). Since the intervals are uniformly definable, this contradicts (WOD) and hence (DC$_L$). □

Example 2.3.13. It is easy to produce examples with (A) or (SA) but without (FMV). For instance, we can consider the Example 2.3.3.

2.4 Forking independence and δ-independence

Definition 2.4.1. Let $\overline{a}$ be a tuple and A, B be countable subsets of M. We say that $\overline{a}$ *is δ-independent of B over A* (denoted by $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\delta}_{A} B$) if $\delta(\overline{a}/AB) = \delta(\overline{a}/A)$.

Remark 2.4.2. With $\overline{a}, A, B$ as in the previous definition, then $\overline{a} \not\perp^{\delta}_{A} B$ there is a formula $\theta(\overline{x}) \in \mathrm{tp}(\overline{a}/AB)$ such that for all $\psi \in \mathrm{tp}(\overline{a}/A)$, $\delta(\theta(\overline{x})) < \delta(\psi(\overline{x}))$.

If $\overline{a} \not\perp^{\delta}_{A} B$, then we would have

$$\begin{aligned}\delta(\overline{a}/AB) &= \inf\ \{\delta(\phi(\overline{x})) : \phi(\overline{x}) \in \mathrm{tp}(\overline{a}/AB)\} \\ &< \delta(\overline{a}/A) = \inf\{\delta(\psi) : \psi \in \mathrm{tp}(\overline{a}/A)\}\end{aligned}$$

then there is $\theta(\overline{x}) \in \mathrm{tp}(\overline{a}/AB)$ such that $\delta(\theta(\overline{x})) < \delta(\overline{a}/A)$, and so, $\delta(\theta(\overline{x})) < \delta(\psi(\overline{x}))$ for all $\psi(\overline{x}) \in \mathrm{tp}(\overline{a}/A)$.

Remark 2.4.3. Note that in the proof of case $k = 1$ in Proposition 2.3.1, we actually found a number $N = N(\epsilon)$ such that if $A_1, \ldots, A_N$ have measure at least ϵ, there are $1 \leq i < j \leq N$ such that $\mu(A_i \cap A_j) \geq \epsilon^3$.

We are interested in the properties of δ-independence, and we want to see until which extent the δ-independence satisfies standard properties of the non-forking independence in simple theories.

Lemma 2.4.4 (Additivity for δ)**.** *Assume* (DC_L) *and* (FMV)*, and let A be a countable set of parameters from $M \models T$. Let $\overline{a} \in M^r, \overline{b} \in M^s$, then $\delta(\overline{a}\overline{b}/A) = \delta(\overline{a}/A\overline{b}) + \delta(\overline{b}/A)$.*

Proof. Since A is countable, we may assume without loss of generality $A = \emptyset$. Let $(\phi_n(\overline{y}) : n < \omega)$ enumerate the formulas in $\mathrm{tp}(\overline{b})$ and $(\psi_n(\overline{x}, \overline{b}) : n < \omega)$ enumerate $\mathrm{tp}(\overline{a}/\overline{b})$. We may suppose that $\psi_{n+1} \to \psi_n$ and $\phi_{n+1} \to \phi_n$ for each $n < \omega$.

Let P be the set of realizations of $\mathrm{tp}(\overline{a}\overline{b})$ in M ($P \subseteq M^r \times M^s$), and put $\epsilon_n := \delta(\phi_n(\overline{y}))$, $\gamma_n := \delta(\psi_n(\overline{x}, \overline{b}))$. By ($\mathrm{DC}_L$) and (FMV), for each n there is a formula $\rho_n(\overline{y}) \in L$ expressing that $\delta(\psi_n(\overline{x}, \overline{y})) = \gamma_n$, as follows:

Let $\chi(\overline{y}_1, \overline{y}_2)$ be an L-formula such that $M \models \chi(\overline{b}_1, \overline{b}_2) \Leftrightarrow \delta(\psi_n(\overline{x}, \overline{b}_1)) \leq \delta(\psi_n(\overline{x}, \overline{b}_2))$. By (FMV), there are finitely many values for the set $\{\delta(\psi_n(\overline{x}, \overline{b}')) : \overline{b}' \in M^s\}$ (say k values), so if γ_n is the j-th of these values, then $\delta(\psi_n(\overline{x}, \overline{b}')) = \gamma_n$ if and only if

$$\begin{aligned} M \models \exists y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_k \quad & \left(\bigwedge_{h<i\in\{1,\ldots,k\}-\{j\}} (\chi(\overline{y}_h, \overline{y}_i) \wedge \neg\chi(\overline{y}_i, \overline{y}_h)) \right) \\ \wedge\, (\chi(\overline{y}_{j-1}, \overline{b}') \wedge \neg\chi(\overline{y}_{j-1}, \overline{b}')) & \wedge (\neg\chi(\overline{y}_{j+1}, \overline{b}') \wedge \chi(\overline{b}', \overline{y}_{j+1})) \end{aligned}$$

The last formula is $\rho_n(\overline{y})$.

Since $\rho_n(\overline{y}) \in \mathrm{tp}(\overline{b})$, there is a formula ϕ_{m_n} such that $\phi_{m_n} \vdash \rho_n$. By refining the sequence, we can suppose that $\phi_n \vdash \rho_n$. Let P_n be the set defined by $\phi_n(\overline{y}) \wedge \psi_n(\overline{x}, \overline{y})$.

Claim: $\delta(P_n) = \epsilon_n + \gamma_n$ for each n.

Proof of the claim: Given $\overline{b}'_n \models \phi(\overline{y})$, since $\delta(\psi_n(\overline{x}, \overline{b}')) = \delta(\psi_n(\overline{x}, \overline{b}))$, there exists an integer $N \in \mathbb{N}$ (uniform, by compactness and saturation) such that

$$\frac{1}{N}|\psi_n(\overline{x}, \overline{b})| \leq |\psi_n(\overline{x}, \overline{b}')| \leq N \cdot |\psi_n(\overline{x}, \overline{b})|$$

By counting in the finite structures M_i, there are $\overline{b}_i^1, \overline{b}_i^2 \in \phi_n(M_i)$ such that

$$|\psi_n(M_i^r, \overline{b}_i^1)| \cdot |\phi_n(M_i)| \leq P_n(M_i) \leq |\psi_n(M_i^r, \overline{b}_i^2)| \cdot |\phi_n(M_i)|.$$

So, for $\mathcal{U}$-almost all i, we have

$$\frac{1}{N}|\psi_n(\overline{M}_i^r, \overline{b}_i)| \cdot |\phi_n(M_i)| \leq |P_n(M_i)| \leq N \cdot |\psi_n(M_i^r, \overline{b}_i)| \cdot |\phi_n(M_i)|$$

$$\log \frac{1}{N} + \log|\psi_n(M_i^r, \overline{b}_i)| + \log|\phi_n(M_i)| \leq \log|P_n(M_n)| \leq \log \frac{1}{N} + \log|\psi_n(M_i^r, \overline{b}_i)| + \log|\phi_n(M_i)|$$

By taking limits and quotient by Conv($\mathbb{Z}$) we obtain

$$\epsilon_n + \gamma_n = \delta(\psi_n(\overline{x}, \overline{b})) + \delta(\phi_n(\overline{y})) \leq \delta(P_n) \leq \delta(\psi_n(\overline{x}, \overline{b})) + \delta(\phi_n(\overline{y})) = \epsilon_n + \gamma_n. \quad \checkmark$$

Note that $P = \bigcap_{n<\omega} P_n$, so

$$\begin{aligned}\delta(\text{tp}(\overline{a}\overline{b}/A)) &= \delta(P) = \inf_n(\delta(P_n)) = \inf_n(\epsilon_n + \gamma_n) \\ &= \inf_n(\epsilon_n) + \inf_n(\gamma_n) = \delta(\text{tp}(\overline{b})) + \delta(\text{tp}(\overline{a}/A\overline{b})).\end{aligned}$$

□

Proposition 2.4.5. *The following are properties for the δ-independence.*

1. Existence: *Given countable sets $A \subseteq B$ and $p \in S_r(A)$ (for any $r \in \mathbb{N}$), there is $\overline{a} \models p$ such that $\overline{a} \mathop{\downarrow}^{\delta}_{A} B$.*
2. Monotonicity and Transitivity: *if $A \subseteq D \subseteq B$, then*
$$\overline{a} \mathop{\downarrow}^{\delta}_{A} B \Leftrightarrow \left(\overline{a} \mathop{\downarrow}^{\delta}_{A} D \text{ and } \overline{a} \mathop{\downarrow}^{\delta}_{D} B\right)$$
3. Finite character: *If $\overline{a} \mathop{\not\downarrow}^{\delta}_{A} B$ then there is a finite subset $\overline{b} \subseteq B$ such that $\overline{a} \mathop{\not\downarrow}^{\delta}_{A} \overline{b}$.*

Proof.

2. If $\delta(\overline{a}/AB) = \delta(\overline{a}/A)$, we have $\delta(\overline{a}/AB) \leq \delta(\overline{a}/D) \leq \delta(\overline{a}/A) = \delta(\overline{a}/AB)$, and so, $\delta(\overline{a}/AD)\delta(\overline{a}/A)$. Similarly, $\delta(\overline{a}/AB) \leq \delta(\overline{a}/BD) \leq \delta(\overline{a}/AD) \leq \delta(\overline{a}/A) \leq \delta(\overline{a}/AB)$, and so $\delta(\overline{a}/BD) = \delta(\overline{a}/AD) = \delta(\overline{a}/A)$. Thus, $\overline{a} \mathop{\downarrow}^{\delta}_{A} D$ and $\overline{a} \mathop{\downarrow}^{\delta}_{D} B$.

   The converse follows from $\delta(\overline{a}/AB) = \delta(\overline{a}/AD) = \delta(\overline{a}/A)$.

1. Given a partial type q over B and a formula $\phi(\overline{x}, \overline{b})$ over B, note that if $\delta(q) = \delta_0$ then either $\delta(q \cup \{\phi(\overline{x}, \overline{b})\}) = \delta_0$ or $\delta(q \cup \{\neg\phi(\overline{x}, \overline{b})\}) = \delta_0$. Otherwise, there would exists a formula $\psi \in q$ such that $\delta(\psi \wedge \neg\phi) < \delta_0$ and $\delta(\psi \wedge \phi) < \delta_0$ and we would have $\delta(q) = \delta_0 \leq \delta(\psi) = \delta((\psi \wedge \phi) \cup (\psi \wedge \neg\phi)) = \max\{\delta(\psi \wedge \phi, \delta(\psi \wedge \neg\phi)\} < \delta_0$, a contradiction.

3. Suppose $\overline{a} \not\mathop{\smile}\limits^{\delta}_{A} B$, then $\delta(\overline{a}/AB) < \delta(\overline{a}/A)$, so there is a formula $\phi(\overline{x}, \overline{b})$ over B such that $\delta(\mathrm{tp}(\overline{a}/A) \cup \{\phi(\overline{x}, \overline{b})) < \delta(\overline{a}/A)$. Then, $\overline{a} \not\mathop{\smile}\limits^{\delta}_{A} \overline{b}$.

$\square$

Proposition 2.4.6. *Under further assumptions, we have*

4. Local character: *(uses* (A)*) For every $\overline{a}$ and $B \subseteq M$ there is a countable set $A \subseteq B$ such that $\overline{a} \mathop{\smile}\limits^{\delta}_{A} B$.*

5. Invariance: *(uses* (DC$_L$)*) If $\alpha \in \mathrm{Aut}(M)$, then $\overline{a} \mathop{\smile}\limits^{\delta}_{A} B$ if and only if $\alpha(\overline{a}) \mathop{\smile}\limits^{\delta}_{\alpha(A)} \alpha(B)$.*

6. Symmetry: *(uses* (DC$_L$) *and* (FMV)*) $\overline{a} \mathop{\smile}\limits^{\delta}_{A} \overline{b}$ if and only if $\overline{b} \mathop{\smile}\limits^{\delta}_{A} \overline{a}$.*

7. Algebraic closure: *(uses* (DC$_L$)*) If $A \subsetneq B \subseteq \mathrm{acl}^{eq}(A)$, then $\delta(\overline{a}/A) = \delta(\overline{a}/B)$.(where δ is defined in the natural way for formulas with parameters in M^{eq})*

Proof. 4. Let $p := \mathrm{tp}(\overline{a}/B)$. By (A), for each formula $\phi(\overline{x}, \overline{y}) \in L$ there is a ϕ-formula $\psi_\phi(\overline{x}, \overline{b}_\phi)$ (collection of ϕ-instances) such that $\delta^\phi(\overline{a}/B) := \delta(\mathrm{tp}_\phi(\overline{a}/B)) = \delta(\psi_\phi(\overline{x}, \overline{b}_\phi))$.

Let A be the collection of all elements in tuples $\overline{b}_\phi$, when ϕ varies. Then, $|A| \leq \aleph_0$ and $\overline{a} \mathop{\smile}\limits^{\delta}_{A} B$.

5. Suppose $\overline{a} \mathop{\smile}\limits^{\delta}_{A} B$ and $\alpha \in Aut(M)$. Note that for every formula $\phi(\overline{x}, \overline{b}) \in \mathrm{tp}(\overline{a}/AB)$ there is $\psi(\overline{x}, \overline{c}) \in \mathrm{tp}(\overline{a}/A)$ such that $\delta(\psi(\overline{x}, \overline{c})) \leq \delta(\phi(\overline{x}, \overline{b}))$.

By (DC$_L$), there is an L-formula $\chi_{\psi,\phi}$ such that $M \models \chi_{\psi.\phi}(\overline{b}, \overline{c})$, and since it is invariant under automorphisms of M, we have $M \models \chi_{\psi,\phi}(\overline{c}, \overline{b})$, hence, $\delta(\psi(\overline{x}, \alpha(\overline{c}))) \leq \delta(\phi(\overline{x}, \alpha(\overline{b})))$, and we conclude that $\alpha(\overline{a}) \mathop{\smile}\limits^{\delta}_{\alpha(A)} \alpha(B)$.

6. It suffices to show that $\overline{a} \mathop{\smile}\limits^{\delta}_{A} \overline{b}$ implies $\overline{b} \mathop{\smile}\limits^{\delta}_{A} \overline{a}$. By additivity of δ (which uses (FMV) and (DC$_L$)) we have

$$\begin{aligned}
\delta(\overline{a}/A) + \delta(\overline{b}/A\overline{a}) &= \delta(\overline{a}\overline{b}/A) \\
&= \delta(\overline{b}/A) + \delta(\overline{a}/A\overline{b}) \\
&= \delta(\overline{b}/A) + \delta(\overline{a}/A) \qquad \text{(because } \overline{a} \mathop{\smile}\limits^{\delta}_{A} \overline{b}\text{)}
\end{aligned}$$

So, $\delta(\overline{b}/A\overline{a}) = \delta(\overline{b}/A)$, which implies $\overline{b} \mathop{\smile}\limits^{\delta}_{A} \overline{a}$.

7. Suppose $\psi(\overline{x}, \overline{b})$ is an algebraic formula such that $\psi(\overline{a}, \overline{b})$ holds, where $\overline{b}$ is possibly an imaginary tuple in $\mathrm{acl}^{eq}(A)$, and let $\overline{b} = \overline{b}_1, \overline{b}_2, \ldots, \overline{b}_k$ be the conjugates of $\overline{b}$ over A. Then, $\delta(\psi(\overline{x}, \overline{b}_i)) = \delta(\psi(\overline{x}, \overline{b}))$ for each $i = 1, \ldots, k$ by (DC$_L$), and so there is a formula $\rho(\overline{x}) \in \mathrm{tp}(\overline{a}/A)$ such that $\rho(x) \equiv \bigvee_{i=1}^{n} \psi(\overline{x}, \overline{b}_i)$, and thus,

$$\delta(\rho(\overline{x})) = \max_i\{\delta(\psi(\overline{x}, \overline{b}_i))\} = \delta(\psi(\overline{x}, \overline{b})).$$

$\square$

Remark 2.4.7. If we assume (SA), then for local character we have that the subset $A \subseteq B$ can be taken to be finite: there is a single formula $\phi(\overline{x}, \overline{b}) \in \text{tp}(\overline{a}/B)$ such that $\delta(\overline{a}/B) = \delta(\phi(\overline{x}, \overline{b}))$, and we can take A to be the tuple $\overline{b}$.

2.4.1 Simplicity and forking

Definition 2.4.8. Let T be a complete theory and M an ω_1-saturated model of T, from which the parameters will be taken.

1. A formula $\phi(\overline{x}, \overline{y})$ has the *tree property* (with respect to T) if there is $k < \omega$ and a sequence $(\overline{a}_\mu : \mu \in \omega^{<\omega})$ such that:
   (a) For every $\mu \in \omega^{<\omega}$, the set $\{\phi(\overline{x}, \overline{a}_{\mu^\frown i} : i < \omega\}$ is k-inconsistent.
   (b) For every $\sigma \in \omega^\omega$, the set $\{\phi(\overline{x}, \overline{a}_{\sigma\restriction i} : i \in \omega\}$ is consistent
2. The theory T is *simple* if no formula ϕ has the tree property with respect to T.
3. A *dividing chain of length* α *for* ϕ is a sequence $(\overline{a}_i : i < \alpha)$ such that $\bigcup_{i<\alpha} \phi(\overline{x}, \overline{a}_i)$ is consistent and $\phi(\overline{x}, \overline{a}_i)$ divides over $\{\overline{a}_j : j < i\}$ for all $i < \alpha$.
4. A simple theory T is *low* if for every formula ϕ there is $n_\phi < \omega$ such that there is no dividing chain of length n_ϕ for ϕ.

Lemma 2.4.9. *Let D be an A-definable subset of M^r in the language L, and let $\phi(\overline{x}, \overline{y})$ be an L-formula with $|\overline{x}| = r$ and $|\overline{y}| = s$. Let $(\overline{a}_i : i \in I)$ be an L^+-indiscernible sequence over A of elements of M^s.*

Put $D_i := \phi(M^r; \overline{a}_i)$ for each $i \in I$, and suppose $D_i \subseteq D$ and $(D_i : i \in I)$ is inconsistent. Then, there is some $i \in I$ such that $\delta(D_i) < \delta(D)$.

Proof. Suppose otherwise. By L^+-indiscernibility of the sequence $(\overline{a}_i : i \in I)$, there is some $n \in \mathbb{N}$ such that $|D_i| \geq \frac{1}{n}|D|$. The rest of the proof follows as in the proof of Theorem 2.3.2. □

Theorem 2.4.10. *1. Assume* (A) *holds. then T is simple and low.*

2. If (A) *and* (DC$_L$) *hold, then* (FMV) *holds.*

Proof. For (1), it is showed in ([29],Proposition 2.8.6) that $\phi(\overline{x}, \overline{y})$ has the tree property then ϕ has a dividing chain of arbitrary length. We will show that for every L-formula $\phi(\overline{x}, \overline{y})$ there is $m := m_\phi < \omega$ such that ϕ does not have dividing chains of length m.

Suppose for a contradiction that there is a sequence $(\overline{a}_j : 1 \leq j \leq m+1)$ such that each $\phi(\overline{x}, \overline{a}_j)$ divides over $\{\overline{a}_i : i < j\}$. Since (A) holds, by Lemma 2.3.9 to obtain a contradiction it will suffice to show that there is a sequence $(\overline{b}_j : 1 \leq j \leq m+1)$ such that $\text{tp}_L(\overline{a}_j : 1 \leq j \leq m+1) = \text{tp}_L(\overline{b}_j : 1 \leq j \leq m+1)$ and

$$\delta(\phi(\overline{x}, \overline{b}_1) \wedge \cdots \wedge \phi(\overline{x}, \overline{b}_k) \wedge \phi(\overline{x}, \overline{b}_{k+1})) < \delta(\phi(\overline{x}, \overline{b}_1) \wedge \cdots \wedge \phi(\overline{x}, \overline{b}_k)).$$

We construct the sequence $(\overline{b}_j : 1 \leq j \leq m+1)$ by induction, starting with $\overline{b}_1 = \overline{a}_1$. Suppose now that $\overline{b}_1, \dots, \overline{b}_k$ have been constructed. As $\mathrm{tp}_L(\overline{b}_1, \dots, \overline{b}_k) = \mathrm{tp}_L(\overline{a}_1, \dots, \overline{a}_k)$, there is $\overline{c} \in M$ such that $\mathrm{tp}(\overline{b}_1, \dots, \overline{b}_k, \overline{c}) = \mathrm{tp}_L(\overline{a}_1, \dots, \overline{a}_k, \overline{a}_{k+1})$. Put now $A = \{\overline{b}_i : i \leq k\}$ and $D = \phi(\overline{x}, \overline{b}_1 \wedge \cdots \wedge \phi(\overline{x}, \overline{b}_k)$, and let $\{\overline{d}_i : i < \omega\}$ be an A-indiscernible sequence witnessing the dividing of the formula $\phi(\overline{x}, \overline{c})$ over A. Using Erdös-Rado, compactness and ω_1-saturation of M, we may assume that $(\overline{d}_i : i < \omega)$ is A-indiscernible in the language L^+. Put now, $D_i = D \wedge \phi(\overline{x}, \overline{d}_i)$. By Lemma 2.4.9 there is $i < \omega$ such that $\delta(D_i) < \delta(D)$. Put $\overline{b}_{k+1} = \overline{d}_i$.

For (2), notice that by Lemma 2.3.11, if (DC_L) holds and (FMV_ϕ) fails, T is not simple. However, this contradicts (1). □

Theorem 2.4.11. *Let A, B be countable subsets of M, and $\overline{a}$ a tuple from M.*

1. *Assume* (A) *and* (DC_L) *hold, then* $\overline{a} \perp_A B \Rightarrow \overline{a} \perp^{\delta}_A B$.

2. *Assume* (SA) *and* (DC_L). *Then* $\overline{a} \perp^{\delta}_A B \Rightarrow \overline{a}_A B$.

In particular, (SA) *and* (DC_L) *imply that δ-independence is equivalent to non-forking independence.*

Proof. Recall that any independence relation satisfying existence, monotonicity, transitivity, finite character, local character, invariance, symmetry and algebraic closure, is implied by forking-independence, as it was proved in [19].

Note that all of them hold by Propositions 2.4.5, 2.4.6 and the fact that (A) and (DC_L) implies (FMV).

However, we show the proof of (1) here: Suppose $\overline{a} \not\perp^{\delta}_A B$. Hence $\delta(\overline{a}/AB) < \delta(\overline{a}/A)$, and so there is a formula $\phi(\overline{x}, \overline{b}) \in \mathrm{tp}(\overline{a}/AB)$ such that $\delta(\phi(\overline{x}, \overline{b})) < \delta(\overline{a}/A)$.

We must show that $\overline{a} \not\perp_A B$, but it suffices to show that $\overline{a} \not\perp_A \overline{b}$, i.e., that $\mathrm{tp}(\overline{a}/A\overline{b})$ forks over A.

Suppose for a contradiction that $\overline{a} \not\perp_A \overline{b}$. By Existence for δ, and ω_1-saturation, there is a sequence $(\overline{b}_i : i < \omega)$ in M such that $\overline{b}_i \models \mathrm{tp}(\overline{b}/A)$ and $\overline{b}_i \perp^{\delta}_A \{\overline{b}_j : j < i\}$ for all i (constructed by induction, of course).

By ω_1-saturation and Erdös-Rado, we may even assume that $(\overline{b}_i : i < \omega)$ is A-indiscernible in L^+.

Let $p := p(\overline{x}, \overline{b}) := \mathrm{tp}(\overline{a}/A\overline{b})$. Since $\overline{a} \perp_A \overline{b}$, the set $\{\phi(\overline{x}, \overline{b}_i) : i \in \omega_1\}$ is consistent, realized by $\overline{a}'$ say. By (DC_L) and (FMV), $\delta(\phi(\overline{x}, \overline{b}_i)) = \delta(\phi(\overline{x}, \overline{b}))$ and $\delta(\overline{a}/A) = \delta(\overline{a}'/A)$. Thus,

$$\overline{a} \not\perp^{\delta}_{A} \overline{b}_i \text{ for all } i. \qquad (\dagger)$$

On the other hand, by local character there is $i \in \omega_1$ such that $\overline{a}' \perp^{\delta}_{A\overline{b}_{<i}} \{\overline{b}_j : j \in \omega_1\}$. In particular, $\overline{a}' \perp^{\delta}_{A\overline{b}_{<i}} \overline{b}_i$. However, using symmetry and transitivity of $\perp^{\delta}$, we obtain a

contradiction as follows:

$$\begin{aligned}
\overline{a}' \mathop{\downarrow}^{\delta}_{A\overline{b}_{<i}} \overline{b}_i \Rightarrow \overline{b}_i \mathop{\downarrow}^{\delta}_{A\overline{b}_{<i}} \overline{a}' \text{ and } \overline{b}_i \mathop{\downarrow}^{\delta}_{A} \overline{b}_{<i} & \qquad \text{(by symmetry, and construction)} \\
\Rightarrow \overline{b}_i \mathop{\downarrow}^{\delta}_{A} \overline{a}' & \qquad \text{(by transitivity for } \delta) \\
\Rightarrow \overline{a}' \mathop{\downarrow}^{\delta}_{A} \overline{b}_i & \qquad \text{(by symmetry)}
\end{aligned}$$

contradicting (†).

For (2), suppose $\overline{a} \not\downarrow_A B$. We must show that $\delta(\overline{a}/AB) < \delta(\overline{a}/A)$. By (SA), there is a formula $\psi(\overline{x}) \in \text{tp}(\overline{a}/A)$ such that $\delta(\psi(\overline{x})) = \delta(\text{tp}(\overline{a}/A))$. Since $\overline{a} \downarrow_A B$, there is a formula $\phi(\overline{x}, \overline{b}) \in \text{tp}(\overline{a}/AB)$ that forks over A, and we may suppose $\phi(M^r; \overline{b}) \subseteq \psi(M^r)$.

By (A), T is simple, and so dividing and forking agree. Hence, by Theorem **??**, there is some $\overline{b}' \models \text{tp}(\overline{b}/A)$ such that $\delta(\phi(\overline{x}, \overline{b}')) < \delta(\psi(\overline{x}))$. Now, by (DC$_L$), if α is an automorphism of M fixing A such that $\alpha(\overline{b}) = \alpha(\overline{b}')$, then $\phi(\alpha(\overline{a}, \overline{b}'))$ holds, which implies $\alpha(\overline{a}) \not\downarrow^{\delta}_A \overline{b}'$, and by invariance, $\overline{a} \not\downarrow^{\delta}_A \overline{b}$. □

Remark 2.4.12.

(a) The above proof shows, assuming (SA) and (DC$_L$), that a formula $\phi(\overline{x}, \overline{b})$ forks over A if and only if for every $L(A)$-formula $\psi(\overline{x})$ that is consistent with ϕ we have $\delta(\phi \wedge \psi) < \delta(\psi)$. The direction ($\Leftarrow$) only requires (A) and (DC$_L$).

(b) Also, just under (SA), the proof of (2) yields the following: Suppose $\overline{a} \not\downarrow_A B$. Then there is $\overline{a}'B'$ such that $\text{tp}_L(\overline{a}B/A) = \text{tp}_L(\overline{a}'B'/A)$ and $\overline{a}' \not\downarrow^{\delta}_A B'$.

Note here that δ-dimension is not part of the L-type, and is not preserved in general under automorphisms of the ultraproduct (unless (DC$_L$ holds, of course).

Theorem 2.4.13. *Assume (SA). Then T is supersimple.*

Proof. Suppose for a contradiction that T is not supersimple. Then there are countable sets $B_0 \subseteq B_1 \subseteq \cdots$ and a type p over $B = \bigcup_{i<\omega} B_i$ such that for all $i < \omega$, $p \upharpoonright_{B_{i+1}}$ forks over B_i. Let $\overline{a}$ be a realization of p.

Claim: *For every $n < \omega$, we can build sets $B'_0 \subseteq B'_1 \subseteq \cdots \subseteq B'_n$ along with tuples $\overline{a}_n$ such that $\text{tp}(\overline{a}_n Bn') = \text{tp}(\overline{a}B_n)$ and $\delta(\overline{a}_n/B'_{i+1}) < \delta(\overline{a}_n/B'_i)$ for every $i < n$.*

Proof of the Claim: By induction, suppose these tuples and sets have been found for a fixed $n < \omega$. Notice that there is a set B^*_n such that $\delta(\overline{a}_n/B'_0 \ldots B'_n B^*_n) = \delta(\overline{a}/B_0 \ldots B_n B_{n+1})$. Thus, $\overline{a}_n \not\downarrow_{B'_{\leq n}} B^*_n$, so by Remark 2.4.12 there is $\overline{a}_{n+1}B'_{n+1}$ such that

$$\text{tp}(\overline{a}_{n+1}B'_{n+1}/B'_0 \ldots B'_n) = \text{tp}(\overline{a}B_{n+1}/B_0 \ldots B_n)$$

and $\delta(\overline{a}_{n+1}/B'_{n+1}) < \delta(\overline{a}_{n+1}/B'_n)$. $\square_{\text{Claim}}$

Let $B' := \bigcup_{n<\omega} B'_n$, $p_n := \text{tp}(\overline{a}_n/B'_n)$ and $p' = \bigcup_{n<\omega} p_n$. Then $p' \in S(B')$, and $\delta(p' \upharpoonright_{B_{n+1}}) < \delta(p' \upharpoonright_{B'_n})$ for each n, contradicting Lemma 2.3.10 □

2.5 Fine pseudofinite dimension and stability

In the following proposition, we characterize among ultraproducts M satisfying (A) when M is stable (and NIP). This characterization if made locally, at the level of formulas $\phi(\overline{x}, \overline{y})$.

Theorem 2.5.1. *Assume* (A^*_ϕ) *holds. Then the following are equivalent:*

1. $\phi(\overline{x}, \overline{y})$ *has the independence property*
2. $\phi(\overline{x}, \overline{y})$ *is unstable.*
3. *For some* $\overline{d}$ *there is a* $\overline{d}$*-definable sets* $D \subseteq M^r$ *and a sequence* $(\overline{a}_i : i \in \omega)$ L^+*-indiscernible over* $\overline{d}$ *such that*

$$\delta(D) = \delta\left(D \wedge \bigwedge_{i\in\omega} \phi(\overline{x}, \overline{a}_i)\right)$$

and $\mu_D(\phi(\overline{x}, \overline{a}_i) \wedge \phi(\overline{x}, \overline{a}_j)) < \mu_D(\phi(\overline{x}, \overline{a}_i))$ *for all* $i < j$.

Proof. (1) $\Rightarrow$ (2) is clear: if $\phi(\overline{x}, \overline{y})$ has the independence property witnessed by the sequences $(\overline{a}_i : i < \omega), (\overline{b}_W : W \subseteq \omega)$ then we have $\phi(\overline{a}_i, \overline{b}_W)$ holds if and only if $i \in W$. By taking $\overline{b}'_j := \overline{b}_{\{1,\ldots,j\}}$, we obtain $\phi(\overline{a}_i, \overline{b}'_j)$ holds if and only if $i \leq j$. Thus, $\phi(\overline{x}, \overline{y})$ is unstable.

(2) $\Rightarrow$ (3) : If $\phi(\overline{x}, \overline{y})$ is unstable, there are sequences $(\overline{b}_i : i < \omega)$ and $(\overline{a}_j : j < \omega)$ such that $M \models \phi(\overline{b}_i, \overline{a}_j)$ if and only if $i > j$. By (A^*_ϕ), there is a number m_ϕ such that there are not finite partial ϕ-types $D_1 \supseteq D_2 \supseteq \cdots \supseteq D_{m_\phi}$ such that $\delta(D_1) > \delta(D_2) > \cdots > D_{m_\phi}$.

So, we can find a set D, defined by a partial ϕ-type such that

- $\{\overline{b}_i : i < \omega\} \subseteq D$.
- If $\{\overline{b}_i : i \in \omega\} \subseteq D'$ and D' is a finite partial ϕ-type, then $\delta(D') \geq \delta(D)$.

Suppose (2) is false. Let $\overline{d}$ be the parameters needed to define D and take $(\overline{a}_i\overline{b}_i : i \in \omega + 1)$ an L^+-indiscernible sequence over $\overline{d}$, with $\overline{b}_i \in D$ for all $i \in \omega + 1$ and such that $M \models \phi(\overline{b}_i, \overline{a}_j)$ if and only if $i > j$.

From L^+-indiscernibility, it follows that $\mu_D(\phi(\overline{x}, \overline{a}_i))$ is constant. Also, by the minimality in the choice of D, we have that $\delta(D) = \delta(D \wedge \phi(\overline{x}, \overline{a}_0))$, and so there is a natural number M such that $|D \cap \phi(\overline{x}, \overline{a}_0)| \geq \frac{|D|}{M}$. Again, by L^+-indiscernibility, $|D \cap \phi(\overline{x}, \overline{a}_i)| \geq \frac{|D|}{M}$ for each $i < \omega$.

Since (2) is false, and by L^+-indiscernibility, we must have $\mu_D(\phi(\overline{x}, \overline{a}_i) \wedge \phi(\overline{x}, \overline{a}_j)) = \mu_D(\phi(\overline{x}, \overline{a}_i)$ for every $i < j < \omega + 1$, which implies $\mu_D(\phi(\overline{x}, \overline{a}_i) \wedge \neg\phi(\overline{x}, \overline{a}_j)) = 0$, and so we have $\delta(D \wedge \phi(\overline{x}, \overline{a}_i)) > \delta(D \wedge \phi(\overline{x}, \overline{a}_i) \wedge \neg\phi(\overline{x}, \overline{a}_j))$ whenever $i < j < \omega + 1$.

Finally, put $D' = D \cap \phi(M^r, \overline{a}_0) \cap \neg\phi(M^r, \overline{a}_\omega)$. Notice that D' is defined by a finite partial ϕ-type, $\{\overline{b}_i : i < \omega\} \in D'$ and $\delta(D') < \delta(D)$. This contradicts the minimality of D.

(3) $\Rightarrow$ (1) :Let D be a $\overline{d}$-definable set, $(\overline{a}_i : i \in \omega)$ be an L^+-indiscernible sequence over $\overline{d}$ with $\phi(\overline{x}, \overline{a}_i) \subseteq D$ and such that

- $\delta\left(D \wedge \bigwedge_{i<\omega} \phi(\overline{x}, \overline{a}_i)\right) = \delta(D)$.

- $\mu_D(\phi(\overline{x}, \overline{a}_i) \wedge \phi(\overline{x}, \overline{a}_j)) < \mu_D(\phi(\overline{x}, \overline{a}_i)$ for all $i < j$.

Put $D_i := D \wedge (\phi(\overline{x}, \overline{a}_{2i}) \wedge \neg\phi(\overline{x}, \overline{a}_{2i+1}))$ for $i < \omega$. Since $\mu_D(\phi(\overline{x}, \overline{a}_{2i}) \wedge \phi(\overline{x}, \overline{a}_{2i+1})) < \mu_D(\phi(\overline{x}, \overline{a}_{2i}))$, we have $\mu(D_i) > 0$. Moreover, since the sequence $(\overline{a}_i : i < \omega)$ is L^+-indiscernible over $\overline{d}$ and D is $\overline{d}$-definable, we may assume that $\mu(D_i) = \mu$ for some constant real $\mu > 0$.

By Proposition 2.3.1 and L^+-indiscernibility, we conclude that $\mu(D_1 \cap \cdots \cap D_k) > 0$ for any $k < \omega$. In particular, this shows that the alternation number of the formula $\phi(\overline{x}, \overline{y})$ in the sequence $(\overline{a}_i : i < \omega)$ is infinite, hence $\phi(\overline{x}, \overline{y})$ has the independence property.

□

Example 2.5.2 (Supersimplicity does not imply (SA))**.** Let $L = \{P_i : i < \omega\}$ be the language consisting of countably many unary predicates. For each $n < \omega$, let M_n be a finite structure with domain $[n^n] := \{1, \ldots, n^n\}$ such that $P_i(M_n) \supseteq P_{i+1}(M^n)$ and $|P_i(M_n)| = n^{n-i}$ for each $1 \leq i \leq n$, and $P_i(M_n) = \emptyset$ for $i > n$.

Notice that for every $a \in M$ and $A \subseteq M$, $\mathrm{tp}(a/A)$ is completely characterized by the set $\{i \in \omega : M \models P_i(a)\}$. Thus, we have elimination of quantifiers, and this also shows that the only dividing formulas are the algebraic ones (defined by disjunction of formulas of the form $x = a$). So, $\mathrm{Th}(M)$ is superstable of U-rank 1.

However, (SA) does not hold since $\delta(P_i(M)) > \delta(P_{i+1}(M))$ for all $i < \omega$.

Example 2.5.3 (Superstability does not imply A_ϕ)**.** For each $n < \omega$, let (M_n, E) be the structure where M_n has $i = 1^n n^i$ elements and E is interpreted as an equivalence relation with a class of size n^i for each $1 \leq i \leq n$. Let $a_{n,i} \in M_n$ be an element in the class of size n^{n-i}.

Notice that the theory $\mathrm{Th}(M)$ is simply the theory of an equivalence relation with infinitely many infinite classes, thus $\mathrm{Th}(M)$ is superstable of U-rank 2.

Consider the formula $\phi(x, y) := \neg(xEy)$. We will show that attainability fails for the formula ϕ. Consider the elements $(b_i := [a_{n,i}]_{n \in \mathcal{U}} : i \in \omega)$, and consider the positive ϕ-type $p := \{\phi(x, b_i) : i < \omega\} = \{\neg(xEb_i) : i < \omega\}$.

Claim: *For each $t < \omega$, $\delta(xEb_t) > \delta(p)$.*

Proof of the Claim: Otherwise, $\delta(xEb_t) \leq \delta(p) = \inf_{k<\omega} \delta\left(\bigwedge_{i \leq k} \neg(xEb_k)\right)$. In particular, this implies that

$$\delta(xEb_t) \leq \delta\left(\bigwedge_{i \leq t+1} \neg(xEb_i)\right).$$

So, there is a constant $N \in \mathbb{N}$ such that, for $\mathcal{U}$-almost all n, we have

$$\begin{aligned}
&\log |xEa_{n,t}| - \log \left| \bigwedge_{i \leq t+1} \neg(xEa_{n,i}) \right| \leq N \\
&\log(n^{n-t}) - \log \left(\sum_{t+2 \leq i \leq n} n^{n-i} \right) \leq N \\
&\log(n^{n-t}) - \log(n \cdot n^{n-(t+2)}) \leq N \\
&(n-t)\log n - \log n - (n-(t+2))\log n \leq N \\
&(\log n) \cdot (n - t - 1 - n + t + 2) = \log n \leq N
\end{aligned}$$

which is clearly false for sufficiently large n. $\square_{\text{Claim}}$.

From the Claim, we conclude that given finitely many $b_{i_1}, \ldots, b_{i_k}$, the set defined by $\bigwedge_{j=1}^{k} \phi(x, b_{i_j})$ contains xEb_t for $t > i_1, \ldots, i_k$ and we have

$$\delta \left(\bigwedge_{j=1}^{k} \phi(x, b_{i_j}) \right) \geq \delta(xEb_t) > \delta(p)$$

Thus, (A_ϕ) does not hold.

Lecture 3

Applications of pseudofinite structures to combinatorics

3.1 Introduction

The ultraproducts of finite structures can be used to provide a bridge between infinite and finite structures, and have been used to establish some results in *asymptotic combinatorics* (or *extremal combinatorics*, as it is most commonly refer), which is a branch of combinatorics that studies the properties of classes of finite structures that contain arbitrarily large finite structures, and in particular those properties that hold "in the limit" of the classes.

The use of ultraproducts in this context have produced striking results whose proofs rely on tools from "infinitary" mathematics. As examples of this results we can mention new proofs of Szemerédi's Regularity Lemma in graph theory and Szemerédi's Theorem in number theory (Goldbring and Towsner, [10]), a striking progress in the so-called *Freiman's problem* in multiplicative combinatorics (Hrushovski, [15]), and an algebraic regularity lemma that yield to a classification of expandes in finite fields (Tao, [26]).

Roughly speking, we can describe the general strategy as follows: Suppose $\mathcal{C}$ is a class of finite structures (e.g. graphs, groups, fields, etc.) and we would like to show that certain property $\mathcal{P}$ holds for every sufficiently large structure in $\mathcal{C}$. For a contradiction, assume otherwise. Then, there are structures $\langle M_n : n < \omega \rangle$ in $\mathcal{C}$ such that:

- $\lim\limits_{n\to\infty} |M_n| = \infty$.
- M_n does not satisfy the property $\mathcal{P}$ for each $n \in \omega$.

This is usually referred to as a *sequence of counterexamples* of the property $\mathcal{P}$. From that we can take the ultraproducts $M = \prod_{\mathcal{U}} M_n$ of the structures M_n with respect to a non-principal ultrafilter $\mathcal{U}$.

If it is possible to give a first-order description of the property $\mathcal{P}$ (possibly in an extended language), we would have by Łoś' theorem we also have that M does not satisfy

$\mathcal{P}$ [1]. On the other hand, if it is possible using methods from infinitary mathematics (e.g. measure theory, functional analysis, comparison finite vs. infinite cardinalities, etc.) that the infinite structure M must satisfy the property $\mathcal{P}$.

To illustrate this method a little bit further, we give a proof of the celebrated Ramsey's Theorem.

Theorem 3.1.1 (Ramsey, 1930)**.** *For every $k \in \mathbb{N}$ there is a number $R(k) \in \mathbb{N}$ such that every graph G with $|V(G)| \geq R(k)$ contains either a clique or an anticlique of size k.*[2]

Proof. If the statement of the theorem does not hold, then for a fix $k \in \mathbb{N}$ and for every $n < \omega$ there is a graph G_n such that $|V(G_n)| \geq n$ and G_n does not contain neither cliques nor anticliques of size k.

In this particular case, the property $\mathcal{P}$ we want to show can be easily stated in the language of graphs $L = \{R\}$:

$$\mathcal{P} := \exists x_1, \ldots, x_k \left(\bigwedge_{1 \leq i < j \leq k} x_i \neq x_j \wedge \left(\bigwedge_{1 \leq i < j \leq k} x_i R x_j \vee \bigwedge_{1 \leq i < j \leq k} \neg(x_i R x_j) \right) \right)$$

Let $\mathcal{U}$ be a non-principal ultrafilter over $\mathbb{N}$, and take $G = \prod_{\mathcal{U}} G_n$. Since $|G_n| \geq n$ for every n and $\langle G_n : n < \omega \rangle$ was a sequence of counterexamples, we know that G is infinite and by Łoś' theorem, $G \models \neg\mathcal{P}$.

The tool from infinitary mathematics we will use in this case is simply the comparison between finite and infinite cardinalities. Let $a_0 \in G$ be any element in G. Studying the edges from a to $G \setminus \{a\}$, we can write

$$G \setminus \{a\} = \{x \in G \setminus \{a\} : G \models xRa\} \cup \{x \in G \setminus \{a\} : G \models \neg(xRa)\}$$

and one of the two sets in the partition must be infinite. Let A_0 be such set. We can define recursively sets $A_1, \ldots, A_{2k}$ and elements $a_1, \ldots, a_{2k}$ by choosing a_{i+1} to be any element in A_i and A_{i+1} to be an infinite set in the partition of $A_i \setminus \{a\}$ given by two sets, $A_i \setminus \{a_i\} = \{x \in (A_i \setminus \{a_i\}) : G \models xRa_i\} \cup \{x \in (A_i \setminus a_i) : G \models \neg(xRa_i)\}$, for $i = 1, \ldots, 2k$.

By construction, we have $A_0 \supseteq A_1 \supseteq \cdots \supseteq A_{2k}, a_{i+1} \in A_i \setminus A_{i+1}$ for each $i < 2k$, and furthermore the truth value of the formula (xRa_i) remains constant for $x \in A_{i+1}$. In particular, for each i, the truth value of $R(a_i; a_j)$ remains constant for $i < j \leq 2k$. If there are indices $0 \leq j_1 < \ldots < j_k < 2k$ such that $R(a_{j_\ell}; a_{j_\ell+1})$, for each ℓ, then we obtain a clique given by $\{a_{j_1}, \ldots, a_{j_k})$. Otherwise, there are indices $0 \leq j_1 < \ldots < j_k < 2k$ realizing $\neg R(a_{j_\ell}, a_{j_\ell+1})$ and we obtain an anticlique of size k. This contradicts the fact that $G \models \neg\mathcal{P}$.

□

[1]Note that we have not written $M \models \neg\mathcal{P}$, as the "first-order description" of $\mathcal{P}$ does not necessarily imply that $\mathcal{P}$ can described by a sentence in the extended language.

[2]This version is equivalent to the version of Ramsey's theorem with two colors, but the proof given here can be easily generalized to the case of finitely many colors.

3.2 Furstenberg's correspondence principle

One application of ultraproducts of finite structures is to give a general framework for *correspondence principles* between finite structures and dynamical systems. We know present a proof of the original correspondence argumend given by Furstenberg in [7], following closely the proof given by I. Goldbring and H. Towsner in [10]. Since the first paper of Furstenberg, many variations and generalizations have ben produced (see, for instance, [?],[2]) and the method presented here extends naturally to these generalizations.

Definition 3.2.1. Let $E \subseteq \mathbb{Z}$. The *upper Banach density of* E *is defined as*

$$\overline{d}(E) = \limsup_{m-n\to\infty} \frac{|[n,m]\cap E|}{m-n} = \inf_{n\in\mathbb{N}} \sup_{m\geq n} \frac{|[n,m]\cap E|}{m-n}$$

For example, it is easy to see that $\overline{d}(2\mathbb{N}) = \frac{1}{2}$ and $\overline{d}(\mathbb{P}) = 0$ if $\mathbb{P}$ is the set of prime numbers. (The second example will be clear after remember that $\pi(x) \sim \frac{x}{log\, x}$ when $x \to \infty$

Definition 3.2.2. A *dynamical system* is a tuple $(Y, \mathfrak{B}, \mu, T)$ such that $(Y, \mathfrak{B}, \mu)$ is a probability space and $T : Y \longrightarrow Y$ is a invertible measure-preserving transformation, that is, $T : Y \longrightarrow Y$ is a bimeasurable bijection for which $\mu(A) = \mu(TA)$ for all $A \in \mathfrak{B}$.

Theorem 3.2.3. ***(Furstenberg's correspondence theorem)*** *Let* $E \subseteq \mathbb{Z}$ *with positive upper Banach density be given. Then there is a dynamical system* $(Y, \mathfrak{B}, \mu, T)$ *and a set* $A \in \mathfrak{B}$ *with* $\mu(A) = \overline{d}(E)$ *such that for any finite set of integers* U*,*

$$\overline{d}\left(\bigcap_{i\in U}(E-i)\right) \geq \mu\left(\bigcap_{i\in U} T^{-i}A\right).$$

Proof. Let $\langle \epsilon_N : N \in \mathbb{N}\rangle$ is a increasing sequence of positive rational numbers such that $sup_N \epsilon_N = \overline{d}(E)$. Then for each $N \in \mathbb{N}$ there is some n such that

$$\epsilon_N < \sup_{m\geq n} \frac{|[n,m]\cap E|}{m-n}.$$

That is, there are $m, n \in \mathbb{N}$, $m - n > N$ with $\frac{|[n,m]\cap E|}{m-n} \geq \epsilon_N$.

Define the functions

$$\begin{aligned} f_{n,m} :& [n,m] \longrightarrow [n,m] \\ & x \longmapsto f_{n,m}(x) = \begin{cases} x+1 & \text{if } n \leq x < m \\ n & \text{if } x = m \end{cases} \end{aligned}$$

Now, consider the classical measured structures $\mathcal{M}_{n,m} = ([n,m], E\cap[n,m], f_{n,m})$ equipped with the normalized counting measure. Let $\mathfrak{M}_{n,m}$ be the AML-structure associated to $\mathcal{M}_{n,m}$.

Let $(Y, A, T) = \prod_{\mathcal{U}} \mathfrak{M}_{n,m}$, $\mathfrak{B} = \sigma(Def_1(M)$ and μ the canonical measure over $\mathfrak{B}$.

- $(Y, \mathfrak{B}, \mu, T)$ *is a dynamical system:* It is clear by construction that $(Y, \mathfrak{B}, \mu)$ is a probability space. Now, since each $f_{n,m}$ is a bijection in the finite structure $([n,m], E \cap [n,m], f_{n,m})$, is an invertible measure-preserving transformation. By Łoś'Theorem, we obtain that $T : Y \longrightarrow Y$ is an invertible measure-preserving transformation.

By construction, $\mu(A) = \overline{d}(E)$. Let U be a finite subset of $\mathbb{Z}$. We will show that $\overline{d}\left(\bigcap_{i\in U}(E-i)\right) \geq \mu\left(\bigcap_{i\in U} T^{-i}A\right)$. Let δ be a rational number such that $\delta < \mu(\bigcap_{i\in U} T^{-i}(A)$ (if there is no such a δ, there is nothing to show).

By Łoś' Theorem again, for each N we can find $m, n \in \mathbb{N}$, $m - n > N$ such that

$$\frac{|[n,m] \cap \bigcap_{i\in U} f_{n,m}^{-i}(E \cap [n,m])|}{m,n} \geq \delta.$$

So, for a fixed $\gamma > 0$, and for N sufficiently large, we have that

$$\frac{|[n,m] \cap \bigcap_{i\in U}(E-i)|}{m,n} > \delta - \gamma$$

(Note here that $f_{n,m}^{-i}(E \cap [n,m]) \subseteq (E-i)$ for all $i \in U$.) Therefore,

$$\overline{d}\left(\bigcap_{i\in U}(E-i)\right) = \inf_{n\in\mathbb{N}} \sup_{m\geq n}\left(\frac{|[n,m] \cap \bigcap_{i\in U}(E-i)|}{m,n}\right) > \delta - \gamma$$

Since γ is arbitrary, we get that

$$\overline{d}\left(\bigcap_{i\in U}(E-i)\right) \geq \delta$$

and the conclusion of the theorem holds. □

3.3 Szemerédi's Regularity Lemma

The purpose of this section is to present a proof of the Szemerédi's Regularity Lemma using measure-theoretic aspects of the ultraproducts of finite structures.

At first, the regularity lemma was the main technical lemma to prove the more celebrated Szemerédi's theorem: every subset of the integers with positive density contains arbitrarily large arithmetic progressions.

Since then, the regularity lemma has played a key role in extremal combinatorics, as well as in other areas such as number theory, algebra, ergodic theory, etc.

More recently, people have been discovering some improved versions of the regularity lemma in different contexts. For instance, Malliaris and Shelah discovered a version of the regularity lemma for stable graphs where there are no irregular pairs, and Tao has proved the so-called of Algebraic regularity lemma in the context of finite fields with sizes going to infinity.

Definition 3.3.1. A graph G is a pair $G = (V, E)$ where E is a symmetric subset of $V \times V$

Definition 3.3.2. Let $G = (V, E)$ be a graph (undirected, simple) and let A, B be two disjoint subsets of V.

1. The edge *density* between A, B is the proportion of edges between vertices in A and vertices in B, that is, $d(A, B) = \dfrac{|E \cap (A \times B)}{|A| \cdot |B|}$.
2. For $\epsilon > 0$, we say that the pair (A, B) is *ϵ-regular* if for all $A' \subseteq A$, $B' \subseteq B$ with $|A'| \geq \epsilon|A|, |B'| \geq \epsilon|B|$ we have $|d(A, B) - d(A', B')| < \epsilon$.

Example 3.3.3. *Every pair (A, B) is $\epsilon = 1$-regular.* If $A' \subseteq A, B' \subseteq B$ satisfy $|A'| \geq |A|$ and $|B'| \geq |B|$, then $A' = A, B' = B$ and so

$$|d(A', B') - d(A, B)| = |d(A, B) - d(A, B)| = 0 < 1.$$

Every pair of singletons is ϵ-regular for every $\epsilon > 0$. Let $A = \{a\}, B = \{b\}$. If $A' \subseteq A, B' \subseteq B$ satisfy $|A'| \geq \epsilon|A| = \epsilon > 0$, $|B'| \geq \epsilon|B| > 0$ then again $A = A'$ and $B = B'$.

Example 3.3.4. Put $A = \{a\}$ and $B = B_1 \cup B_2$ with $|B_1| = |B_2|$ and $\{x \in B : aEx\} = B_1$. Then the pair is ϵ-regular for $\epsilon > \frac{1}{2}$: notice that

$$d(A, B) = \frac{E \cap (A \times B)}{|A| \cdot |B|} = \frac{|B_1|}{2|B_1|} = \frac{1}{2}.$$

So, if $B' \subseteq B$ satisfy $|B'| \geq \epsilon|B|$ and $|A'| \geq \epsilon|A|$, we obtain

$$|d(A, B) - d(A', B')| = \left|\frac{1}{2} - d(A, B')\right| \leq \frac{1}{2} < \epsilon$$

because $d(A', B') \in [0, 1]$.

Theorem 3.3.5 (Szemerédi's Regularity Lemma)**.** *For any $k \in \mathbb{N}$ and $\epsilon > 0$ there is $K \geq k$ such that the following holds:*

Whenever (G, E) is a finite graph, there is $n \in [k, K]$ and a partition $G = U_1 \cup \cdot \cup U_n$ such that

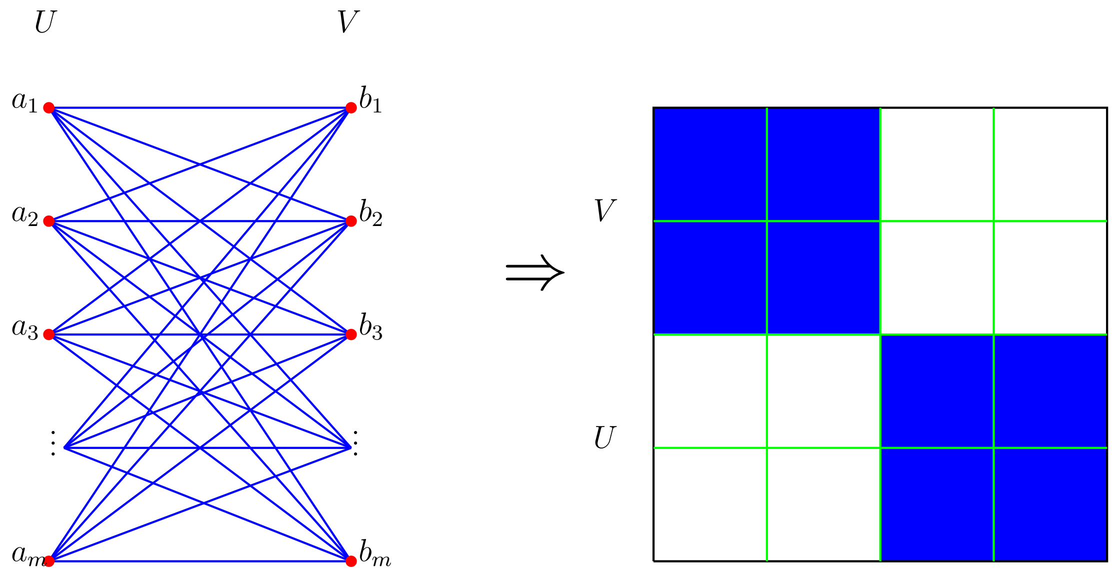
$$\left|\bigcup\{U_i \times U_j : (U_i, U_j) \text{ is not } \epsilon\text{-regular}\}\right| \leq \epsilon|G|^2$$

This theorem establishes that every graph can be divided into subsets (and there is an absolute bound on the number of subsets it is divided into) so that the edges between different subsets behave pseudo-randomly for most of the pairs. (at most an ϵ-fraction of the pairs are not ϵ-regular).

Before going into the proof, let us analyze the following important examples:

Example 3.3.6 (Bipartite complete graph)**.** Suppose U, W are disjoint sets of size n, and we define the graph G with $V(G) = U \cup W$ and $E(G) = \{(u, w) : u \in U, w \in W\}$.

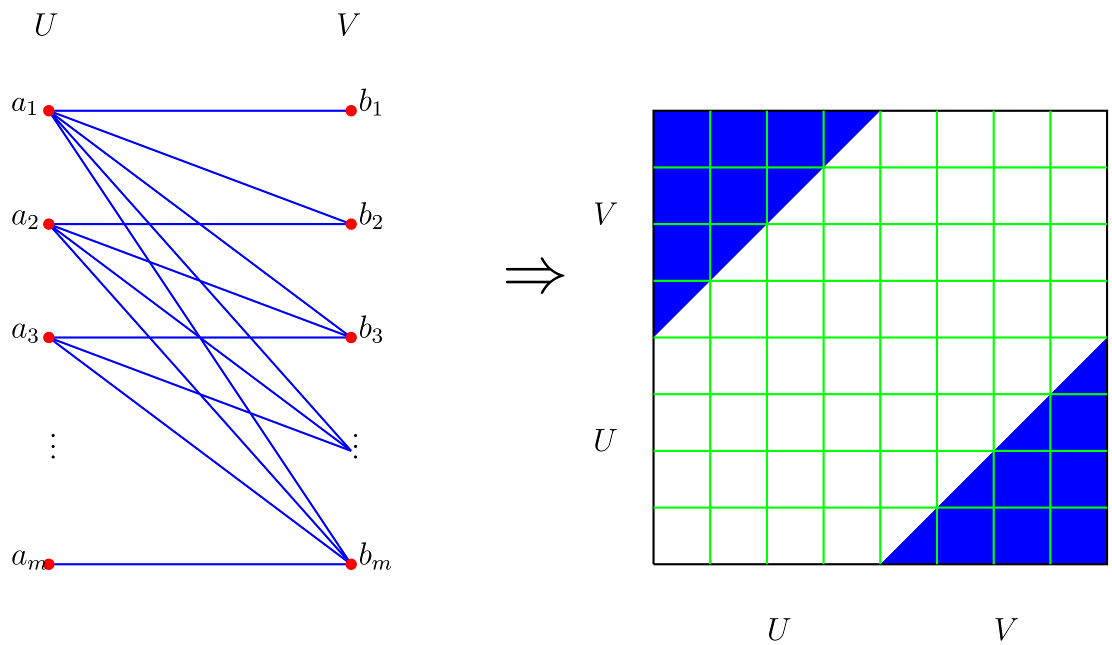
Then, for every $\epsilon > 0$, the partition given by $G = U \cup V$ has only regular pairs, with uniform densities $d(U, U) = 0, d(U, V) = 1$. If necessary, we can consider further partitions of U and V to decompose G into more sets, still having uniform densities either 0 or 1.



Example 3.3.7 (Half-graph)**.** Put $G = \{a_1, \ldots, a_m\} \cup \{b_1, \ldots, b_m\}$ with the edge relation given by $a_i E b_j$ if and only if $i \leq j$. In this case we can start again with the partition $G = U \cup V$, which will produce pairs (U, U) and (V, V) with uniform density 0, and ϵ-irregular pairs (U, V). By partitioning further the sets U and V we get more regular pairs of uniform densities 0 and 1, together with some irregular pairs (The half-blue squares in the representation below).

This example shows that, in general, irregular pairs will always appear. However, notice that if the partition of G has n equal pieces (partitioning U and V in $n/2$ equal pieces), then total number of pairs is n^2 and the total number of irregular pairs is n. We still have that the proportion of irregular pairs tends to

$$\lim_{n\to\infty} \frac{\#\text{ irregular pairs}}{\#\text{ pairs}} = \lim_{n\to\infty} \frac{n}{n^2} = 0.$$



The following proof of Szemerédi's Regularity Lemma using ultraproducts is due to I. Goldbring and H. Towsner, and appear in [10].

Proof. Towards a contradiction, assume that the conclusion fails. Then there are $k \in \mathbb{N}, \epsilon > 0$ such that for every $K \geq k$, there is a finite graph (G_K, E_K) with no partition into $k \leq n \leq K$ satisfying the conclusion of the theorem.

Notice that since any pair of singletons is ϵ-regular, we must have $|G_K| \geq K$ for all K.

We will now choose inductively a suitable language to construct an ultraproduct. First, put $L_0 = \{E\}$, a binary relation. Now, suppose that $L_0 \subseteq \cdots L_n$ have already been constructed. For any two formular $\phi(x), \psi(x)$ (at least one not in L_{n-1}), we add unary predicates $R_{\varphi,\psi}(x)$ and $S_{\varphi,\psi}(x)$ to the language L_{n+1}. Finally, put $L := \bigcup_{n<\omega} L_n$.

Every graph (G_K, E_K) may be seen as an L-structure by interpreting E as the binary relation E_K, and supposing φ, ψ have been interpreted in G_K, we may set $U = \phi(G_K)$, $U' = \psi(G_K)$ (note that φ and ψ may have parameters).

If the pair (U, U') is ϵ-regular, then set $R_{\varphi,\psi} = \emptyset = S_{\varphi,\psi}$. Otherwise, there are $V \subseteq U$ and $V' \subseteq V'$ such that $|V| \geq \epsilon|U|, |V'| \geq \epsilon|U'|$ and $|d(V, V') - d(U, U')| > \epsilon$. Put $R_{\varphi,\psi} = V$ an $S_{\varphi,\psi}(V')$ in this case.

Consider now the ultraproduct of L-structures given by $G = \prod_{\mathcal{U}} G_K$, with the canonical product measure on G^2 generated by $\mu(X) = st\left(\frac{|X|}{|G|}\right)$.

Let $h := \mathbb{E}(\chi_E|\mathcal{B}_{2,1})$, the conditional expectation of the characteristic function χ_E with respect to the σ-algebra $\mathcal{B}_{2,1}$.

Since h can be approximated by simple function, we may write h as

$$h = \sum_{i,j} \alpha_{i,j} \chi_{C_i \times D_j} + h',$$

where $||h'||_{L^2} < \frac{\epsilon^4}{4}$ and C_i, D_i are L-definable subsets of G.

Let $U_1, \ldots, U_n$ be the atoms of the finite algebra generated by $\{C_i\} \cup \{D_j\}$. Then we have a refinement given by $h = \sum_{(i,j) \leq n} \alpha_{ij} \chi_{U_i \times U_j} + h''$ with $||h''||_{L^2} \leq ||h'||_{L^2}$, and $U_1, \ldots, U_n$ is a finite partition of G into definable sets.

Let B be the collection of pairs (i, j) such that $R_{U_i,U_j} \neq \emptyset \neq S_{U_i,U_j}$ (i.e., the collection of irregular pairs), and define

$$\alpha_{ij} = \frac{\mu(E \cap (U_i, U_j))}{\mu(U_i) \cdot \mu(U_j)} \quad \text{and} \quad \beta_{ij} = \frac{\mu(E \cap (R_{U_i,U_j} \times S_{U_i,U_j})}{\mu(R_{U_i,U_j}) \cdot \mu(S_{U_i,U_j})}.$$

By Łoś' theorem, for every $(i, j) \in B$ we know that, since R_{U_i,U_j}, S_{U_i,U_j} where the respective subsets of U_i, U_j contradicting ϵ-regularity, the following inequality holds:

$$|d(R_{ij}; S_{ij}) - d(U_i, U_j)| \geq |\beta_{ij} - \alpha_{ij}| \geq \epsilon.$$

Suppose now that $\mu\left(\bigcup_{(i,j)\in B} U_i \times U_j\right) \geq \frac{\epsilon}{2}$, and let $B^+ = \{(i,j) : \alpha_{ij} - \beta_{ij} \geq \epsilon\}$. Without loss of generality, we may assume that $\mu\left(\bigcup_{(i,j)\in B^+} U_i \times U_j\right) \geq \frac{\epsilon}{4}$, since otherwise we can carry out a similar argument with $B^- = B \setminus B^+$ instead of B^+.

Again, by Łoś' theorem, we have $\mu(R_{ij}) \geq \epsilon\mu(U_i)$ and $\mu(S_{ij}) \geq \epsilon\mu(U_j)$. Put $Z = \bigcup_{(i,j)\in B^+} R_{ij} \times S_{ij}$. Note that $Z \in \mathcal{B}_{2,1}$, so we have

$$\begin{aligned}
0 &= \int (\chi_E - h) \cdot \chi_Z \, d\mu \\
&= \int \chi_E - \cdot \chi_Z - h\chi_Z \, d\mu \\
&= \int \left(\chi_E \cdot \chi_Z - \sum_{i,j\leq n} \alpha_{ij}\chi_{U_i \times U_j}\chi_Z\right) d\mu - \int h''\chi_Z \, d\mu
\end{aligned}$$

which implies

$$\begin{aligned}
\left|\int h'' \cdot \chi_Z \, d\mu\right| &= \left|\int \sum_{i,j\leq n} \alpha_{ij}\chi_{U_i\times U_j}\chi_Z - \chi_E\chi_Z \, d\mu\right| \\
&= \left|\int \left[\sum_{i,j\leq n} \alpha_{ij}\chi_{U_{ij}} \left(\sum_{(k,\ell)\in B^+} \chi_{R_{k,\ell}\times S_{k,\ell}}\right)\right] - \left[\chi_E \cdot \left(\sum_{(k,\ell)\in B^+} \chi_{R_{k,\ell}\times S_{k,\ell}}\right)\right] d\mu\right| \\
&= \left|\sum_{(k,\ell)\in B^+} \left(\int \left[\sum_{i,j\leq n} \alpha_{ij} \cdot \chi_{U_i\times U_j} \cdot \chi_{R_{k,\ell}\times S_{k,\ell}} - \chi_E \cdot \chi_{R_{k,\ell}\times S_{k,\ell}}\right] d\mu\right)\right| \\
&= \left|\sum_{(k,\ell)\in B^+} \left(\int \alpha_{k,\ell} \cdot \underbrace{\chi_{U_k\times U_\ell} \cdot \chi_{R_{k,\ell}\times S_{k,\ell}}}_{=\chi_{R_{k,\ell}\times S_{k,\ell}}} - \chi_E \cdot \chi_{R_{k,\ell}\times S_{k,\ell}} \, d\mu\right)\right| \\
&= \left|\sum_{(k,\ell)\in B^+} (\alpha_{k,\ell} \cdot \mu(R_{k,\ell}) \cdot \mu(S_{k,\ell}) - \mu(E \cap (R_{k,\ell} \times S_{k,\ell})))\right| \\
&= \left|\sum_{(k,\ell)\in B^+} (\alpha_{k,\ell} - \beta_{k,\ell})\mu(R_{k,\ell})\mu(S_{k,\ell})\right| \\
&\geq \epsilon \sum_{(k,\ell)\in B^+} \mu(R_{U_k,U_\ell}) \cdot \mu(S_{U_k,U_\ell}) \geq \epsilon^3 \sum_{(k,\ell)\in B^+} \mu(U_k) \cdot \mu(U_\ell) \geq \frac{\epsilon^4}{4},
\end{aligned}$$

but this is a contradiction, because by Cauchy-Schwartz inequality we have

$$\left|\int h'' \cdot \chi_Z \, d\mu\right| = |\langle h'', \chi_Z\rangle| \leq ||h''||_{L^2} \cdot ||\chi_Z||_{L^2} < \frac{\epsilon^4}{4} \cdot 1.$$

It follows then that $\mu\left(\bigcup_{(i,j)\in B} U_i \times U_j\right) \leq \frac{\epsilon}{2}$.

Note that until here we have only found a partition which satisfies the regularity conditions *in the infinite model*, so we have to pull down this partition into some finite model G_K, obtaining a contradiction.

By choosing $K \geq n$ large enough, we may ensure that the following hold:

- Whenever $(i,j) \in B$, $R_{U_i,U_j}(G_K) = \emptyset$.
- $\mu(U_i(G_K)) \leq \sqrt{2}(\mu(U_i(G)))$

This implies that

$$\mu\left(\bigcup_{(i,j)\in B} U_i(G_K) \times U_j(G_K)\right) \leq 2\mu\left(\bigcup_{(i,j)\in B} U_i(G) \times U_j(G)\right) \leq \epsilon,$$

that is,

$$\left|\bigcup_{(i,j)\in B} U_i(G_K) \times U_j(G_K)\right| \geq \epsilon|G|^2,$$

contradicting that G_K was a counterexample of the existence of an ϵ-regular decomposition. □

3.4 The Erdös-Hajnal property for stable graphs.

The Erdös-Hajnal conjecture is an interesting problem in graph theory whose statement can be easily stated in terms of the normalized pseudofinite dimension described in Section 2.2.11. In this lecture we will present the main definitions regarding the Erdös-Hajnal conjecture, and a proof of the Erdös-Hajnal property due to A. Chernikov and S. Starchenko, presented in [6]. Recall the following definitions:

Definition 3.4.1. Let G be a graph. A *clique* in G is a set of vertices all pairwise adjacent, and an *anticlique* in G is a set of vertices that are all pairwise non-adjacent.

Definition 3.4.2. Let H be a graph. We say that another graph G is *H-free* if G does not contain an induced subgraph isomorphic to H.

One of the first successful applications of the probabilistic method is due to Erdös, who use it to show the following:

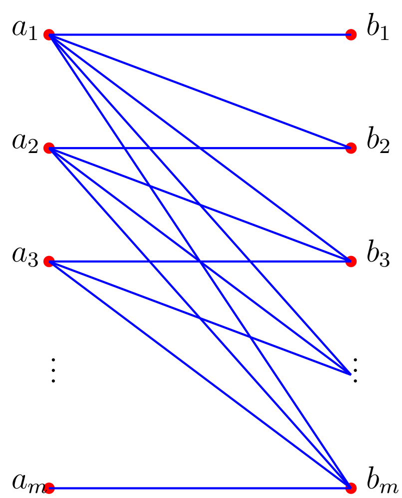
Theorem 3.4.3 (Erdös, 1947). *For every integer $k \geq 3$, the Ramsey number of k satisfies $R(k) > 2^{k/2}$. That is, for every $k \geq 3$ there is a graph G_k of size with more than $2^{k/2}$ vertices such that G_k does not contain neither a clique nor an anticlique of size k.*

The corollary of Erdös theorem show that, in general, for a graph of n vertices we cannot expect neither cliques nor anticliques of size around $O(\log n)$. However, the way the probabilistic method was used in its proof suggested that the graphs used as examples satisfy (in the limit) the same properties as the random graph, so in particular, every graph H can be found with probability 1 in the graphs G_k. Erdös and A. Hajnal conjectured in 1989 that the situation was very different for "structured" graphs:

Conjecture 3.4.4 (Erdös-Hajnal conjecture, 1989)**.** *For every finite graph H there is a constant $\delta = \delta(H) > 0$ such that every finite H-free graph $G = (V, E)$ contains either a clique or an anticlique of size at least $|V(G)|^\delta$.*

This conjecture is known to be true for several classes of graphs H, as well as for every graph of size $|V(H)| \leq 4$.

Definition 3.4.5. For $m \in \mathbb{N}$ let H_m be the *half-graph* on $2m$ vertices, i.e., H_m is a bipartite graph whose vertex set is a disjoint union of $\{a_1, \ldots, a_m\} \cup \{b_1, \ldots, b_m\}$ such that $a_i E b_j$ if and only if $i \leq j$.



The following theorem is a corollary of ([23], Theorem 3.5)

Definition 3.4.6. For every $m \in \mathbb{N}$, we say that a graph G is H_m-free$*$ if G does not contain H_m as a subgraph.

Remark 3.4.7. It is important to notice the subtle difference with the concept of an H_m-free graph. The main difference, is that an H_m-free$*$ graph does not have to avoid H_m only as an *induced* subgraph, but rather as a subgraph. That is, it cannot contain vertices $a_1, \ldots, a_m, b_1, \ldots, b_m$ with the edge relation $a_i E b_j \Leftrightarrow i \leq j$, regardless the edge relation inside the sets $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_m\}$. In other words, to be H_m-free$*$, it is necessary to avoid a finite family of graphs (namely, all graphs with the m-order property).

The following theorem is a striking example that model-theory can be meaningful in this contexts and shed some lights on the Erdös-Hajnal property for different classes of graphs.

Theorem 3.4.8 (Chernikov-Starchenko, 2015)**.** *For every $m \in \mathbb{N}$ there is a constant $\delta(m) > 0$ such that every finite graph that does not contain H_m as a subgraph $G = (V, E)$ contains either a clique or an anticlique of size at least $|V|^{\delta(m)}$.*

Remark 3.4.9. This is not the same as proving the Erdös-Hajnal conjecture for the graph H_m. Rather, it shows that if a graph G does not have the *m-order property* for the formula xRy, then it contains either a clique or an anticlique of size at least $|V|^{\delta(m)}$.

In what follows, we will present a proof of Theorem 3.4.8 following closely the presentation given in [6].

Proposition 3.4.10. *Let $Y \subseteq M \times M^m$ and $Z \subseteq M$ definable. Assume that $\delta(Z) = \alpha$ and for all pairwise distinct $a_1, \ldots, a_m \in Z$, we have $\delta(\{x \in M : (x, a_1, \ldots, a_m) \in Y\}) \leq \beta$. Then,*

$$\delta\left(\left\{x \in M : \exists z_1, \ldots, z_m \in Z \left(\bigwedge_{i \neq j} z_i \neq z_j \wedge (x, z_1, \ldots, z_m) \in Y\right)\right\}\right) \leq m \cdot \alpha + \beta$$

Proof. Consider $Z^{(m)} = \{(z_1, \ldots, z_m) \in Z^m : z_1, \ldots, z_m \text{ are pairwise different}\}$ and define

$$Y' := \{(x, \overline{a}) \in Y : \overline{a} \in Z^{(m)}\}.$$

First, let $f : Y' \to M$ be the projection on the first coordinate. Then we have,

$$f(Y') = \left\{x \in M : \exists z_1, \ldots, z_m \in Z \left(\bigwedge_{i \neq j} z_i \neq z_j \wedge (x, z_1, \ldots, z_m) \in Y\right)\right\}.$$

On the other hand, we can consider the function $g : Y' \to Z^{(m)}$ to be the projection on the last m coordinates. Notice that for every $\overline{a} \in Z^{(m)}$, there is a bijection between $g^{-1}(\overline{a})$ and $\{x \in M : (x, \overline{a}) \in Y\}$, and by hypothesis we have

$$\delta(g^{-1}(\overline{a}) = \delta(\{x \in M : (x, \overline{a}) \in Y\}) \leq \beta$$

for all $\overline{a} \in Z^{(m)}$.

So, by Proposition 2.2.7, we have that

$$\begin{aligned} \delta(f(Y')) &\leq \delta(Y') \leq \delta(Z^{(m)}) + \beta \\ &\leq \delta(Z^m) + \beta = m \cdot \delta(Z) + \beta \\ &= m \cdot \alpha + \beta. \end{aligned}$$

□

Theorem 3.4.8 is implied by the following non-standard version:

Theorem 3.4.11. *Let V be a pseudofinite set and $E \subseteq V \times V$ a definable symmetric subset. Assume that the graph (V, E) is H_m-free* for some $m \in \mathbb{N}$. Then there is a definable $A \subseteq V$ such that $\delta(A) > 0$ and either $(a, a') \in E$ for all $a \neq a'$ in A, or $(a, a') \notin E$ for all $a \neq a'$ in A.*

We know show that this result implies Theorem 3.4.8:

Proof. Assume 3.4.8 is false. Then, there is $m \in \mathbb{N}$ such that for every $n < \omega$ there is a H_m-*-free graph G_n such that G_n does not contain neither a clique nor an anticlique of size $|V(G_n)|^{\frac{1}{n}}$.

Let $\mathcal{U}$ be a non-principal ultrafilter on ω, and put $G = \prod_{\mathcal{U}} G_n$. We have:

1. *G is infinite:* Notice that any pair of points in G_n is either a clique or an anticlique. So we have $2 < |G_n|^{1/n}$ which implies $2^n < |G_n|$ for any $n < \omega$, and $\lim_{n\to\mathcal{U}} |G_n| = +\infty$.

Note also that G is H_m-*-free graph, because of Łoś' theorem and the fact that

$$G_n \models \neg\exists x_1, \ldots, x_m, y_1, \ldots, y_m \left(\bigwedge_{i<j\leq m} x_i \neq x_j \wedge y_i \neq y_j \wedge \bigwedge_{i\leq j} xEy_j \wedge \bigwedge_{i>j} \neg x_i E y_j \right).$$

By Theorem 3.4.11, there is a definable subset $A \subseteq V$ with $\delta(A) > 0$, which is either a clique or an anticlique. Let us write $A = \prod_{\mathcal{U}} A_n$, and say $\delta(A) = \epsilon > 0$.

Assume without loss of generality that A is a clique. Then, we have

$$G \models \forall x, y\, (x \neq y \wedge A(x) \wedge A(y) \to xEy)\,.$$

By Łoś' theorem, this means that for $\mathcal{U}$-almost all $n < \omega$,

$$G_n \models \forall x, y\, (x \neq y \wedge A_n(x) \wedge A_n(y) \to xEy)\,,$$

which means that $A_n \subseteq G_n$ is a clique for $\mathcal{U}$-almost all n.

By the choice of the graphs G_n, we have that $|A_n| \leq |G_n|^{\frac{1}{n}}$, which implies that $\frac{\log|A_n|}{\log|G_n|} \leq \frac{1}{n}$. So,

$$\delta(A) = \lim_{n\to\mathcal{U}} \frac{\log|A_n|}{\log|G_n|} \leq \lim_{n\to\mathcal{U}} \frac{1}{n} = 0 < \delta(A),$$

a contradiction. □

We will see a more general proof of Theorem 3.4.11 that works for hypergraphs. Assume throughout the rest of this section that $V = \prod_{\mathcal{U}} V_i$ and $E = \prod_{\mathcal{U}} E_i$ is a definable subset of V^n, which is symmetric (i.e., closed under permutation of the coordinates.).

Definition 3.4.12.

1. For $v_1, \ldots, v_{n-1} \in V$ and a subset $X \subseteq V$, we define

$$E(v_1, \ldots, v_{n-1}, X) := \{x \in X : V \models E(v_1, \ldots, v_n, x).\}$$

2. A *partitioned formula* is a formula $\phi(x_1, \ldots, x_k; y_1, \ldots, y_\ell)$ with two distinguished groups of variables $\overline{x}, \overline{y}$. We often refer to the variables $\overline{y}$ as the *parameter variables*.

3. A partitioned formula $\phi(\overline{x},\overline{y})$ is *stable* if the graph $(V^{|x|}\times V^{\ell},R)$ with $R:=\{(\overline{a},\overline{b})\in V^{|x|}\times V^{|y|}: V\models\phi(\overline{a},\overline{b})\}$ is H_m-*-free for some $m\in\mathbb{N}$.

   That is, for some m there are no tuples $\overline{a}_1,\dots,\overline{a}_m\in V^k, \overline{b}_1,\dots,\overline{b}_m\in V^{\ell}$ such that $V\models\phi(\overline{a}_i,\overline{b}_j)$ if and only if $i\leq j$.

4. We say that a definable set $X\subseteq V$ is *large* if $\delta(X)>0$, and we say that X is small if $\delta(X)\leq 0$.

We will use some basic local stability such as definability of types, and Shelah's 2-rank. We will discuss these as we go through the proof of the following result:

Proposition 3.4.13. *Assume that $E(x_1;x_2,\dots,x_n)$ is stable. Then there is a definable set $A\subseteq V$ such that $\delta(A)>0$ and either $(a_1,\dots,a_n)\in E$ for all pairwise distinct $a_1,\dots,a_n\in A$, or $(a_1,\dots,a_n)\notin E$ for all pairwise distinct $a_1,\dots,a_n\in A$.*

Given the assumption of stability and symmetry of E and basic properties of stable formulas, such as:

- Closure of stability under boolean combinations.
- For every stable formula $\phi(\overline{x},\overline{y})$, all ϕ-types over models are defined by boolean combinations of instances of $\psi(\overline{y}:\overline{x}):=\phi(\overline{x};\overline{y})$.

Claim: *We can choose a finite set of partitioned formulas Δ such that:*

1. *$E(x_1,\dots,x_n)\in\Delta$.*
2. *Δ is closed under negation and permutation of variables.*
3. *If $\phi(x_1,\dots,x_s)\in\Delta$ then $\phi(x_1;x_2,\dots,x_s)$ is stable.*
4. *If $\phi(x;\overline{y})\in\Delta$, then every ϕ-type in x over V is defined by an instance of some formula in Δ.*

Proof. Take $\Delta=\{E(x_1,\dots,x_n),\neg E(x_1,\dots,x_n)\}$, then: (1) is clear, (2) follows from the symmetry of E, and (3) follows from the assumption that $E(x_1;x_2,\dots,x_n)$ is stable.

Finally, since $E(x_1;x_2,\dots,x_n)$ is stable, every E-type has a definition of the form

$$\bigvee_{I\subseteq\{1,\dots,2k\},|I|=k}\left(\bigwedge_{i\in I}E(a_i;x_2,\dots,x_m)\right)$$

for some $a_1,\dots,a_{2k}\in V$. □

As every formula in Δ is stable, $R_\Delta(x=x)$ is finite. Let $S\subseteq V$ be a large definable subset of the smalles R_Δ-rank (among large subsets): since $R_\Delta(x=x)$ is finite, we can simply take

$$\ell=\min\{R_\Delta(S):S\subseteq V, S\text{large}\}$$

and pick $S\subseteq V$ such that $R_\Delta(S)=\ell$.

By Proposition 2.2.7(4), S cannot be covered by finitely many definable sets of smaller R_Δ-rank: If $S_1, \ldots, S_k \subseteq S$ have R_Δ-rank smaller than ℓ, then $S_1, \ldots, S_k$ are small (by the minimality of ℓ) and we would have

$$\delta\left(\bigcup_{i=1}^{k} S_k\right) = \max\{\delta(S_i) : i \leq k\} = 0 < \delta(S)$$

So, $\bigcup_{i=1}^{k} S_i$ does not cover S.

Hence, by compactness, there is a complete Δ-type p over V such that $R_\Delta(\{S\} \cup p) = R_\Delta(S)$: Let $\pi(x) := \{S\} \cup \{\neg\psi : R_\Delta(\psi \wedge S) < \ell\}$. By the previous statement, $\pi(x)$ is consistent, and any completion p of $\pi(x)$ will satisfy $R_\delta(\{S\} \cup p) = R_\Delta(S)$.

As p is definable, say using parameters from some countable $V_0 \subseteq V$, for every k we have a complete well-defined type $p^{(k)}(x_1, \ldots, x_k)$ over V given by

$$p^{(k)}(x_1, \ldots, x_k) = \bigcup\{\mathrm{tp}_\Delta(a_k, \ldots, a_1/V') : V_0 \subseteq V' \subseteq V,\ V' \text{ is countable, and } a_{i+1} \models p \restriction_{V' a_0 \ldots a_i} \text{ for } i$$

We know show the type $p^{(k)}$ is well-defined: Let $a_1, \ldots, a_k$ and $a'_1, \ldots, a'_k$ be tuples such that $a_{i+1} \models p \restriction_{V' a_1 \ldots a_i}$ and $a'_{i+1} \restriction_{V' a'_1 \ldots a'_i}$ for $i < k$. If $\phi(x_1, \ldots, x_k; b) \in \mathrm{tp}_\Delta(a_1, \ldots, a_k)$, we have:

$$\begin{aligned}
&\phi(x_1, \ldots, x_k; \overline{b}) \in \mathrm{tp}_\Delta(a_1, \ldots, a_k) \\
&\Rightarrow \phi(a_1, \ldots, a_{k-1}, x_k; b) \in p \restriction_{V' a_1, \ldots a_{k-1}} \\
&\Rightarrow (a_1, \ldots, a_{k-1}, b) \models d_p\phi(x_k; x_1, \ldots, x_{k-1}, \overline{y}) \in p^{k-1} \\
&\Rightarrow (a_1, \ldots, a_{k-1}) \models d_p\phi(x_k; x_1, \ldots, x_{k-1}, \overline{b}) \\
&\Rightarrow (a'_1, \ldots, a'_{k-1}) \models d_p\phi(x_k; x_1, \ldots, x_{k-1}, \overline{b}) \qquad \text{(Induction hypothesis)} \\
&\Rightarrow d_p\phi(x_k; a'_1, \ldots, a'_{k-1}, \overline{b}) \in p \restriction_{V' a'_1 \ldots a'_{k-1}} \\
&\Rightarrow \phi(x_1, \ldots, x_k; \overline{b}) \in \mathrm{tp}_\Delta(a'_1, \ldots, a'_k).
\end{aligned}$$

It is clear that $p^{(k)}$ is a complete k-type.

Claim: *For any formula $r(x_1, \ldots, x_k) \in \Delta$, if $p^{(k)} \vdash r(x_1, \ldots, x_k)$ then there is a large definable subset $A \subseteq S$ such that $\models r(a_1, \ldots, a_k)$ holds for any pairwise distinct $a_1, \ldots, a_k$ from A.*

Proof. We prove this by induction on k. For $k = 1$, suppose $p(x) \vdash r(x_1)$ for $r(x_1) \in \Delta$, and take $A = r(S)$. It is necessary only to show that A is large.

By the choice of p, the R_Δ-ranks of S and $S \wedge r$ are equal. Therefore, $R_\Delta(S \wedge \neg r) < R_\delta(S)$, and by the minimality of $R_\Delta(S)$ among large sets, we have that $\delta(S \wedge \neg r) = 0$.

Thus, $\delta(A) = \delta(S \wedge r) = \delta(S) > 0$, and we conclude that A is large.

Assume now the result for $k > 1$. By the choice of Δ, there is some formula $\psi(x_1, \ldots, x_{k-1}) \in \Delta$ such that the type $p \restriction_{r(x_1, \ldots, x_{k-1}; x_k)}$ is defined by $\psi(x_1, \ldots, x_{k-1})$, that is,

$$r(v_1, \ldots, v_{k-1}; x_k) \in p(x_k) \Leftrightarrow V \models \psi(v_1, \ldots, v_{k-1}).$$

By induction hypothesis, there is a large definable set $B \subseteq S$ such that $V \models \psi(b_1, \ldots, b_{k-1})$ for all pairwise distinct $b_1, \ldots, b_{k-1} \in B$.

As B is definable, B can be written as $B = \prod_{\mathcal{U}} B_i$ for some $B_i \subseteq S_i$. For each $i < \omega$, let $A_i \subseteq B_i$ be maximal under inclusion such that $r(a_1, \ldots, a_k)$ holds for all pairwise distinct $a_1, \ldots, a_k$ from A_i, and let $A = \prod_{\mathcal{U}} A_i$. We have:

(i) $A \subseteq B$.

(ii) $V \models r(a_1, \ldots, a_k)$ for any pairwise distinct $a_1, \ldots, a_k \in A$.

(iii) For any $b \in B \setminus A$ there are some pairwise distinct $a_1, \ldots, a_{k-1} \in A$ such that $V \models \neg r(a_1, \ldots, a_{k-1}, b)$. This follows from the maximality of $A_i \subseteq B_i$, and by Łoś' Theorem.

We will show that A is large. In fact, we will show that $\delta(A) \geq \dfrac{1}{k-1}\delta(B)$.

Assume otherwise, and say $\delta(A) = \alpha_1 < \dfrac{1}{k-1}\delta(B)$. Since $A \subseteq B$, we have that for all pairwise distinct $a_1, \ldots, a_{k-1} \in A$, $V \models \psi(a_1, \ldots, a_{k-1})$. So, $r(a_1, \ldots, a_{k-1}; x_k) \in p$.

By the choice of p, the R_Δ-rank of $S(x_k) \wedge r(a_1, \ldots, a_{k-1}; x_k)$ is equal to the R_Δ-rank of S, so the R_Δ-rank of $S(x_k) \wedge \neg r(a_1, \ldots, a_{k-1}; x_k)$ is smaller than $R_\Delta(S)$.

This implies that $B \setminus r(a_1, \ldots, a_{k-1}; S)$ is small by the choice of S, that is, $\delta(B \setminus r(a_1, \ldots, a_{k-1}; B)$=0. By condition (iii), we have

$$B \setminus A \subseteq \bigcup_{a_1,\ldots,a_{k-1} \in A \text{distinct}} (B \setminus r(a_1, \ldots, a_{k-1}; B)$$

and by Proposition 2.2.7(6), $\delta(B \setminus A) \leq (k-1)\delta(A) + 0 = (k-1)\alpha_1$. This implies by Proposition 2.2.7(4)

$$\delta(B) = \max\{\delta(A), \delta(B \setminus A)\} \leq \max\{\alpha_1, (k-1)\alpha_1\} \leq (k-1)\alpha_1 < \delta(B),$$

a contradiction. □

Finally, as both $E(x_1, \ldots, x_n)$ and $\neg E(x_1, \ldots, x_n)$ are in Δ and $p^{(n)}$ is complete, either $p^{(n)} \vdash E(x_1, \ldots, x_n)$ or $p^{(n)} \vdash \neg E(x_1, \ldots, x_n)$. This concludes the proof of Theorem 3.4.13. The case $n = 2$ is the Theorem 3.4.11

Lecture 4

Strongly minimal pseudofinite structures

Strongly minimal sets control uncountably categorical theories and more generally ω-stable theories of finite Morley rank. The model theory of strongly minimal sets is very well-known and is the most accesible special case of general stability theory.

The basic examples of strongly minimal structures are algebraically closed fields (in the ring language), infinite vector spaces over division rings, and infinite free G-sets in the language $L = \{g \cdot * : g \in G\}$.

In this chapter we will present the following result, which is presumably a folklore result that can be traced back to Zilber. The proof we present here follows the exposition of [24].

In the first section we will collect some standard definitions and results about strongly minimal structures. The reader with background in this subject can simply skip to Section 4.2 where these results are applying for pseudofinite structures.

4.1 Strongly minimal structures

Definition 4.1.1. A compete 1-sorted theory T in a language L is said to be *strongly minimal* if every definable subset $X \subseteq M^1$ (possibly defined with parameters) of any model $M \models T$ is either finite or cofinite.

For this chapter, we fix a complete strongly minimal L-theory T, a saturated model D of T. The cartesian powers of D are the sets D^n for $n \geq 1$, and we fix an auxiliary point $D^0 = \{*\}$.

Definition 4.1.2.

1. Let $X \subseteq D^n$ be a definable set. Then $\dim(X)$ is the least $k \leq n$ such that we can write X as a finite union of definable sets $X_1 \cup \cdots \cup X_r$ such that for each i there is a projection $\pi_i : D^n \to D^k$ such that $\pi_i \restriction_{X_i}: X_i \to D^k$ is finite-to-one.

2. Let $X \subseteq D^n$ be a definable set of dimension k. Then $\mathrm{mult}(X)$ is the greatest natural number m (if one exists) such that X can be written as a disjoint union of definable sets $X_1, \ldots, X_m$ such that $\dim(X_i) = k$ for each i.

3. A *k-cell* is a definable set $X \subseteq D^n$ for some $n \geq k$ such that for some $r \neq 0$ there is a projection $\pi : D^n \to D^k$ such that $\dim(\pi(X)) = k$ and $\pi \restriction_X$ is r-to-1.

Remark 4.1.3. Clearly $\dim(X)$ as defined in (1) exists, because the projection identity $\pi : D^n \to D^n$ is one-to-one.

Lemma 4.1.4. *If $X = X_1 \cup \cdots \cup X_m$ are all definable sets, then $\dim(X) = \dim(X_i)$ for some $i \leq m$.*

Proof. Suppose $X \subseteq D^n$. Note that if for some $Y \subseteq D^n$ we have a projection $\pi : D^n \to D^k$ such that $\pi|_Y$ is finite-to-one, then whenever $k \leq k' \leq n$ there is a projection $\tilde{\pi} : D^n \to D^{k'}$ that is also finite-to-one, simply by adding more variables on which take the projection (that is, if $\pi^{-1}|_Y(\overline{a})$ is finite, certainly $\tilde{\pi}^{-1}|_Y(\overline{a}, b_1, \ldots, b_{k'-k})$ is finite too).

Suppose now that $\dim(X_i) < \dim(X)$ for all $i \leq m$. Consider for every $i \leq m$ the decomposition $X_i = Y_i^1 \cup \cdots \cup Y_i^{r_i}$ with projections $\pi_i^j : Y_i^j \to D^{\dim(X_i)}$ which are finite-to-one, witnessing that X_i has dimension $\dim(X_i)$. We can adjust these projection to find other projections $\tilde{\pi}_i^j : D^n \to D^{\dim(X)-1}$ such that $\tilde{\pi}_j^i|_{Y_i^j}$ is finite-to-one, obtaining a decomposition $X = \bigcup_{1 \leq i \leq m, 1 \leq j \leq r_i} Y_i^j$ together with projections $\tilde{\pi}_i^j$ showing that X has dimension less than or equal to $\dim(X) - 1$. A contradiction. ∎

Proposition 4.1.5.

1. $\mathrm{mult}(X)$ *exists for any definable X.*
2. *For any $n \geq 0$, D^n has dimension n and multiplicity* 1.
3. *Any k-cell has dimension k. Moreover, any definable set X is a finite disjoint union of cells, i.e., k-cells for possibly varying k.*
4. *For $X \subseteq D^n$ definable,* $\dim(X) = 0$ *if and only if X is finite.*

Proof.

4. Recall that D^0 is an auxiliary point $D^0 = \{*\}$. Then, X is finite if and only if the map $\pi : X \to \{*\}$ has finite domain iff $\pi = \pi^0|_X$ is finite-to-one, (when π^0 is the constant projection $D^n \to D^0$) iff $\dim(X) = 0$.

2. By induction on n. For $n = 0$, $D^0 = \{*\}$ is finite and by (4) we have that $\dim(D^0) = 0$. Also, since D^0 has a single point, if D^0 is written as a disjoint union of non-empty sets, the union has to contain only one set. Thus, $\mathrm{mult}(D^0) = 1$.

   For $n = 1$, notice that $X = D^1 \subseteq D^1$, so $\dim(D^1) \leq 1$ as witnessed by the identity projection $\pi : D^1 \to D^1$. Now, since D^1 is saturated, in particular it is infinite, and so $\dim(D^1) \neq 0$ by (4). So, $\dim(D^1) = 1$.

   For the multiplicity, let us assume that $\mathrm{mult}(D^1) = m \geq 2$. Then there are disjoint definable sets $X_1, \ldots, X_m$ such that $D^1 = X_1 \cup \cdots \cup X_m$ and $\dim(X_i) = 1$ for $i = 1, \ldots, m$. Consider the definable set $Y = X_2 \cup \cdots \cup X_m$. Since D is strongly minimal, either $Y \subseteq X_2$ or $Y^c = X_1$ is finite, but this is a contradiction because both X_1, X_2 have dimension 1, and in particular they are infinite by (4).

Thus, we conclude that $\mathrm{mult}(D^1) = 1$.

Now, we assume as induction hypothesis that $\dim(D^n) = n$ and $\mathrm{mult}(D^n) = 1$. Suppose for a contradiction that $\dim(D^{n+1}) \leq n$. Then there are definable sets $X_1, \ldots, X_m \subseteq D^{n+1}$ and projections $\pi_i : D^{n+1} \to D^k$ so that $k \leq n$, $\pi_i|_{X_i}$ is finite-to-one and $D^{n+1} = X_1 \cup \cdots X_m$.

Given $\overline{a} \in D^n$, we can define the set $\ell_{\overline{a}} := \{(\overline{a}, y) : y \in D\}$ ("the line above $\overline{a}$). Given X_i, we can consider the definable set $\Gamma_i = \{y \in D : (\overline{a}, y)\} \subseteq D^1$. By strong minimality, Γ_i is either finite or cofinite. If finite, then $X_i \cap \ell_{\overline{a}}$ is finite. If Γ_i is cofinite, then $\ell_{\overline{a}} \setminus X_i$ is finite.

So, we have showed that for every X_i either $X_i \cap \ell_{\overline{a}}$ is finite or $\ell_{\overline{a}} \setminus X_i$ is finite, and we conclude that every $\ell_{\overline{a}}$ is "almost contained" in at least one set $X_{i_{\overline{a}}}$ with $1 \leq i_{\overline{a}} \leq m$.

(i.e., $\ell_{\overline{a}} \setminus X_{i_{\overline{a}}}$ is finite for some index $1 \leq i_{\overline{a}} \leq m$).

Notice that if $\ell_{\overline{a}}$ is "almost contained" in X_i, then the projection π_i sends the variables $(x_1, \ldots, x_n, x_{n+1})$ to a tuple of variables $(x_{i_1}, \ldots, x_{i_{k-1}}, x_{n+1})$ with $1 \leq i_1 < \cdots < i_{k-1} \leq n$, since otherwise the map $\pi_i|_{X_i}$ would not be finite-to-one.

Consider now the sets $Y_i := \{\overline{a} \in D^n : \ell_{\overline{a}}$ is almost contained in $X_i\}$. The sets Y_i are definable, because of the uniform bound of $|\ell_{\overline{a}} \setminus X_i|$ while $\overline{a}$ varies. We then have that $D^n = Y_1 \cup \cdots \cup Y_m$, and we can consider the projections $\tilde{\pi}_i = \pi \circ \pi_i$, where π is the projection forgetting only the last variable.

The maps $\tilde{\pi}_i$ are projections from D^n to D^{k-1}, and $\tilde{\pi}_i|_{Y_i} = (\pi \circ \pi_i)|_{X_i}$ are finite-to-one. Since $\dim(D^n) = n$, one of the projections $\tilde{\pi}_i$ uses n variables, and so the corresponding projection $\pi_i : D^{n+1} \to D^k$ uses $n+1$ variables. So, $k = n+1$. A contradiction.

e now show that $\mathrm{mult}(D^{n+1}) = 1$. Assume $D^{n+1} = X_1 \cup X_2$, where X_1, X_2 are disjoint non-empty definable subsets of D^{n+1} of dimension $n+1$. Define $Y_j = \{\overline{a} \in D^n : \ell_{\overline{a}}$ is almost contained in $X_j\}$ for $j = 1, 2$. Then $D^n = Y_1 \cup Y_2$ and Y_1, Y_2 are disjoint, which implies that one of them must have dimension lower than n. Assume without loss of generality that $\dim(Y_2) \leq n - 1$.

If $Y_2 = Z_1 \cup \cdots \cup Z_m$ is a decomposition witnessing that $\dim(Y_2) = k < n$, with projections $\pi_i : D^n \to D^k$, then

$$X_2 = (Z_1 \times D) \cup \cdots \cup (Z_m \times D) \cup \pi^{-1}|_{X_2}(Y_1),$$

with π being the projection on the first n coordinates, is a decomposition showing that $\dim(X_2) \leq n$, a contradiction.

Thus, $\mathrm{mult}(D^{n+1}) = 1$, and we conclude the proof of (2).

1. Let $X \subseteq D^n$ be a definable set with $\dim(X) = k$, and suppose there are infinitely many disjoint sets $(X_i : i < \omega)$ such that $\dim(X_i) = k$ and $X_i \subseteq X$ for every $i < \omega$.

Since $\dim(X) = k$, there are definable sets $Y_1, \ldots, Y_r$ and projections $\pi_j : D^n \to D^k$ such that $X = Y_1 \cup \cdots \cup Y_r$ and $\pi_j \restriction_{Y_j}$ is finite-to-one. Then, for every $i < \omega$, we can consider the sets $X_i^j := X_i \cap Y_j$ for $j = 1, \ldots, r$. By Lemma 4.1.4, for every X_i there is a unique minimal index $j_i \leq r$ such that $X_i \cap Y_{j_i}$ has the same dimension as X_i (that is, $\dim(X_i \cap Y_{j_i}) = k$).

By the piggeonhole principle, there are infinitely many indices $i < \omega$ such that $X_i \cap Y_j$ has dimension k, for a fixed $j \leq r$, and by restricting our attention to those indices we have that the following:

- The projection $\pi_j : D^n \to D^k$ satisfies that $\pi_j|_{Y_j}$ is finite-to-one.
- $\dim(X_i \cap Y_j) = k$ for all $i < \omega$.
- $X_i \cap X_{i'} = \emptyset$ for all $i \neq i'$.

It is clear that for every $i < \omega$, $\pi_j(X_i \cap Y_j)$ has dimension k, since otherwise we would be able to extend the projections π_j to projections from D^n witnessing $\dim(X_i \cap Y_j) < k$. For simplicity, let us write $\pi = \pi_j$, $Y = Y_j$ and $X_i = X_i \cap Y_j$.

Since $\mathrm{mult}(D^k) = 1$ (by (4)), there are not disjoint subsets Z_1, Z_2 of D^k such that $\dim(Z_1) = \dim(Z_2) = k$. Moreover, if Z_1, Z_2 are subsets of D^k of dimension k, then $\dim(Z_1 \cap Z_2) = k$.

Consider the type $p(\overline{y}) := \{\exists \overline{x}(X_m(\overline{x}) \wedge \pi(\overline{x}) = \overline{y}) : m < \omega\}$. This type is finitely consistent because $\dim\left(\bigcap_{i \leq m} \pi(X_i)\right) = k$. So, by saturation of D, there is $\overline{a} \in D$ realizing $p(\overline{y})$, and this implies that $\pi^{-1}(\overline{a}) \cap X_i \neq \emptyset$ for all $i < \omega$, and since the sets X_i are disjoint, we conclude that $\pi^{-1}(\overline{a})$ is infinite. This contradicts the fact that π is finite-to-one.

3. Let X be a k-cell, that is, $X \subseteq D^n$ and for some $r \neq 0$ there is a projection $\pi : D^n \to D^k$ such that $\dim(\pi(X)) = k$ and $\pi|_X$ is r-to-1.Then, π and $Y_1 = X$ serve as a decomposition that shows that $\dim(X) = k$.

   Now let X be an arbitrary definable set of dimension k. Note that if $X = Y_1 \cup \cdots \cup Y_r$ is a union of disjoint sets of dimension k, then by decomposing each Y_i into cells we obtain a decomposition of X. So, we may assume without loss of generality that $\mathrm{mult}(X) = 1$.

   Let $X = Y_1 \cup \cdots Y_\ell$ and $\pi_i : Y_i \to D^k$ be finite-to-one projections. By using intersections and complements, and possibly repeating projections, we may assume that $Y_1, \ldots, Y_\ell$ are disjoint. By reordering, we can also assume that $\dim(X) = \dim(Y_1)$ and $\dim(Y_i) < \dim(X)$ for $i = 2, \ldots, m$. By induction on $\dim(X)$, we may also assume that every Y_i can be written as a union of cells.

   Consider for $r < \omega$ the set $Y_{1,r} = \{y \in Y : |\pi_i^{-1}(\pi_i(y))| = r\}$. By strong minimality, there is $r < \omega$ such that $Y_{1,t} = \emptyset$ for $t > r$. Then we have the disjoint union $Y_1 = Y_{1,1} \cup \cdots \cup Y_{1,r}$, and since $\dim(Y_1) = k$, one of the sets mut have dimension k (say $Y_{1,s}$) and the rest have dimension less than k.

   So, since $\pi_i : Y_{1,1} \to \pi_i(Y_{1,1})$ is s-to-one, we have that $\pi_i(Y_{1,s}) \subseteq D^k$ with $\dim(\pi_i(Y_{1,s})) = RM(\pi_i(Y_{1,s})) = RM(Y_{1,s}) = k$. Thus, $Y_{1,s}$ is itself a k-cell, and the result follows.

□

The main point is to deduce "good behavior" for definable sets in higher ambient spaces, using some basic lemmas about algebraic closure in strongly minimal sets.

For example, for $b_1, \dots, b_n \in D$ and A a small subset of D, we say that $\{b_1, \dots, b_n\}$ is *algebraically independent over* A if $b_i \in \operatorname{acl}(A, b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n)$ for each i. This is equivalent to $b_i \in \operatorname{acl}(A, b_1, \dots, b_{i-1})$ for all $i = 1, \dots, n$, by the exchange property.

Proposition 4.1.6 (Exchange principle)**.** *Suppose D is strongly minimal, $A \subseteq D$ and $b, c \in D$. If $b \in \operatorname{acl}(Ac) \setminus \operatorname{acl}(A)$, then $c \in \operatorname{acl}(Ab)$.*

Proof. Let $\phi(x, \overline{a}, c)$ be the algebraic formula witnessing that $b \in \operatorname{acl}(Ac)$, and assume that $|\phi(D, \overline{a}, c)| = n$. Consider the formula $\psi(y) := \exists^{=n} x(\phi(x, \overline{a}, y)$. We have that $D \models \psi(c)$. If $\phi(b, \overline{a}, y) \wedge \psi(y)$ defines a finite set, this would imply that $c \in \operatorname{acl}(Ab)$ and we are done. Otherwise, by strong minimality, $|D \setminus (\phi(b, \overline{a}, D) \wedge \psi(D))| = \ell$ for some $\ell \in \mathbb{N}$. Consider now the formula $\chi(x) := \exists^{=\ell} y(\neg(\phi(x, \overline{a}, y) \wedge \psi(y))$. Notice that $D \models \chi(b)$, so since $b \notin \operatorname{acl}(A)$ we have that $\chi(x)$ defines an infinite set in D.

Let $b_1, \dots, b_{n+1}$ be different elements in $\chi(D)$. We have that $\phi(b_i, \overline{a}, D) \wedge \psi(D)$ is cofinite for each $1 \leq i \leq n+1$, and so we can take $c' \in D$ such that $c' \models \bigwedge_{i=1}^{n+1}(\phi(b_i, \overline{a}, y) \wedge \psi(y))$, which is a contradiction because $c' \models \psi(y)$ implies $n = |\phi(D, \overline{a}, c')| \geq |\{b_1, \dots, b_{n+1}\}| = n + 1$.

Therefore, we conclude that $c \in \operatorname{acl}(Ab)$. □

Proposition 4.1.7. *A set $\{b_1, \dots, b_n\}$ is algebraically independent over A if and only if $b_i \notin \operatorname{acl}(A, b_1, \dots, b_{i-1})$ for all $i = 1, \dots, n$.*

Proof. The direction ($\Rightarrow$) is clear by definition. Suppose now that $\{b_1, \dots, b_n\}$ are not algebraically independent over A, and let $i \leq n$ minimal such that

$$b_i \in \operatorname{acl}(A, b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n).$$

If $i = n$, or $b_i \in \operatorname{acl}(A, b_1, \dots, b_{i-1})$, we are done. Otherwise, let $j \geq i$ minimal such that $b_i \in \operatorname{acl}(A, b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_{j+1})$, and put $\tilde{A} = A \cup \{b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_j\}$. By minimality, $b_i \in \operatorname{acl}(\tilde{A} b_{j+1}) \setminus \operatorname{acl}(\tilde{A})$, so by the exchange principle, $b_{j+1} \in \operatorname{acl}(\tilde{A} b_i) = \operatorname{acl}(A, b_1, \dots, b_{i-1}, b_i, b_{i+1}, \dots, b_j)$. □

Definition 4.1.8. For $\overline{b}$ a tuple from D, $\dim(\overline{b}/A)$ denotes the cardinality of some (equivalently any) maximal subtuple of $\overline{b}$ algebraically independent over A.

Proposition 4.1.9. *For a subset $X \subseteq D^n$, definable over A,*

$$\dim(X) = \max\{\dim(\overline{b}/A) : \overline{b} \in X\}.$$

Proof. By induction on n. If $X = \phi(D, \overline{a})$ is an A-definable subset of D^1, we have by strong minimality that X is either finite or cofinite. If X is finite, then for every $b \in X$ we have $b \in \operatorname{acl}(\overline{a}) \subseteq \operatorname{acl}(A)$, so $\max\{\dim(\overline{b}/A) : b \in X\} = 0 = \dim(X)$.

If X is cofinite, consider the type

$$p(x) = \{\phi(x, \overline{a})\} \cup \{\neg\psi(x) : \psi(x) \in L(A) \text{ and } \psi(D) \text{ is finite}\}.$$

This type is finitely consistent in D, since the finite union of finite sets is finite, and by saturation there is $b \models p(x)$. Note that $b \in X \setminus \mathrm{acl}(A)$, so

$$1 = \max\{\dim(b/A) : b \in X\}$$

On the other hand, since $X \cup (X^c) = D^1$ and X^c is finite, we have that $\dim(D^1) = 1 = \max\{\dim(X), \dim(X^c)\} = \max\{\dim(X), 0\} = \dim(X)$.

Now we proceed with the induction step. Let $X = \phi(D^{n+1}, \overline{a})$ be an A-definable subset of D^{n+1}. If $\dim(X) = n+1$, consider the type

$$p(\overline{x}) := \{\phi(\overline{x}, \overline{a})\} \cup \{\neg\psi(\overline{x}) : \psi(\overline{x}) \in L(A) \text{ and } \dim(\psi(D^{n+1}) < n+1\}.$$

If $\Gamma_0 \subseteq \{\phi(\overline{x}, \overline{a}\} \cup \{\neg\psi_i(\overline{x}) : i \leq m\}$ is a finite subset of $p(\overline{x})$, then $\phi(\overline{x}, \overline{a}) \wedge \bigwedge_{i \leq m} \neg(\psi(\overline{x}))$ is non-empty, since otherwise we would have $\phi(\overline{x}, \overline{a}) \vdash \bigvee_{i \leq m} \psi_i(\overline{x})$ which implies $\dim(X) = \dim(\phi(D^{n+1}, \overline{a})) \leq \max\{\dim(\psi_i(D^{n+1})) : i \leq m\} \leq n$, a contradiction.

By saturation, there is a tuple $\overline{b} = (b_1, \ldots, b_{n+1}) \in D^{n+1}$ realizing the type $p(\overline{x})$. It remains to show that $\{b_1, \ldots, b_{n+1}\}$ is an A-independent set. Suppose not, and let k be minimal such that $b_{k+1} \in \mathrm{acl}(A, b_1, \ldots, b_k)$, begin $\psi(y, \overline{a}, b_1, \ldots, b_k)$ be an algebraic formula realized by b_{k+1}.

Consider the formula

$$\tilde{\psi}(x_1, \ldots, x_{n+1}; \overline{a}) := \psi(x_{k+1}; \overline{a}, x_1, \ldots, x_k).$$

We have that $\tilde{\psi}(D) \subseteq D^{n+1}$, and we can consider the projection $\pi : D^{n+1} \to D^n$ that omits the $(k+1)$-th component. Since $\psi(x_{k+1}; \overline{a}, x_1, \ldots, x_k)$ is an algebraic formula, the map $\pi|_{\tilde{\psi}(D)}$ is finite-to-one, which implies that $\dim(\psi) \leq n$. However, this contradicts the fact that $\overline{b} \models p(\overline{x})$. This contradiction allows us to conclude that $\max\{\dim(\overline{b}/A) : \overline{b} \in X\} = n+1 = \dim(X)$.

Suppose now that $\dim(X) \leq n$. Then there is a decomposition $X = Y_1 \cup \cdots \cup Y_\ell$ into disjoint cells, with $\dim(Y_1) = \dim(X)$. Since Y_1 is a cell, there is a projection $\pi : D^{n+1} \to D^{\dim(X)}$ such that $\dim(\pi(X)) = \dim(X)$ and $\pi|_{Y_1}$ is r-to-1. By induction hypothesis, there is $\overline{b}' \in \pi(X)$ such that $\dim(X) = \dim(\pi(X)) = \dim(\overline{b}'/A)$.

Let $\overline{b}$ be an element in Y_1 such that $\pi(\overline{b}) = \overline{b}'$. Since $\overline{b}' \in D^{\dim(X)}$ and $\dim(\overline{b}'/A) = \dim(X)$, we also know that the tuple $\overline{b}'$ is algebraically independent over A. So, since $\overline{b}'$ is a subtuple of $\overline{b}$, we have $\dim(\overline{b}/A) \geq \dim(\overline{b}'/A) = \dim(X)$.

On the other hand, since $\overline{b}$ satisfies the algebraic formula "$\overline{y} \in Y_1 \wedge \pi(\overline{y}) = \overline{b}'$", we also have $\dim(\overline{b}/A) \leq \dim(\overline{b}'/A) = \dim(X)$. ☐

What we have called dimension is the same thing as Morley rank (or RM), and what we called multiplicity is the same thing as Morley degree (or dM), all in the special case of strongly minimal theories.

Below, we recall the main definition and results about Morley rank and Morley degree.

Definition 4.1.10 (Morley rank, Morley degree)**.** 1. For a definable set $X \subseteq D^n$, we define inductively on the ordinal α the relation $RM(X) \geq \alpha$ as follows:

- $RM(X) \geq 0$ if and only if $X \neq \emptyset$.
- If α is a limit ordinal, $RM(X) \geq \alpha$ if and only if $RM(X) \geq \beta$ for all $\beta < \alpha$.
- $RM(X) \geq \alpha + 1$ if and only if there is a family of disjoint definable subsets of X $(X_i : i < \omega)$ such that $RM(X_i) \geq \alpha$ for every $i < \omega$.

2. We say that the *Morley rank of X is α* if $RM(X) \geq \alpha$ but $RM(X) \not\geq \alpha + 1$.

3. If $RM(X) = \alpha$, we define the *Morley degree of X* as

$$dM(X) := \max\{n < \omega : \text{There are } X_1, \ldots, X_n \subseteq X, \text{ pairwise disjoint, of Morley rank } \alpha\}$$

Definition 4.1.11 (Morley rank for types)**.** 1. Let $p \in S_n(A)$. Then $RM(p) := \inf\{RM(X) : X \in p\}$ and $dM(p) := \{dM(X) : X \in p \text{ and } RM(X) = RM(p)\}$.

2. If $\overline{a} \in D^n$, we define $RM(\overline{a}/A) := RM(\text{tp}(\overline{a}/A))$.

Note that from these definitions it is clear that given a type p there is a formula $\phi_p \in p$ such that $(RM(p), dM(p)) = (RM(\phi_p), dM(\phi_p))$.

Lemma 4.1.12. *If $X \subseteq D^n$ is A-definable, then $RM(X) = \sup\{RM(\overline{a}/A) : \overline{a} \in X\}$.*

Proof. By definition, $RM(\overline{a}/A) \leq RM(X)$ for all $\overline{a} \in X$. Thus, $\sup\{RM(\overline{a}/A) : \overline{a} \in X\} \leq RM(X)$. conversely, we can consider the type $p(\overline{x}) = \{X(\overline{x})\} \cup \{\neg\phi(\overline{x}) : RM(\phi) < RM(X)\}$.

If $\Gamma \subseteq \{X, \neg\phi_1, \ldots, \phi_n\}$ is a finite subset of Γ, then we have that

$$RM\left(\bigvee_{i=1}^{n} \phi_i\right) = \max\{RM(\phi_i) : 1 \leq i \leq n\} < RM(X),$$

and therefore X cannot be contained in $\bigcup_{i=1}^{n} \phi_i(D)$. That is, there is $b_\Gamma \models \Gamma$.

By saturation, there is $\overline{a} \in D^n$ such that $\overline{a} \in p(\overline{x})$, and by construction, we will have that $RM(\overline{a}/A) = RM(X)$ ☐

Lemma 4.1.13. *If $b \in acl(A, \overline{a})$, then $RM(\overline{a}, b/A) = RM(\overline{a}/A)$.*

Proof. Clearly $RM(\overline{a}, b/A) \geq RM(\overline{a}/A)$. We know prove by induction on ordinals that $RM(\overline{a}, b/A) \geq \alpha$ implies $RM(\overline{a}/A) \geq \alpha$.

For $\alpha = 0$ or for α a limit ordinal, this is clear. Suppose now that $RM(\overline{a}, b/A) \geq \alpha + 1$. Let $\theta(\overline{x}) \in \text{tp}(\overline{a}/A)$ be a formula such that $RM(\overline{a}/A) = RM(\theta(\overline{x})) = \alpha$ and $dM(\phi) = 1$, and let $\psi(y; \overline{a}) \in \text{tp}(b/A\overline{a})$ be a formula such that $|\psi(y; \overline{a}) = r$, with r minimal. We consider now the formula $\phi(\overline{x}, y) := \theta(\overline{x}) \wedge \psi(y; \overline{x}) \wedge \exists^{=r} y(\psi(y, \overline{x}))$. Since $\phi(\overline{x}, y) \in \text{tp}(\overline{a}, b/A)$, we know that $RM(\phi(\overline{x}, y)) \geq RM(\overline{a}, b/A) \geq \alpha + 1$.

Let $(X_i : i < \omega)$ be a collection of pairwise disjoint definable subsets of $\phi(D^{n+1})$ contained in $\phi(\overline{x}, y)$, with $RM(X_i) \geq \alpha$, and consider the formula $\chi_i(\overline{x}) := \exists y(X_i(\overline{x}, y))$.

1. $RM(\chi_i(\overline{x})) \geq \alpha$*:* since X_i has Morley rank at least α, there are $\overline{c}, d$ such that $\models X_i(\overline{c}, d)$ and $RM(\overline{c}, d) \geq \alpha$. By induction hypothesis, since d will b algebraic over $A\overline{c}$, we have that $RM(\overline{c}) \geq \alpha$, and so, $RM(\chi_i(\overline{x})) \geq RM(\overline{c}/A) \geq \alpha$.

2. $RM(\chi_1 \wedge \cdots \wedge \chi_m) \geq \alpha$ *for all* m*:* If not, let m be minimal such that $RM(\chi_1 \wedge \cdots \wedge \chi_m) < \alpha$. Then, $RM(\chi_1 \wedge \cdots \wedge \chi_{m-1}) = \alpha = RM(\chi_m)$, and since we have

$$\chi_m = \left(\chi_m \cap \bigcap_{i=1}^{m-1} \chi_i\right) \cup \left(\chi_m \cap \neg \bigcap_{i=1}^{m-1} \chi_i\right)$$

and the first part of the disjoint have rank less than α, we have that $RM\left(\chi_m \wedge \neg \bigwedge_{i=1}^{m-1} \chi_i\right) \geq \alpha$.

But then, both $\bigwedge_{i=1}^{m-1} \chi_i$ and $\neg \bigwedge_{i=1}^{m-1} \chi_i$ have rank at least α, and are included in $\theta(\overline{x})$, contradicting the fact that $dM(\theta)$.

Since D is saturated, there exists $\overline{c} \in D$ such that $\chi_i(\overline{c})$ for all $i < \omega$. So, for each i, there is d_i such that $\chi_i(\overline{c}, d_i)$ holds, but all those elements have to be different because the sets X_i are disjoint. Also, $\chi_i(\overline{c}, d_i)$ implies $\psi(\overline{d_i c})$. However, $X_i \subseteq \phi(\overline{x}, y)$, so there cannot be more than r elements d satisfying $\psi(y, \overline{c})$, a contradiction.

Thus, $RM(\overline{a}/A) \geq \alpha + 1$, and the proof is complete.

□

Theorem 4.1.14. *Let D be a strongly minimal saturated structure, $A \subseteq D$ and $\overline{a} \in D$. Then* $\dim(\overline{a}/A) = RM(\overline{a}/A)$ *and* $dM(\overline{a}/A) = \mathrm{mult}(\overline{a}/A)$.

Proof. We will show by induction on k that if $a_1, \ldots, a_k$ are independent over A, then $RM((a_1, \ldots, a_k)/A) = k$. For $k = 1$, let $a_1 \notin \mathrm{acl}(A)$ and take ϕ_1 to be the formula in $\mathrm{tp}(a_1/A)$ with $RM(a_1/A) = RM(\phi_1)$. Clearly, $\phi_1(D)$ is infinite, and so we have $RM(\phi_1(D)) \geq 1$. Now, since D is strongly minimal, there cannot infinite disjoint definable subsets $X_1, X_2 \subseteq D$. In particular, $RM(\phi_1(D)) \ngeq 2$, and we conclude that $RM(a_1/A) = RM(\phi_1) = 1$.

Suppose now that $a_1, \ldots, a_{k+1}$ are independent over A, and let $\phi_{k+1}(x_1, \ldots, x_k, x_{k+1})$ and $\psi_{k+1}(x_{k+1}; a_1, \ldots, a_k) \in \mathrm{tp}(a_{k+1}/Aa_1 \ldots a_k)$ be formulas satisfying $RM(\phi_{k+1}) = RM(a_1, \ldots, a_{k+1}$ and $RM(\psi_{k+1}) = RM(a_{k+1}/Aa_1 \ldots a_k)$.

By the case $k = 1$, ψ_{k+1} is infinite. Let $\{b_i : i < \omega\}$ be different elements realizing $\mathrm{tp}(a_{k+1}/Aa_1, \ldots, a_k)$. Now, let $\phi_k(x_1, \ldots, x_k) \in \mathrm{tp}(a_1, \ldots, a_k/A)$ be a formula witnessing that $RM(\phi_k) = RM(a_1, \ldots, a_k/A) = k$, by induction hypothesis. Then, since Morley rank is invariant under definable bijections, we have that the formulas defined by $\theta_i(x_1, \ldots, x_{k+1}) :=$ "$\phi_k(x_1, \ldots, x_k) \wedge x_{k+1} = b_i$" have all Morley rank k.

Also, since there is a unique type over A of any algebraically independent $k+1$-tuple, we have that

$$\mathrm{tp}(a_1, \ldots, a_k, a_{k+1}/A) = \mathrm{tp}(a_1, \ldots, a_k, b_i/A).$$

This implies that $\theta_i \subseteq \phi_{k+1}$ for every $i < \omega$, and since they are disjoint we conclude that $RM(a_1, \ldots, a_{k+1}/A) = RM(\phi_{k+1}) \geq k + 1$.

On the other hand, since $dM(D^{k+1}) = 1$, there are no disjoint subsets of D^{k+1} with rank $\geq k + 1$. This completes the proof that $RM(a_1, \ldots, a_{k+1}/A) = k + 1$.

We have shown the result for independent tuples of elements. Suppose now that $\overline{a} \in D^n$ is an arbitrary tuple, and assume without loss of generality that $a_1, \ldots, a_k$ are independent and $a_{k+1}, \ldots, a_n \in \mathrm{acl}(A, a_1, \ldots, a_k)$. Then, $\dim(\overline{a}/A) = k = \dim(a_1, \ldots, a_k/A) =$

$RM(a_1,\ldots,a_k/A)$ and by Lemma 4.1.13 we have

$$\begin{aligned}\dim(\overline{a}/A) &= k = \dim(a_1,\ldots,a_k/A)\\ &= RM(a_1,\ldots,a_k/A) = RM(a_1,\ldots,a_k,a_{k+1}/A)\\ &= \cdots = RM(\overline{a}/A).\end{aligned}$$

□

Lemma 4.1.15. *Let $X_1,\ldots,X_n$ be definable sets. Then $RM\left(\bigcup_{i=1}^n X_i\right) = \max\{RM(X_i) : 1 \le i \le n\}$*

Proof. First not that $X \subseteq Y$ implies $RM(X) \le RM(Y)$ since any collection of non-empty disjoint subsets of X can be taken to be subsets of Y too. By induction on ordinals, we prove that if $RM\left(\bigcup_{i=1}^n X_i\right) \ge \alpha$ for subsets $X_1,\ldots,X_n$ of D^m then $RM(X_i) \ge \alpha$ for some $i \le n$.

The case $\alpha = 0$ is clear: if the union is not empty, then one of the sets must be also non-emtpy. If α is a limit ordinal, it follows from induction hypothesis and the fact that, since the union is finite, one of the sets X_i must be mentioned infinitely often and thus must have Morley rank bigger than β for all $\beta < \alpha$.

Suppose now that $RM\left(\bigcup_{i=1}^n X_i\right) \ge \alpha + 1$, and let $(A_j : j < \omega)$ be a collection of disjoint subsets of $\bigcup_{i=1}^n X_i$ with $RM(A_j) \ge \alpha$ for all $j < \omega$.

Therefore, for each $j < \omega$, we have $A_j = \bigcup_{i=1}^n (A_j \cap X_i)$, and by induction hypothesis there is an index $i_j \in \{1,\ldots,n\}$ such that $RM(A_j \cap X_i) \ge \alpha$. By piggeonhole principle, there is $i \le n$ such that $i = i_j$ for infinitely many j, and $(X_i \cap A_j : i_j = i)$ would be an infinite collection of pairwise disjoint subsets of X_i of Morley rank at least α. Thus, $RM(X_i) \ge \alpha + 1$. □

Remark 4.1.16. Note that if $X \subseteq D^n$, then $\dim(X) = \max\{\dim(\overline{b}/A) : \overline{b} \in X\} = \max\{RM(\overline{b}/A) : \overline{b} \in X\} = RM(X)$, and clearly $dM(X) = \text{mult}(X)$.

Fact 4.1.17. *Suppose $X \subseteq D^n$ and $Y \subseteq D^m$ are non-empty definable sets. Then,*

1. $RM(X \times Y) = RM(X) + RM(Y)$.
2. $dM(X \times Y) = dM(X) \cdot dM(Y)$.

In particular, if $dM(X) = dM(Y) = 1$ we also have that $dM(X \times Y) = 1$.

Proof.

1. We first show by induction on α that if $RM(Y) \ge \alpha$ then $RM(X \times Y) \ge RM(X) + \alpha$.
   - *For $\alpha = 0$:* If $RM(Y) \ge 0$, then $Y \ne \emptyset$, and since X is also non-empty we would have, for a fixed $y \in Y$, $RM(X \times Y) \ge RM(X \times \{y\}) = RM(X) = RM(X) + 0$.
   - *For α limit ordinal:* If $RM(Y) \ge \alpha$ then $RM(Y) \ge \beta$ for all $\beta < \alpha$, and by induction hypothesis, we would have $RM(X \times Y) \ge RM(X) + \beta$ for all $\beta < \alpha$. Thus, $RM(X \times Y) \ge RM(X) + \alpha$.

- Suppose $RM(Y) \geq \alpha+1$, and let $(Y_i : i < \omega)$ be a collection of disjoint subsets of Y with $RM(Y_i) \geq \alpha$. Then, $RM(X \times Y_i) \geq RM(X) + \alpha$ by induction hypothesis, and since the sets $(X \times Y_i : i < \omega)$ are all disjoint, we have $RM(X \times Y) \geq RM(X) + \alpha + 1$.

So, we have proved that $RM(X \times Y) \geq RM(X) + RM(Y)$.

Notice that the inequality $RM(X \times Y) \geq RM(X) + RM(Y)$ is not always true: if $RM(X) = 1$ and $RM(Y) = \omega$, then $RM(X \times Y) = RM(Y \times X)$ since there is a definable bijection, but $RM(X \times Y) \geq RM(Y \times X) \geq \omega + 1 > 1 + \omega = RM(X) + RM(Y)$.

However, if D is a saturated strongly minimal structure, the equality will follow from the fact that $\dim(X \times Y) = \dim(X) + \dim(Y)$. We have proved already that $\dim(X \times Y) \geq \dim(X) + \dim(Y)$. Now we show that $\dim(X \times Y) \leq \dim(X) + \dim(Y)$.

Let $X = X_1 \cup \cdots \cup X_r$ and $Y = Y_1 \cup \cdots \cup Y_s$ be decomposition of X and Y respectively, with projections $\pi_i^X : D^n \to D^{\dim(X)}$ such that $\pi_i^X|_{X_i}$ is finite-to-one, and respectively π_j^Y for Y. Then, $X \times Y = \bigcup_{1\leq i\leq r, 1\leq j\leq s} X_i \times Y_j$, and it suffices to show that $\pi_{ij} := \pi_i^X \times \pi_j^Y : D^{n+m} \to D^{\dim(X)+\dim(Y)}$ given by $\pi_{ij}(\overline{x}, \overline{y}) = (\pi_i^X(\overline{x}), \pi_j^Y(\overline{y}))$ is finite to one when restricted to $X_i \times Y_j$. But this is easy, since $\pi_{ij}^{-1}(\overline{c}, \overline{d}) = \{\overline{a} \in D^n : \pi_i^X(\overline{a}) = \overline{c}\} \times \{\overline{b} \in D^m : \pi_j^Y(\overline{b}) = \overline{d}\}$ is a cartesian product of finite sets. Thus, since the projections $\pi_{ij}|_{X_i \times Y_j}$ are finite-to-one, we conclude that $\dim(X \times Y) = \dim(X) + \dim(Y)$ as desired.

2. If $X = X_1 \cup \cdots \cup X_{dM(X)}$ and $Y = Y_1 \cup \cdots \cup Y_{dM(Y)}$ be maximal decompositions for X and Y into disjoint sets of maximal rank. Then, $X \times Y = \bigcup_{i,j} X_i \times Y_j$ is a decomposition of $X \times Y$ into $dM(X) \cdot dM(Y)$ sets of rank $RM(X_i \times Y_j) = RM(X_i) + RM(Y_j) = RM(X) + RM(Y) = RM(X \times Y)$. Thus, $dM(X \times Y) \geq dM(X) + dM(Y)$.

□

Definition 4.1.18. Let $B \subseteq M$. We consider the *transcendence dimension of B over A* to be the cardinality of a maximal subset of B independent over A. We denote this by $\dim_{tr}(B/A)$.

Theorem 4.1.19. *Suppose T is a strongly minimal theory. If $M, N \models T$ then $M \cong N$ if and only if $\dim_{tr}(M) = \dim_{tr}(N)$.*

More generally, if $\phi(\overline{x}, \overline{a})$ is a strongly minimal formula with parameters from A with $A \subseteq M, N$, and $\dim_{tr}(\phi(M)) = \dim_{tr}(\phi(N))$ then there is a bijective partial elementary map $f : \phi(M) \to \phi(N)$.

Proof. Let B be a transcendence basis for $\phi(M)$ and C be a transcendence basis for $\phi(N)$ over A (i.e., maximal subsets of $\phi(M)$ and $\phi(N)$ independent over A).

Since $|B| = \dim_{tr}(\phi(M)) = \dim_{tr}(\phi(N)) = |C|$, we can find a bijection $f : B \to C$, and such bijection is actually A-elementary: if $M \models \phi(b_1, \ldots, b_n)$ then ϕ has the form $\neg\psi(x_1, \ldots, x_n)$ for some formula $\psi(x_1, \ldots, x_n)$ with dimension $< n$. Given that

$f(b_1), \ldots, f(b_n)$ are different elements in the transcendence basis C ,we have $N \models \neg\psi(f(b_1), \ldots, f(b_n))$ and so $N \models \phi(f(b_1), \ldots, f(b_n))$.

Now, let I be the collection of all partial A-elementary maps $g : B' \to C'$ with $f \subseteq g$, $B \subseteq B' \subseteq \phi(M)$, $C \subseteq C' \subseteq \phi(N)$. It is easy to show by Zorn's Lemma that there is an element $g \in I$ which is maximal under inclusion. We have $g : B' \to C'$ is a partial A-elementary map, and now we show that $B' = \phi(M)$ and $C' = \phi(N)$, completing the proof.

If $b \in \phi(M)\backslash B'$, then there is an algebraic formula $\psi(x, \overline{d})$ isolating the type $\mathrm{tp}_M(b/B')$ (the formula in $\mathrm{tp}(b/B')$ with the least possible number of solutions). Then $M \models \exists x(\psi(x, \overline{d}))$ for some $\overline{d} \subseteq B'$, and so $N \models \exists x(\psi(x, g(\overline{d})))$. Let $c \in N$ be an element satisfying $\psi(c, g(\overline{d}))$. By the minimality of $\psi(x, \overline{d})$, we have $\mathrm{tp}_M(b/B') = \mathrm{tp}_N(c/C')$, and the map $g \cup \{(b, c)\}$ would be a partial elementary map, contradicting the maximality of g.

Thus $B' = \phi(M)$, and a similar argument shows that $C' = \phi(N)$, concluding that $g : \phi(M) \to \phi(N)$ is a partial elementary map.

□

Corollary 4.1.20. *If T is a countable strongly minimal theory, then T is κ-categorical for all $\kappa \geq \aleph_1$, and T has at most $\aleph_0$ non-isomorphic models of size $\aleph_0$ (also written as $I(\aleph_0, T) \leq \aleph_0$).*

Proof. If M, N are two models of T of cardinality $\kappa \geq \aleph_1$, then since T is countable we have $\dim_{tr}(M) = \aleph_1 = \dim_{tr}(N)$, and by the previous theorem, $M \cong N$.

If M, N are countable models of T, $M \cong N$ iff $\dim_{tr}(M) = \dim_{tr}(N)$, and so there are as many non-isomorphic models as possible values of the transcendence dimension, which are countably many (namely, $\dim_{tr}(M) \in \mathbb{N} \cup \{\aleph_0\}$). So, $I(T, \aleph_0) \leq \aleph_0$. □

Lemma 4.1.21. *Suppose $\phi(\overline{x}, \overline{y})$ is an L-formula and $\mathrm{tp}(\overline{a}) = \mathrm{tp}(\overline{b})$. Tjen $RM_D(\phi(\overline{x}, \overline{a})) = RM_D(\phi(\overline{x}, \overline{b}))$ (that is, Morley rank is invariant under automorphisms).*

Proof. By the symmetry of the statement, it is enough to show by induction on α that if $\overline{a} \equiv \overline{b}$ then $RM(\phi(\overline{x}, \overline{a}) \geq \alpha$ implies $RM(\phi(\overline{x}, \overline{b})) \geq \alpha$.

For $\alpha = 0$: since $\overline{a} \equiv \overline{b}$, then $D \models \exists \overline{x}(\phi(\overline{x}, \overline{a})) \leftrightarrow \exists \overline{x}(\phi(\overline{x}, \overline{b}))$, and so $RM(\phi(\overline{x}, \overline{a})) \geq 0$ if and only if $RM(\phi(\overline{x}, \overline{b}))$.

For α limit ordinal, note that $RM(\phi(\overline{x}, \overline{a})) \geq \alpha$ iff $RM(\phi(\overline{x}, \overline{a}) \geq \beta$ for all $\beta < \alpha$ iff (by induction hypothesis) $RM(\phi(\overline{x}, \overline{b}) \geq \beta$ for all $\beta < \alpha$ iff $RM(\phi(\overline{x}, \overline{b})) \geq \alpha$.

Now, suppose, that $RM(\phi(\overline{x}, \overline{a})) \geq \alpha + 1$. Then there are definable sets $(\psi_i(D^{|x|}; \overline{c}_i) : i < \omega)$ which are pairwise disjoint subsets of $\phi(D^{|x|}, \overline{a})$, of Morley rank at least α. By saturation of D, there is a sequence $\langle \overline{d}_i : i < \omega \rangle$ such that for every $n < \omega$,

$$\mathrm{tp}(\overline{a}, \overline{c}_1, \ldots, \overline{c}_n) = \mathrm{tp}(\overline{b}, \overline{d}_1, \ldots, \overline{d}_n),$$

and so we have the collection $(\psi_i(D^{|x|}; \overline{d}_i) : i < \omega)$ of pairwise disjoint definable subsets of $\phi(\overline{x}, \overline{b})$, and by induction hypothesis, $RM(\psi_i(\overline{x}, \overline{d}_i)) = RM(\psi_i(\overline{x}, \overline{c}))$. Therefore, $RM(\phi(\overline{x}, \overline{b})) \geq \alpha + 1$. □

Corollary 4.1.22. *Suppose that T is ω-stable, $M \models T$, $X \subseteq M^n$ and $Y \subseteq M^m$ are definable subsets and $f : X \to Y$ is a definable finite-to-one function from X onto Y. Then, $RM(X) = RM(Y)$.*

Proof. Recall that by the previous results, $RM(X) = \max\{RM(\overline{a}/A) : \overline{a} \in X\}$ where A is a set of parameters over which X, Y, f are defined. Let $\overline{a} \in X, \overline{b} \in Y$ and $\overline{a}' \in X$ such that: $RM(X) = RM(\overline{a}/A), RM(Y) = RM(\overline{b}/A)$ and $f(\overline{a}') = \overline{b}$. Since f is finite-to-one, $\overline{a}$ and $f(\overline{a})$ are A-interalgebraic (and also $\overline{a}'$ and $f(\overline{a}') = \overline{b}$), so by Lemma 4.1.13 we have

$$\begin{aligned} RM(X) = RM(\overline{a}/A) &= RM(\overline{a}, f(\overline{a})/A) = RM(f(\overline{a})/A) \\ &\leq RM(Y) = RM(\overline{b}/A) = RM(\overline{b}, \overline{a}'/A) = RM(\overline{a}'/A) \\ &\leq RM(X). \end{aligned}$$

□

Lemma 4.1.23. *Let T be strongly minimal and $\phi(\overline{x}, \overline{y})$ an L-formula. The set*

$$Y_{\phi,k} := \{\overline{a} : RM(\phi(\overline{D}^{|x|}; \overline{a})) \geq k\}$$

is definable for each $k \leq n$.

Proof. By induction on n. For $n = 1$, notice that by strong minimality and saturation there is a number $N \in \mathbb{N}$ such that $|\phi(D; \overline{a})| \geq N$ implies $\phi(D, \overline{a})$ is infinite. Then, $RM(\phi(D; \overline{a})) \geq 1$ if and only if $D \models \exists^{\geq N} x(\phi(x, \overline{a}))$, and so $Y_{\phi,1}(\overline{y}) = \exists^{\geq N} x(\phi(x, \overline{a}))$ and $Y_{\phi,0}(\overline{y}) = \exists x(\phi(x, \overline{y}))$.

Suppose the result for all $s \leq n$. We now prove it for $n+1$ by induction on k.

Consider a formula $\phi(x_1, \ldots, x_{n+1}, \overline{y})$ and put $\Phi(x_1, \ldots, x_n; \overline{y}) = \exists x_{n+1}(\phi(x_1, \ldots, x_n, \overline{y}))$. Clearly, if $RM(\Phi(D^n; \overline{a})) \geq k$ then $RM(\phi(D^{n+1}, \overline{a})) \geq k$, since any decomposition of Φ into disjoint sets provides a similar decomposition of $\phi(D^{n+1}, \overline{a})$ into disjoint sets.

By induction hypothesis, there is a formula $Y_{\Phi,k}(\overline{y})$ such that $Y_{\Phi,k}(\overline{a})$ if and only if $RM(\Phi(D^n; \overline{a})) \geq k$.

Consider now the set

$$\theta_{\overline{a}}(D^n) = \{\overline{b} \in D^n : \phi(\overline{b}, x_{n+1}; \overline{a}) \text{ is infinite}\}.$$

Notice that $\theta_{\overline{a}}(D^n) = \{\overline{b} \in D^n : |\phi(\overline{b}, x_{n+1}; \overline{a})| \geq N\}$ for an appropriate $N \in \mathbb{N}$, and we can consider the formula

$$\theta(x_1, \ldots, x_n; \overline{y}) := \exists^{\geq N} x_{n+1}(\phi(\overline{x}, x_{n+1}, \overline{y})).$$

Let $(\overline{b}, c) \in \phi(D^{n+1}, \overline{a})$ be an element such that $RM(\phi(\overline{x}, x_{n+1}; \overline{a})) = RM(\overline{b}, c/\overline{a})$. Then we have:

$$\begin{aligned} RM(\phi(\overline{x}, x_{n+1}; \overline{a})) &= RM(\overline{b}, c/\overline{a}) \\ &= \dim(\overline{b}, c/\overline{a}) = \dim(\overline{b}/\overline{a}) + \dim(c/\overline{b}\overline{a}). \end{aligned}$$

But $\overline{b} \models \Phi(\overline{x}, \overline{a})$ and $\dim(c/\overline{b}\overline{a})$ depends only on $\theta(\overline{b}, \overline{a})$, and $\Phi(\overline{x}, \overline{a}) \wedge \theta(\overline{x}, \overline{a})$ defines exactly those elements in the projection which have fibers of dimension 1, so we have

$$\begin{aligned} RM(\phi(\overline{x}, x_{n+1}; \overline{a})) \geq k &\Leftrightarrow RM(\phi(\overline{x}; \overline{a})) \geq k \text{ or } RM(\Phi(\overline{x}; \overline{a}) \wedge \theta(\overline{x}; \overline{a})) \geq k - 1 \\ &\Leftrightarrow D \models Y_{\Phi,k}(\overline{a}) \vee Y_{\Phi\wedge\theta,k-1}(\overline{a}). \end{aligned}$$

as desired. □

4.2 Strongly minimal pseudofinite structures

We now take a look to strongly minimal pseudofinite theories. Strongly minimal theories are the "nicest" stable theories in various senses, and are defined/characterized by the fact that any definable subset of the universe of a model of T being either finite or cofinite. As it turns out, the behavior of strongly minimal pseudofinite structures combines effectively the main features of pseudofiniteness and strongly minimal structures, and allows us to provide a good control of the dimension and size of definable sets, as we will see in this section.

Theorem 4.2.1. *Let D be a (saturated) pseudofinite strongly minimal structures, and let $q \in \mathbb{N}^*$ be the pseudofinite cardinality of D ($q = |D|$). Then,*

1. *For any definable (with parameters) set $X \subseteq D^n$, there is a polynomial $P_X(x) \in \mathbb{Z}[x]$ with positive leading coefficient such that $|X| = P_X(q)$. Moreover, $RM(X) = \text{degree}(P_X)$.*

2. *For any L-formula $\phi(\overline{x}, \overline{y})$ there is a finite number of polynomials $P_1, \ldots, P_k \in \mathbb{Z}[x]$ and L-formulas $\psi_1(\overline{y}), \ldots, \psi_k(\overline{y})$ such that:*

   (a) $\{\psi_i(\overline{y}) : i \leq k\}$ is a partition of the $\overline{y}$-space.

   (b) For any $\overline{b}$, $|\phi(D^{|x|}; \overline{b})| = P_i(q)$ if and only if $D \models \psi_i(\overline{b})$.

Proof. First we prove (1) by induction on $RM(X)$. If $RM(X) = 0$, then X is finite and we can simply put $P_X(x) = a_0 = |X|$.

Now, suppose $RM(X) = n \geq 1$ where $X \subseteq D^m$ for some $m \geq n$. If the result is true for cells, then by 4.1.5, there is a decomposition of X into disjoint cells $X = Y_1 \cup \cdots \cup Y_k$, and if by putting $P_X(x) = \sum_{i=1}^k P_{Y_i}(x)$ we would have

$$|X| = \sum_{i=1}^{k} |Y_i| = \sum_{i=1}^{k} P_{Y_i}(q) = P_X(q).$$

So, let us assume that X is an n-cell. Then there is a projection $\pi : D^m \to D^n$ such that $\pi(X)$ has Morley rank n, and $\pi|_X$ is r-to-1 for some $r \in \mathbb{N}$. So, $|X| = r \cdot |\pi(X)|$.

Also note that $|\pi(X)| = |D^n| - |(D^n \setminus \pi(X))|$, and since D^n has multiplicity 1, we have $RM(D^n \setminus \pi(X)) < n$. By induction hypothesis, there is a polynomial $p \in \mathbb{Z}[x]$ such that $|D^n \setminus \pi(X)| = p(q)$. Then, we have

$$\begin{aligned} |X| &= r|\pi(X)| = r(|D^n| - |(D^n \setminus \pi(X))|) \\ &= r(q^n - p(q)) = rq^n - r \cdot p(q) \end{aligned}$$

So, we can put $P_X(x) = r \cdot x^n - r \cdot p(x)$. Also note that $\text{degree}(p(x)) = RM(D^n \setminus \pi(X)) < n$, so we have that $\text{degree}(P_X(x)) = n = RM(X)$.

We now will prove (2). We start with the following claim.

Claim: *For any L-formula $\phi(\overline{x},\overline{y})$ where $\overline{x}=(x_1,\dots,x_n)$ and $\overline{a}$ with $|\overline{y}|=|\overline{a}|$, there is an L-formula $\psi(\overline{y})\in \mathrm{tp}(\overline{a})$ such that for all $\overline{b}\models\psi(\overline{y})$ we have $P_{\phi(\overline{x},\overline{a})}(q)=P_{\phi(\overline{x},\overline{b})}(q)$.*

Proof of the Claim: By induction on $n=|\overline{x}|$. For $n=1$, by strong minimality, $\phi(\overline{x},\overline{a})$ defines a set that is either finite or cofinite. If $D=\phi(D;\overline{a})|=\ell<\omega$, then $P_{\phi(\overline{x},\overline{a})}(q)=\ell$ and we can take $\psi_\phi(\overline{a}):=\exists^{=\ell}x(\phi(x;\overline{a}))$. On the other hand, $|\neg\phi(D,\overline{a})|=q-\ell$, and we can take $\psi_\phi(\overline{a})=\exists^{=\ell}x(\neg\phi(x;\overline{a}))$.

Suppose now the result for all formulas in at most n variables, and take a formula $\phi(x_1,\dots,x_n,x_{n+1};\overline{y})$. As in previous proofs, consider the formula

$$\Phi(x_1,\dots,x_n;\overline{y}):=\exists x_{n+1}(\phi(x_1,\dots,x_n,x_{n+1};\overline{y}).$$

By definability of Morley rankwe have $\phi(\overline{x};\overline{a})\equiv V_0(\overline{x};\overline{a})\dot{\cup}V_1(\overline{x};\overline{a})$ where $V_s(\overline{x};\overline{a})=\{\overline{b}\in D^n:\dim(\phi(\overline{b},x_{n+1};\overline{a}))=s\}$ for $s=0,1$.

Furthermore, by strong minimality we can write $V_s(\overline{x};\overline{a})=\dot{\bigcup}_j W_{s,j}(\overline{x},\overline{a})$ where we have:

$$\overline{b}\in W_{0,j}(\overline{x};\overline{a})\Leftrightarrow V_0(\overline{b};\overline{a})\wedge|\phi(\overline{x};\overline{a})|=j$$
$$\overline{b}\in W_{1,j}(\overline{x};\overline{a})\Leftrightarrow V_1(\overline{b};\overline{a})\wedge|\neg\phi(\overline{x};\overline{a})|=j.$$

By induction hypothesis, there exists formulas $\psi_{s,j}(\overline{y})$ such that $D\models\psi_{s,j}(\overline{a})$, and for all $\overline{c}\in D^{|\overline{y}|}$,

$$D\models\psi_{s,j}(\overline{c})\Leftrightarrow|W_{s,j}(\overline{x};\overline{a})|=|W_{s,j}(\overline{x};\overline{c})|$$

Note that j varies in a finite set of natural numbers by strong minimality and saturation, so we can now take the formula

$$\psi_\phi(\overline{y}):=\left(\phi(\overline{x},\overline{y})\leftrightarrow\bigvee_{s,j}W_{s,j}(\overline{y})\right)$$
$$\wedge\left(\forall\overline{x}\,(W_{s,j}(\overline{x},\overline{y})\leftrightarrow[\dim(\phi(\overline{x},x_{n+1},\overline{y}))=s\wedge|(\phi(\overline{x},x_{n+1},\overline{y}))^s|=j])\wedge\bigwedge_{s,j}\psi_{s,j}(\overline{y})\right)$$

By construction, $D\models\psi_\phi(\overline{a})$. Now, if $\overline{c}\models\psi_\phi(\overline{y})$, then $\models\bigwedge_{s,j}\psi_{s,j}(\overline{y})$ and we have

$$\begin{aligned}|\phi(\overline{x},\overline{c})|&=\sum_j|W_{0,j}(\overline{x},\overline{c})|\cdot j+\sum_j|W_{1,j}(\overline{x},\overline{c})|\cdot(q-j)\\&=\sum_j|W_0(\overline{x},\overline{a})|\cdot j+\sum_j|W_{1,j}(\overline{x},\overline{a})|\cdot(q-j)\\&=|\phi(\overline{x},\overline{a})|\qquad,\end{aligned}$$

as we desired. $\square_{\text{Claim.}}$

Note now that from the proof of the Claim that we may suppose, by induction hypothesis, that for each s,j there are finitely many formulas $\psi^i_{s,j}(\overline{y})$ such that $\{\psi^i_{s,j}(\overline{y})\}_i$ is a partition of the $\overline{y}$-space, and for any $\overline{b}$, $|W^i_{s,j}(\overline{x},\overline{b})|=P_{s,j,i}(q)$ if and only if $\models\psi^i_{s,j}(\overline{b})$.

By varying along all possible combinations of indices s,j,i, we can conclude (2). $\square$

Corollary 4.2.2. *We also have that $dM(X) = \ell c(P_X(q))$, the leading coefficient of $P_X(q)$.*

Proof. Suppose $X \subseteq D^n$, and let us prove the result by induction on n. For $n = 1$, by strong minimality, $X \subseteq D^1$ is either finite or cofinite. If X is cofinite, then $P_X(q) = q - c$, and since $X \subseteq D^1$ has maximal dimension $dM(X) = 1$ because $\text{mult}(X) = 1$. If X is finite, then $|X| = k$ for some k, and clearly $dM(X) = k$ (there are k points).

Suppose now the result for all definable sets in at most n variables, and let $X = \phi(x_1, \ldots, x_{n+1}; \overline{a}) \subseteq D^{n+1}$ and consider $\Phi(x_1, \ldots, x_n; \overline{a})$ as in the proof of Theorem 4.2.1. By realizing the same decomposition we have

$$\begin{aligned} |X| &= |\phi(x_1, \ldots, x_{n+1}; \overline{a})| \\ &= \sum_j j \cdot |W_{0,j}(\overline{x}; \overline{a})| + \sum_j (q - j) \cdot |W_{1,j}(\overline{x}; \overline{a})| \end{aligned}$$

Suppose for the moment that X has Morley degree 1. By taking generic points there is a tuple $(b_1, \ldots, b_{n+1}) \in X$ such that

$$RM(X) = \dim((b_1, \ldots, b_{n+1})/\overline{a}) = \dim((b_1, \ldots, b_n)/\overline{a}) + \dim(b_{n+1}/\overline{a} b_1, \ldots, b_n).$$

Let $\widehat{s}, \widehat{j}$ be indices such that $\overline{b} = (b_1, \ldots, b_n) \models W_{\widehat{s},\widehat{j}}(\overline{x}; \overline{a})$. Then $\dim(b_{n+1}(\overline{a}\overline{b}) = s$, and also $\dim(b_1, \ldots, b_n/\overline{a}) \leq W_{\widehat{s},\widehat{j}}$ because $W_{\widehat{s},\widehat{j}}(\overline{x}, \overline{a})$ is definable over $\overline{a}$. So, $\dim(X) = \dim(W_{\widehat{s},\widehat{j}}) + s$.

Clearly the set $X' := \phi(x_1, \ldots, x_{n+1}; \overline{a}) \wedge W_{\widehat{s},\widehat{j}}(x_1, \ldots, x_n; \overline{a})$ is contained in X and $(b_1, \ldots, b_{n+1}) \in X'$, so we have

$$\dim(X) = \dim(b_1, \ldots, b_{n+1}/\overline{a}) \leq \dim(X') \leq \dim(X) \Rightarrow \dim(X') = \dim(X).$$

Since we have assumed that the Morley degree of X is 1, we have $\dim(X \setminus X') < \dim(X)$. We then have $P_X(q) = |X| = |X'| + |X \setminus X'| = P_{X'}(q) + P_{X \setminus X'}(q)$ and since $\text{degree}(P_{X \setminus X'}(q)) = \dim(X \setminus X') < \dim(X) = \text{degree}(P_X(q))$, we have that $\ell c(P_X(q)) = \ell c(P_{X'}(q))$.

By a straightforward calculation we have

$$P_{X'}(q) = |X'| = \begin{cases} |W_{0,j}(\overline{x}; \overline{a})| \cdot j & \text{if } s = 0 \\ |W_{1,j}(\overline{x}, \overline{a})| \cdot (q - j) \text{if } s = 1 \end{cases} = \begin{cases} j \cdot P_{W_{0,j}}(q) & \text{if } s = 0 \\ (q - j) \cdot P_{W_{1,j}}(q) & \text{if } s = 1 \end{cases}$$

In both cases, note that by induction hypothesis we have $\ell c(P_X(q)) = \ell c(P_{W_{s,j}}(q)) =^{(*)} dM(W_{s,j})$. It remains to show that $dM(W_{s,j}) = 1$.

If $s = 0$, then $\dim(X) = \dim(W_{s,j})$. Also note that i $W_{s,j} = Y_1 \cup Y_2$ for disjoint sets Y_1, Y_2 of dimension $\dim(X)$, then the sets $\pi_X^{-1}(Y_1), \pi_X^{-1}(Y_2)$ would be disjoint, and since $\pi|_X$ is a finite-to-one map, we have $\dim(\pi_X^{-1}(Y_t)) = \dim(Y_t) = \dim(X)$ for $t = 1, 2$, contradicting that $dM(X) = 1$.

Therefore, $dM(W_{s,j}) = 1$. Also note in this case we have $j = 1$ because otherwise we could write X as the union of j-sets of dimension $\dim(X)$, by (*).

Now, if $s = 1$, as in the previous case, if $Y_1 \dot\cup Y_2 = W_{1,j}$ for disjoint sets of dimension equal to $\dim(W_{1,j})$, then by additivity we would have

$$\dim(\pi_X^{-1}(Y_t) = \dim(Y_t) + 1 = \dim(W_{1,j}) + 1 = \dim(X),$$

contradicting that $dM(X) = 1$. $\square$

4.2.1 Unimoduarity

Definition 4.2.3.

1. We say that two tuples $\overline{a}, \overline{b}$ are *interalgebraic* if both $\text{tp}(\overline{a}/\overline{b})$ and $\text{tp}(\overline{b}/\overline{a})$ have a finite number of realizations.

2. When $\text{tp}(\overline{a}/\overline{b})$ we define the *multiplicity of* $\overline{a}$ *over* $\overline{b}$ to be the number of realizations of $\text{tp}(\overline{a}/\overline{b})$.

Definition 4.2.4. A strongly minimal theory is said to be *unimodular* if whenever $\overline{a} = (a_1, \ldots, a_n)$ and $\overline{b} = (b_1, \ldots, b_n)$ are algebraic independent n-tuples, and $\overline{a}$ is interalgebraic with $\overline{b}$, then

$$\text{mult}(\overline{a}/\overline{b}) = \text{mult}(\overline{b}/\overline{a}).$$

This notion is due to Hrushovski in [13], but a clarifications of the definition appears in [18] and [9]. The most important result in [13] is the following:

Theorem 4.2.5 (Hrushovski, 1992)**.** *Unimodular strongly minimal sets are locally modular.*

This result generalizes Zilber's result that ω-categorical strongly minimal sets are locally modular.

Corollary 4.2.6. *A strongly minimal pseudofinite structure D is unimodular. Hence, by Theorem 4.2.5, also locally modular.*

Proof. Suppose $\overline{a}, \overline{b}$ are n-tuples from D which are algebraicalally independent tuples, but $\overline{a}$ is interalgebraic with $\overline{b}$. Write $k = \text{mult}(\overline{b}/\overline{a})$ and $\ell = \text{mult}(\overline{b}/\overline{a})$. We have to prove that $k = \ell$.

Let $\phi(\overline{x}, \overline{y})$ be an L-formula such that $D \models \phi(\overline{a}, \overline{b})$, $\phi(\overline{x}, \overline{b})$ isolates the type $\text{tp}(\overline{a}/\overline{b})$, and $\phi(\overline{a}, \overline{y})$ isolates the type $\text{tp}(\overline{b}/\overline{a})$. (It can be the conjuction of the formulas that isolate each of the types). Let $\phi_1(\overline{x}) := \exists^{=k}\overline{y}(\phi(\overline{x}, \overline{y}))$ and $\phi_2(\overline{y}) := \exists^{=\ell}\overline{x}(\phi(\overline{x}, \overline{y}))$. Let $\chi(\overline{x}, \overline{y}) := \phi(\overline{x}, \overline{y}) \wedge \phi_1(\overline{x}) \wedge \phi_2(\overline{y})$, and take $Z \subseteq D^{2n}$ the set defined by $\chi(\overline{x}, \overline{y})$, and put $X = \pi_{\overline{x}}(Z)$ (the projection of Z on the first n coordinates) and $Y = \pi_{\overline{y}}(Z)$ (the projection of Z on the last n coordinates).

Then, by counting on the finite structures, $|Z| = k \cdot |X| = \ell|Y|$, and this implies by Theorem 4.2.1 that $k \cdot P_X(q) = \ell \cdot P_X(q)$. $(*)$

On the other hand, since $\overline{a} \in X \subseteq D^n$ and $\overline{b} \in Y \subseteq D^n$, we have $RM(X) \leq n, RM(Y)$ and also

$$RM(X) \geq RM(\overline{a}) = \dim(\overline{a}) = n \;\text{ and }\; RM(Y) \geq RM(\overline{b}) = \dim(\overline{b}) = n.$$

So $RM(X) = n = RM(Y)$, and since $\text{mult}(D^n) = 1$, we conclude that $dM(X) = dM(Y) = 1$. This implies that the leading term of $P_X(q)$ and $P_Y(q)$ is q^n, so by comparing the leading terms in both sides of the equality $(*)$ we get $k \cdot q^n = \ell \cdot q^n$, which implies $k = \ell$. ☐

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