You could have invented crossed modules 2*
CROSSED MODULES' USES IN ALGEBRAIC TOPOLOGY.

From last time:

**Def** A crossed module \( G = (\partial : E \rightarrow G, \triangleright) \) is given by a homomorphism \( \partial : E \rightarrow G \) and a group action \( \triangleright : G \times E \rightarrow E \) by automorphisms, where the homomorphism \( \partial \) satisfies the following compatibility axioms:

1. **(Peiffer laws)** \( \forall g \in G, \forall e \in E : \partial(g \triangleright e) = g \partial(e) g^{-1} \)
2. \( \forall e, e' \in E : \partial(e) \triangleright e' = e e' e^{-1} \)

**Exercises:**

1. **Ker \( \partial \) is central (in particular, Ker \( \partial \) is abelian)**
   Suppose \( e \in E \), \( k \in \text{ker} \( \partial \) \)
   \[ \text{ker} \partial^{-1} = \{ g(k) e \partial(k)^{-1} = e \} \Rightarrow \text{ker} \partial = Z(E) \]
   **Remark** Ker \( \partial \) is sometimes denoted by \( \pi_2 G \).

2. **The image of \( E \) is a normal subgroup of \( G \).**
   Take \( g \in \text{Im} \partial \). Then there exists a \( e \in E \) such that \( \partial(e) = g \).
   Let \( \ell \in G \), and consider
   \[ \ell g e^{-1} = \ell \partial(e) e^{-1} \stackrel{(i)}{=} \partial(\ell \triangleright e) \Rightarrow \ell g e^{-1} \in \text{Im} \partial \]
   **Remark** \( G / \text{ker} \partial \) is sometimes denoted by \( \pi_1 G / \text{Im} \partial \).

**Def** Squares in crossed modules.

**Def** Given two crossed modules \( G_\partial = (\partial : E \rightarrow G, \triangleright) \) and \( G'_\partial = (\partial' : E' \rightarrow G', \triangleright) \)
Let \( \lambda : E \rightarrow E' \), and \( \mu : G \rightarrow G' \) be group morphisms satisfying:

1. \( \lambda \triangleright = \partial \circ \mu \)
2. \( \forall e \in E \) and \( \forall g \in G : \lambda(g \triangleright e) = \mu(g) \triangleright \lambda(e) \)

*If you did not get the reference, go and read T.Y Chow's paper "You could have invented spectral sequences" on Notices.
Moreover \( \pi_1(A, x_0) \) acts on \( \pi_2(X, A, x_0) \) as described by:

![Diagram](image)

(resealing the domain)

Denote by \( \triangleright \) this action.

\[
\text{CLAIM} \quad \left( \partial: \pi_2(X, A, x_0) \to \pi_4(A, x_0), \triangleright \right) \text{ is a crossed module.}
\]

we will call this the **FUNDAMENTAL CROSSED MODULE** \( \pi_4(X, A, x_0) \).

**Exercise:** find a proof.

END FOR FRIDAY 15/11/2019

2. AN APPLICATION

2a. The classical Seifert - Van Kampen theorem

Method for computing the fundamental groups of spaces that can be seen as unions of "simpler" spaces, having a path-connected intersection.

**Theorem (Seifert - Van Kampen)**

Let \( X \) be a path-connected topological space, and \( U, V \) path-connected open subsets of \( X \) such that \( X = U \cup V \) and \( U \cap V \) is non-empty and pathwise connected.

Let \( x_0 \in U \cap V \). Then the following diagram is commutative:

\[
\begin{array}{ccc}
\pi_1(U \cap V, x_0) & \xrightarrow{j_1} & \pi_1(U, x_0) \xrightarrow{\psi_1} \pi_1(X, x_0) \\
\downarrow{j_2} & & \downarrow{\psi_2} \\
\pi_1(V, x_0) & \xrightarrow{\psi_3} & \pi_1(V, x_0)
\end{array}
\]

where arrows denote homomorphisms induced by inclusion maps.

This means that

\[
\pi_1(X, x_0) \cong \pi_1(U, x_0) \ast_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0). \quad \text{Amalgamated product.}
\]

Moreover \( \pi_1(X, x_0) \) is the freest possible group to make the diagram commutative, meaning that it respects the following universal property:

\((UP)\) Let \( H \) be a group, and \( \phi_1: \pi_1(U, x_0) \to H \) and \( \phi_2: \pi_1(V, x_0) \to H \) be homomorphisms, such that
\[ \pi_4(U \cup V, x_0) \xrightarrow{\phi_1} \pi_2(U, x_0) \xrightarrow{j_1} \pi_2(X, x_0) \rightarrow \pi_2(V, x_0) \xrightarrow{j_2} \pi_2(Y, x_0) \rightarrow \pi_2(H, x_0) \]

Then there is a unique homomorphism
\[ \Phi : \pi_4(X, x_0) \rightarrow H \]

such that \( \Phi = \phi_1 = \phi_2 \).

**Operationally:**

- **Amalgamated product:**
  
  Let \( G = \langle S_G \mid \mathcal{R}_G \rangle \), \( S_G = \{ s_{a_1}, \ldots, s_{a_k} \} \)
  
  \( H = \langle S_H \mid \mathcal{R}_H \rangle \)
  
  \( J = \langle S_J \mid \mathcal{R}_J \rangle \) a subgroup with inclusions

\[
\begin{array}{c}
\chi_s \\
\chi_t
\end{array}
\]

Then \( G *_J H = \langle S_{G} \cup S_{H} \cup S_{J} \mid \mathcal{R}_{G} \cup \mathcal{R}_{H} \cup \{ j_1^{-1}(s_{a_1}), j_2^{-1}(s_{a_2}), \ldots, j_k^{-1}(s_{a_k}) \} \rangle \)

In particular \( G *_J H = \langle S_{G} \cup S_{H} \cup S_{J} \mid \mathcal{R}_{G} \cup \mathcal{R}_{H} \cup \{ j_1^{-1}(s_{a_1}), j_2^{-1}(s_{a_2}), \ldots, j_k^{-1}(s_{a_k}) \} \rangle \)

**Corollary:** If \( U \cup V \) simply-connected, then there is an isomorphism
\[ \pi_4(U, x_0) \times \pi_4(V, x_0) \cong \pi_4(X, x_0) \]

**Exercise:** \( \pi_4\left(\infty\right) = ? \)

**Limitations:** we can't compute \( \pi_4\left(S^4\right) \)

2.6 The fundamental groupoid