Fuzzy Line Bundles, the Chern Character and Topological Charges over the Fuzzy Sphere

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Abstract

Using the theory of quantized equivariant vector bundles over compact coadjoint orbits we determine the Chern characters of all non-commutative line bundles over the fuzzy sphere with regard to its derivation based differential calculus. The associated Chern numbers (topological charges) arise to be non-integer, in the commutative limit the well known integer Chern numbers of the complex line bundles over the 2–sphere are recovered.

1 Introduction and Overview

Classical gauge field theories exhibit interesting features connected with the geometry and topology of nontrivial fibre bundles (over space or space–time). Examples are monopole and instanton solutions.

The Serre–Swan theorem \cite{1} (cf. \cite{2} and \cite{3}) leads to a complete equivalence between the category of continuous vector bundles over a compact manifold $M$ and the finitely generated projective modules over the unital commutative $C^*$–algebra $C(M)$ of continuous functions over $M$. This circumstance can be generalized to the smooth case \cite{3} and even holds for non compact manifolds. Moreover the geometry of $M$ is encoded in $C(M)$. 

1
In noncommutative geometry one proceeds by defining vector bundles as finitely generated projective—left, right, bi—modules over some algebra, which is thought of the algebra of functions over some noncommutative manifold. Accordingly, the geometrical nontriviality is purely algebraical and encoded solely in projective non free modules over the noncommutative algebra under consideration.

For the fuzzy sphere this case has first been analyzed by one of us (HG) et. al. in [4] (see also [5]), leading to scalar and spinor field configurations in monopole backgrounds. A different approach using spectral triples and their Dirac operator based differential calculi has been used in [6] and [7]. This leads to integer topological charges similar to [4].

We review the definition of the Chern character on projective modules in section 2 and the complex line bundles over the 2–sphere in section 3. In section 4 and 5 we will show that the Chern character of projective modules over the matrix algebra of the fuzzy sphere gives, with respect to its free derivation based differential calculus, rise to non-integer Chern numbers. In the commutative limit the well known Chern characters of complex line bundles over the 2–sphere with its integer topological charges are recovered.

2 The Chern Character of Projective Modules

Let \(\mathcal{A}\) be a complex unital not necessarily commutative \(C^\ast\)–algebra and denote \(\mathcal{A} \otimes \mathbb{C}^n\) by \(\mathcal{A}^n\). Then any projector (selfadjoint idempotent) \(p \in M_n(\mathcal{A})\), where \(M_n(\mathcal{A}) = \mathcal{A} \otimes M_n(\mathbb{C})\) denotes the \(n \times n\)–matrices with coefficients in \(\mathcal{A}\), defines a (finitely generated) projective left \(\mathcal{A}\)–module by \(E = \mathcal{A}^n p\). Elements \(\psi\) of \(E\) can be viewed as \(\psi \in \mathcal{A}^n\) with \(\psi p = \psi\). If \(\mathcal{A}\) is further endowed with a differential calculus \((\Omega^\ast(\mathcal{A}), d)\), the Grassmann connection \(\nabla : E \rightarrow \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} E\) of \(E\) is defined by \(\nabla = p \circ d\). It satisfies \(\nabla(f \psi) = f \nabla \psi + df \otimes_{\mathcal{A}} \psi\) for all \(f \in \mathcal{A}, \psi \in E\). After extending \(\nabla\) to \(\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} E\) one can define the \(\mathcal{A}\)-linear map

\[

\nabla^2 : E \rightarrow \Omega^2(\mathcal{A}) \otimes_{\mathcal{A}} E, \tag{1}

\]

called the curvature of \(\nabla\). For more details see [8] and [9]. Evaluating \(\nabla^2\) one finds \(\nabla^2 = p(dp)(dp)\), whereas if we write \((p_{ij}) = p \in M_n(\mathcal{A})\), \((dp)\) is the \(n \times n\)–matrix with coefficients \(dp_{ij}\) and the entries of \(dp(dp)\) are \(p_{il}dp_{lk}dp_{kj}\).
The curvature can be viewed as $\nabla^2 \in \Omega^2(\mathcal{A}) \otimes_{\mathcal{A}} \text{End}_{\mathcal{A}}(E)$, where $\text{End}_{\mathcal{A}}(E)$ denotes the right $\mathcal{A}$–module of endomorphisms of $E$, i.e. $\mathcal{A}$–linear mappings from $E$ to $E$. Now define

$$F := \text{Tr}(dp)(dp) \in \Omega^2(\mathcal{A}),$$

which is a cocycle, i.e. $dF = \text{Tr}(dp)(dp)(dp) = 0$. Here $\text{Tr}$ is the trace in $\text{End}_{\mathcal{A}}$. So $F$ defines a cohomology class $[F] \in H^2(\mathcal{A})$. More generally, the Chern character of $E$ with respect to $(\Omega^*(\mathcal{A}), d)$ is the set of

$$\text{Ch}_r(p) := \frac{1}{r!} \text{Tr}(dp)^{2r}, \quad r \in \mathbb{N} \cup \{0\},$$

the $\text{Ch}_r(p)$ are called its $r$–th components. They are also cocycles and provide equivalence classes in $H^{2r}(\mathcal{A})$. $\text{Ch}_0(p) = \text{Tr}p$ simply gives the rank of the module.

### 3 Complex Line Bundles over the 2–Sphere

One approach to the construction of the complex line bundles over the 2–sphere is the one given in [10], cf. also [11]. Starting with the complex Hopf fibration $U(1) \hookrightarrow SU(2) \simeq S^3 \twoheadrightarrow S^2$ and the irreducible representations of $U(1)$ on $\mathbb{C}$ labelled by integers $k \in \mathbb{Z}$, one defines the space of smooth equivariant functions $C^\infty_{(k)}(S^3, \mathbb{C}) \ni \varphi : S^3 \to \mathbb{C}$ with $\varphi(x \cdot z) = z^{-k}\varphi(x)$ for $x \in S^3$ and $z \in U(1)$. These are modules over $C^\infty_{(0)}(S^3, \mathbb{C}) \simeq C^\infty(S^2, \mathbb{C})$, and as such are isomorphic the smooth sections $\Gamma^\infty(S^2, L^k)$ of the associated complex line bundles $L^k := S^3 \times_k \mathbb{C}$ over the 2–sphere.

By the Serre–Swan theorem this modules are finitely generated and projective and hence, it is possible to identify $\Gamma^\infty(S^2, L^k)$ with $(C^\infty(S^2, \mathbb{C}))^n p$, where $p \in M_n(C^\infty(S^2, \mathbb{C}))$ is a projector. In [10] the projectors $p$ were explicitly constructed with help of the equivariant functions $C^\infty_{(k)}(S^3, \mathbb{C})$. The integer $n \in \mathbb{N}$ turned out to be $|k| + 1$ and the first Chern numbers where calculated giving

$$c_1(p) := -\frac{1}{2\pi i} \int_{S^2} \text{Tr}(dp)(dp) = -k \in \mathbb{Z}.$$

Let us shortly mention that $k$ is related to the magnetic charge $Q_m$ of a Dirac (point) monopole in $\mathbb{R}^3$ via

$$Q_m = k \frac{hc}{2e},$$

(3)
where $\hbar$ is Planck’s constant, $c$ the vacuum light velocity and $e$ the elementary electrical charge, meaning that $Q_m$ is quantized.

## 4 Fuzzy Line Bundles

### 4.1 General Remarks

We start with the repetition of well known facts about the fuzzy sphere and its free derivation based differential calculus. Then we use the prescription of quantizing equivariant vector bundles over coadjoint orbits to obtain the projectors that define the modules over the matrix algebra of the fuzzy sphere and its Chern characters.

Denote $SU(2)$ by $G$, its Lie algebra $\mathfrak{su}(2)$ by $\mathfrak{g}$ and let $\{X_a\}_{a=1,2,3}$ be the generators of the irreducible spin–$N$ representation of $\mathfrak{g}$ acting on the representation space $[N]$ with $\dim([N]) = 2N + 1$.

The algebra of the fuzzy sphere [12, 13] is the noncommutative algebra $\text{End}([N]) \simeq A_N$, the algebra of $(2N + 1) \times (2N + 1)$–matrices with complex coefficients. $A_N$ is generated by $Y_a = \left(\frac{N(N+1)}{2}\right)^{-1/2}X_a$ which satisfy

$$[Y_a, Y_b] = \frac{i\epsilon_{abc}}{\sqrt{N(N+1)}} Y_c \quad \text{and} \quad \sum_{a=1}^3 (Y_a)^2 = 1. \quad (4)$$

The derivation based differential calculus (cf. [14]) on $A_N$ is defined as follows: One chooses the three derivations (“vector fields”) $e_a$, defined by $e_a(f) := [X_a, f]$. We denote by $\text{Der}_3(A_N)$ the linear subspace of $\text{Der}(A_N)$ generated by the $e_a$’s. Here $\text{Der}(A_N)$ is the $\mathbb{C}$–vector space of all derivations of $A_N$. $A_N$ decomposes into $[0] \oplus [1] \oplus \cdots \oplus [2N]$ as $\mathfrak{g}$– and $G$–module, respectively. The derivations $e_a$ satisfy $[e_a, e_b] = i\epsilon_{abc}e_c$, and are the noncommutative analogue of the vector fields $L_a = i\epsilon_{abc}x_b \partial/\partial x_c$ on the 2–sphere.

The set of $p$–forms $\Omega^p_{(N)}$ over $A_N$ is the free $A_N$–module $\Omega^p_{(N)} = A_N \otimes (\text{Der}_3(A_N)^*) \wedge \cdots \wedge (\text{Der}_3(A_N)^*) \simeq A_N \otimes (\mathfrak{g}_C^* \wedge \cdots \wedge \mathfrak{g}_C^*)$, where $\mathfrak{g}_C \simeq \text{sl}(2, \mathbb{C})$. Note that $\Omega^0_{(N)} = 0$ for $p > 3$. The exterior derivative $d : A_N \to \Omega^1_{(N)}$ is defined by $d\phi(u) = u(\phi)$ for all $\phi \in A_N$ and $u \in \text{Der}_3(A_N)$. It extends to $\Omega^*_{(N)} = \bigoplus_p \Omega^p_{(N)}$ by linearity and the graded Leibniz rule. There is a distinguished one–form $\Theta$ defined by $\Theta(e_a) = -X_a$. $\Theta$ is the analogue of the Maurer–Cartan form satisfying $d\Theta + \Theta^2 = 0$. The exterior derivative of a zero form $\phi \in \Omega^0_{(N)} = A_N$ can with help of $\Theta$ be written as $d\phi = -[\Theta, \phi]$. 

4
One can choose a basis $\Theta_a$ in $\Omega^1_{(N)}$ completely determined by $\Theta_a(\epsilon_b) = \delta_{ab} I$. Then $\Theta = -X_a \Theta_a$ and $d\phi = \epsilon_a(\phi) \Theta_a$.

It follows from the procedure given in [15] (cf. also [16]) that the quantization of equivariant vector bundles over coadjoint orbits is achieved by means of the orthogonal projection

$$p \in [N] \otimes [N]^* \otimes [\nu] \otimes [\nu]^* \cong \mathcal{A}_N \otimes \text{End}[\nu],$$

which projects onto the unique irreducible subrepresentation of $[N] \otimes [\nu]$ with highest (lowest) spin, i.e. onto $[N \pm \nu]$. Here $[\nu]$ is the representation space of the irreducible spin-$\nu$ representation of $g$. In the case $[N - \nu]$ we have to assume that $N > \nu$. So the fuzzy line bundles obtained in this way are of the form $L^k \pm 2\nu := (\mathcal{A}_N \otimes [\nu]) p$, they are isomorphic to the $(2N + 1) \times (2(N + \nu) + 1)$-matrices. Here $p$ acts from the right providing left $\mathcal{A}_N$-modules. The modules $L^k$ approximate in the commutative limit $N \to \infty$ the module of sections of $L^k$.

Let $\pi : G \to [N \pm \nu]$ be the irreducible representation of $G$ with spin $N \pm \nu$ and $|h\rangle$ its highest weight vector. $|h\rangle$ is thought of being embedded in $[N] \otimes [\nu]$ as $|h\rangle \oplus 0 \oplus \cdots \oplus 0$. Denote by $\mu$ the normalized Haar measure on $G$.

**Lemma 4.1.** The projector $p : [N] \otimes [\nu] \to [N \pm \nu]$ defined above is given by

$$p = (2(N \pm \nu) + 1) \int_G \pi(g)|h\rangle\langle h|\pi(g)^{-1} d\mu(g). \quad (5)$$

**Proof.** Denote by $p_1$ the right hand side of equation (5). Then $p_1$ sends every vector in $[N \pm \nu]^+$ to zero. Now the invariance of $\mu$ implies that $\pi(g)p_1\pi(g)^{-1} = p_1 \forall g \in G$, so by the Schur Lemma $p_1$ is proportional to the identity on $[N \pm \nu]$. Since $\text{Tr} p_1 = 2(N \pm \nu) + 1$, $p_1$ restricted to $[N \pm \nu]$ is the identity and $p_1^2 = p_1$. Accordingly, $p = p_1$. \Box

### 4.2 Explicit Calculations

For the sake of simplicity we identify in this section $\text{Der}_3(\mathcal{A}_N)$ with $g_\mathbb{C}$. It is now our aim to calculate the first component of the Chern character determined by $p$, i.e. $F = \text{Tr}_2(p d\phi d\phi) \in \mathcal{A}_N \otimes (g_\mathbb{C}^* \wedge g_\mathbb{C}^*)$, where $d$ acts only on the $\mathcal{A}_N$ part of $p \in \mathcal{A}_N \otimes \text{End}([\nu])$ and $\text{Tr}_2$ is the trace in $\text{End}([\nu])$.

**Lemma 4.2.** $F = f \epsilon_{abc} X_c \Theta_a \wedge \Theta_b$ with $f \in \mathbb{C}I$. 

Proof. Let $\text{Ad}$ be the adjoint representation of $G$ on $\mathfrak{g}_C \ni u \mapsto \text{Ad}_g u = gug^{-1}$ and $\Theta$ as defined in section 4.1. Then $\Theta \otimes \mathbb{I} \in \Omega^1(\mathbb{N}) \otimes \text{End}([\nu])$ transforms as

$$\Theta(\text{Ad}_g u) \otimes \mathbb{I} = (\pi_1(g) \otimes \mathbb{I})(\Theta(u) \otimes \mathbb{I})(\pi_1^{-1}(g) \otimes \mathbb{I})$$

$$= \bar{\pi}(g)(\Theta(u) \otimes \mathbb{I})\bar{\pi}^{-1}(g),$$

where $\bar{\pi} = \pi_1 \otimes \pi_2$ is acting on $[\mathbb{N}] \otimes [\nu]$ and $u \in \mathfrak{g}_C$, i.e. $\Theta \otimes \mathbb{I}$ is invariant under the action of $G$ on $\Omega^1(\mathbb{N}) \otimes \text{End}([\nu])$. This implies for $dp = -[\Theta \otimes \mathbb{I}, p]$ that $dp(\text{Ad}_g u) = \bar{\pi}(g)dp(u)\bar{\pi}^{-1}(g)$ and finally for the first component of the Chern character

$$F(u, v) = \pi_1(g)F(\text{Ad}_g^{-1} u, \text{Ad}_g^{-1} v)\pi_1^{-1}(g)$$

for all $u, v \in \mathfrak{g}_C$. So $F$ is an invariant element of $\mathcal{A}_N \otimes (\mathfrak{g}_C^* \wedge \mathfrak{g}_C^*)$. Reducing this latter space as a $G$–module shows that $[0]$ appears only once, as $\mathfrak{g}_C^* \wedge \mathfrak{g}_C^* \simeq [1]$. Consequently, the subspace of invariant 2–forms is one dimensional and, since $\epsilon_{abc}X_c\Theta_a \wedge \Theta_b$ is invariant, $F$ can be written as claimed.

Note that $f \epsilon_{abc}X_c\Theta_a \wedge \Theta_b$ can also be written as $iq/4 \epsilon_{abc}Y_a dY_b \wedge dY_c$, where $q$ and $f$ are related by

$$q = \frac{4}{i} \frac{(N(N + 1))^{3/2}}{1/2 - N(N + 1)} f,$$

as can be seen by expanding $dY_a = [X_b, Y_a] \Theta_b$. It will turn out later that $q$ can be naturally interpreted as Chern numbers.

What is left to do is to determine $f$, depending on $N$ and $\nu$. For this note first that

$$\text{Tr}_2(dp(e_a) dp(e_b)) = f \epsilon_{abc}X_d.$$

Now multiply this equation with $\epsilon_{abc}X_c$ and take the trace also in $\mathcal{A}_N$ to get

$$f = \frac{\epsilon_{abc} \text{Tr}(dp(e_a) dp(e_b) X_c)}{2N(N + 1)(2N + 1)};$$

where $\text{Tr}$ denotes the trace in $\text{End}([\mathbb{N}]) \otimes \text{End}([\nu])$. This expression can be further simplified by the following lemma:

Lemma 4.3.

$$\epsilon_{abc} \text{Tr}(dp(e_a) dp(e_b) X_c) = (2(N \pm \nu) + 1) \epsilon_{abc} \langle h | [X_a, p] [X_b, p] X_c | h \rangle.$$
Proof. With \( \bar{\pi} \) as above the right hand side is by lemma 4.1

\[
(2(N \pm 1) + 1) \int_G \epsilon_{abc} \langle h|\bar{\pi}^{-1}(g)[X_a, p][X_b, p]X_c\pi(g)|h\rangle d\mu(g).
\]

Now \( \epsilon_{abc}[X_a, p][X_b, p]X_c \) are the components of the equivariant multilinear map

\[
-\text{[}H \otimes \nu, \nu\text{]} - \text{[}\Theta \otimes I, p\text{]}\text{([}\Theta \otimes I, p\text{]}\),
\]

from \( \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g} \simeq [0] \to \mathcal{A}_N \otimes \text{End}(\nu) \) which is constant. This implies the assertion. \( \square \)

Let us denote the expectation value \( \epsilon_{abc}\langle h|\cdot|h\rangle \) appearing in lemma 4.3 by \( B \). Expanding the commutators it is straightforward to see that \( B = C + iD \) with

\[
C = \epsilon_{abc}\langle h|X_a pX_b pX_c|h\rangle \quad \text{and} \quad D = -\langle h|X_a pX_c|h\rangle.
\]

Consider now (cf. [17]) the space of homogenous polynomials \( H_n \) of two complex variables \( z_1 \) and \( z_2 \) of fixed degree \( n \in \mathbb{N} \). We define the following “creation” and “annihilation” operators \( a_i^\dagger = z_i \) and \( a_i = \partial/\partial z_i \), satisfying \( [a_i, a_j^\dagger] = \delta_{ij} \), which give an irreducible representation of \( \mathfrak{g} \) by

\[
X_1 = \frac{1}{2}(a_1^\dagger a_2 + a_2^\dagger a_1), \quad X_2 = -\frac{i}{2}(a_1^\dagger a_2 - a_2^\dagger a_1) \quad \text{and} \quad X_3 = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2)
\]

with spin \( n/2 \) on \( H_n \). To compute \( C \) and \( D \) of equation (8) we realize the \([N] \otimes [\nu] \) representation on \( H_n \otimes H_l \) with \( 2N = n \) and \( 2\nu = l \), respectively. An orthonormal basis of \( H_n \) is given by

\[
|\psi_k\rangle = \left( \begin{array}{c} n \\ k \end{array} \right)^{1/2} z_1^k z_2^{n-k} \quad \text{with} \quad \langle \psi_k|\psi_{k'}\rangle = \delta_{kk'}.
\]

First we analyze the case where \( p \) projects onto \([N+\nu]\). Then the highest weight vector \( |h\rangle \in H_n \otimes H_l \) is given by \( |h\rangle = z_1^n \otimes z_1^l \), i.e. \( (X_3 \otimes I + I \otimes X_3)|h\rangle = (n+l)|h\rangle/2 \) and \( (X_+ \otimes I + I \otimes X_+)|h\rangle = 0 \), with \( ||h|| = 1 \). Define \( |w\rangle := X_1|h\rangle \), then \( X_2|h\rangle = i|w\rangle \). Since \( X_3|h\rangle = n/2|h\rangle \) a lengthy but straightforward calculation yields

\[
B = 2i(n-1)\langle w|p|w\rangle - 2i\langle w|pX_3 p|w\rangle - \frac{in^2}{4}.
\]

Now because \( |w\rangle \) and \( |v\rangle := (X_- \otimes I + I \otimes X_-)|h\rangle \) have the same eigenvalue \( (n+l-2)/2 \) of \( X_3 \otimes I + I \otimes X_3 \), we know that \( p|w\rangle = \lambda|v\rangle \). Accordingly, \( \lambda \) can be evaluated

\[
\lambda = \frac{\langle v|w\rangle}{\langle v|v\rangle} = \frac{1}{2} \frac{n}{n+l}.
\]
This gives
\[ \langle w|p|w \rangle = \frac{1}{4} \frac{n^2}{n+l} \quad \text{and} \quad \langle w|pX_3p|w \rangle = \frac{1}{8} \frac{n^2}{(n+l)^2} (n(n-2) + nl). \quad (11) \]

Inserting (11) into (10) this gives finally for \( f \) in equation (6) expressed in terms of \( N \) and \( \nu \)
\[ f = -i N \nu \frac{(N + \nu + 1)(N + \nu + 1/2)}{(N + \nu)^2(2N + 1)(N + 1)}. \quad (12) \]

The case where \( p \) projects onto \([N - \nu]\) is more involved, since we first have to determine the highest weight vector. The ansatz
\[ |h\rangle = \sum_{k=0}^{l} a_k z_2^k z_1^{n-k} \otimes z_1^{k-l} \]
leads through \((X_+ \otimes I + I \otimes X_+) |h\rangle = 0\) to the recursion relation \((k-l) a_k = (k+1) a_{k+1}\) which is solved by
\[ a_k = (-1)^k a_0 \binom{l}{k}. \quad (13) \]

The remaining \( a_0 \) is determined by the normalization condition \(||h|| = 1\) and gives for \( a_k \)
\[ a_k = (-1)^k \sqrt{\frac{n-l+1}{n+1}} \binom{l}{k} \quad \text{for } k = 0, \ldots, l, \quad (14) \]
where we have used the formula
\[ \sum_{k=0}^{l} \binom{l}{k} \binom{n}{k}^{-1} = \frac{n+1}{n-l+1}. \quad (15) \]

Proceeding analogously we define \(|w\rangle = X_1|h\rangle\), but now \(|w\rangle = |w_+\rangle + |w_-\rangle\) with
\[ (X_3 \otimes I + I \otimes X_3) |w_\pm\rangle = \left( \frac{n-l}{2} \pm 1 \right) |w_\pm\rangle. \]

From this it follows that \( pX_1|h\rangle = \lambda |w_-\rangle \) and \( pX_2|h\rangle = \lambda i |w_-\rangle \). Using
\[ \sum_{k=0}^{l} \binom{l}{k} \binom{n}{k+1}^{-1} = \frac{n+1}{(n-l)(n+1-l)} \quad \text{and} \]
\[ \sum_{k=0}^{l} \binom{l}{k} \binom{n}{k} \]
\[
\sum_{k=0}^{l} (n-k) \binom{l}{k} \binom{n}{k+1}^{-1} = \frac{(n+1)(n+2)}{(l-n-1)(l-n-2)}
\] (16)

one finds with \((X_- \otimes I + I \otimes X_-)|h\rangle = |v\rangle\) for the proportionality factor \(\lambda\)

\[
\lambda = \frac{\langle v|w_- \rangle}{\langle v|v \rangle} = \frac{1}{2} \frac{n+2}{n-l+2}.
\] (17)

Applying \(X_3\) to \(|h\rangle\) yields \(n/2|h\rangle - |K\rangle\) with

\[|K\rangle = \sum_{k=1}^{l} a_k k z_2^k z_1^{n-k} \otimes z_1^k z_2^{l-k}.
\]

Since \((X_3 \otimes I + I \otimes X_3)|K\rangle = (n-l)|K\rangle/2\) it follows that \(pX_3|h\rangle = (n/2-\mu)|h\rangle\), where \(\mu\) is given by

\[
\mu = \langle h|K \rangle = \frac{l}{n+2-l},
\]

for which a formula similar to (16) has been used. Now we have enough ingredients to write \(B\) as

\[
B = 2i\lambda^2 \left( (n-2\mu-1)(v|v \rangle - \langle v|X_3|v \rangle) - i \left( \frac{n}{2} - \mu \right)^2 \right).
\] (18)

It is left is to determine \(\langle v|X_3|v \rangle\). One finds

\[
\langle v|X_3|v \rangle = \frac{n+1-l(n-l)^2}{n+1} \frac{1}{2} \sum_{k=0}^{l} \binom{l}{k} \binom{n}{k+1}^{-1} (n-2k-2)
\]

\[
= \frac{(n+2)(n-l-2)(n-l)}{2(n-l+2)}.
\]

We collect the results to obtain \(f\) in terms of \(N\) and \(\nu\)

\[
f = \frac{i\nu(N+1)(N-\nu)(2N-2\nu+1)}{2N(2N+1)(N-\nu+1)^2}.
\] (19)

5 Results and Commutative Limit

Summarizing the calculations of the previous chapter we get for the Chern character \(F\) of the modules \(L_k\), with \(k = \pm 2\nu\), the formula

\[
F = \frac{i\epsilon_{abc}}{4} Y_a dY_b \wedge dY_c,
\] (20)
where

\[ q = \frac{4 \, (N(N + 1))^{3/2}}{i \, \sqrt{1/2 - N(N + 1)}} f, \quad (21) \]

and

\[
\begin{align*}
  f &= \frac{i \nu(N + 1)(N - \nu)(2N - 2\nu + 1)}{2N(2N + 1)(N - \nu + 1)^2} \quad \text{for } k = 2\nu > 0, \quad (22) \\
  f &= -i\nu \frac{(N + \nu + 1)(N + \nu + 1/2)}{(N + \nu)^2(2N + 1)(N + 1)} \quad \text{for } k = -2\nu < 0. \quad (23)
\end{align*}
\]

What is needed to obtain the associated Chern numbers is a certain notion of integration over 2–forms. Thus for \( \phi \in \mathcal{A}_N \) and

\[
\omega := \frac{\varepsilon_{abc} Y_a}{8\pi} dY_b \wedge dY_c \in \Omega^2(\mathcal{A}_N)
\]

we define the integral by

\[
\int^{\star} \phi \omega = \text{Tr}_N(\phi), \quad \text{where } \text{Tr}_N(\cdot) = \frac{1}{2N + 1} \text{Tr}(\cdot), \quad (24)
\]

with \( \int^{\star} \omega = 1 \). The 2-form \( \omega \) is the noncommutative volume form, which in the commutative limit converges to the normalized volume form on \( S^2 \). Consequently, the first Chern numbers of the fuzzy line bundles determined by \( p \) are given by

\[
c_1(p) := -\frac{1}{2\pi i} \int^{\star} F = -q. \quad (25)
\]

In the commutative limit we find for the topological charges

\[
k = \lim_{N \to \infty} c_1(p) = \mp 2\nu \in \mathbb{Z}, \quad (26)
\]

where the minus sign corresponds to the projection onto \([N + \nu]\) and the plus sign to \([N - \nu]\). Some topological charges \( q \) and their commutative limits \( k \) are shown in Figure 1.

6 Conclusions

We constructed projective modules over the matrix algebra \( \mathcal{A}_N \) of the fuzzy sphere using the prescription of quantizing equivariant vector bundles given
in [13], leading to fuzzy line bundles. With respect to the free derivation based differential calculus \((\Omega^*_{(N)}, \partial)\) on \(\mathcal{A}_N\) we calculated the Chern character \(F \in \Omega^2_{(N)}\). Since \(F\) was seen to be \(SU(2)\)–invariant, i.e. an \(SU(2)\)–equivariant mapping from \(sl(2, \mathbb{C}) \wedge sl(2, \mathbb{C})\) to \(\mathcal{A}_N\), it was unique up to a factor. The determination of this factor \(f \in \mathbb{C}\) was achieved and with help of a certain notion of integration the Chern numbers \(q\) associated with \(F\) were calculated. These turned out to be non-integer, becoming integers in the commutative limit \(N \to \infty\).

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References


Figure 1: Topological charges $q$ with commutative limit $k$ between $-4$ and $4$ as function of the fuzzyness $1/N$. These can be viewed as the (fuzzy) magnetic charges of a Dirac monopole living on the fuzzy sphere.