Microlocal analysis of quantum fields on curved spacetimes: Analytic wavefront sets and Reeh-Schlieder theorems

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Abstract

We show in this article that the Reeh-Schlieder property holds for states of quantum fields on real analytic curved spacetimes if they satisfy an analytic microlocal spectrum condition. This result holds in the setting of general quantum field theory, i.e. without assuming the quantum field to obey a specific equation of motion. Moreover, quasifree states of the Klein-Gordon field are further investigated in the present work and the (analytic) microlocal spectrum condition is shown to be equivalent to simpler conditions. We also prove that any quasifree ground- or KMS-state of the Klein-Gordon field on a stationary real analytic spacetime fulfills the analytic microlocal spectrum condition.

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1 Introduction

One of the remarkable features of quantum field theory is the ubiquity of fluctuations and, connected with that, the generic appearance of long-range correlations. What is even more remarkable is the fact that, using suitable selective operations and applying them, say, in an arbitrary spacetime region to the vacuum, one may produce in this way any given state in any other causally separated spacetime region up to arbitrary precision. This is known as the Reeh-Schlieder theorem [40]. Let us recall its statement in the setting of the operator-algebraic approach to quantum field theory. Suppose that we are given a spacetime manifold \((M, g)\)\(^1\) and a family (“local net”) \(\{A(O)\}_{O \subset M}\) of von Neumann algebras, all acting on a Hilbert-space \(\mathcal{H}\); the family is indexed by the open, relatively compact subsets of \(M\), and is subject to the conditions of isotony and locality:

\[
O_1 \subset O \Rightarrow A(O_1) \subset A(O) \quad \text{and} \quad O_1 \subset O^\perp \Rightarrow A(O_1) \subset A(O)' .
\]

Here, \(O^\perp\) denotes the causal complement of \(O\), i.e. the set of all points in \(M\) which cannot be connected to \(O\) by any causal curve, and \(A(O)\)\(^\prime\) denotes the commutant algebra of \(A(O)\) in \(\mathcal{B}(\mathcal{H})\). These conditions are the minimal assumptions in order that \(\{A(O)\}_{O \subset M}\) may be viewed as a net of local observable algebras of a (relativistic) physical system situated in \(M\), see [22] for discussion. One now says that a unit vector \(\Omega \in \mathcal{H}\) satisfies the Reeh-Schlieder property with respect to the region \(O \subset M\) if \(\Omega\) is cyclic for the algebra \(A(O)\) of observables localized in \(O\), that is, the set of vectors \(A(O)\Omega = \{A\Omega : A \in A(O)\}\) is dense in \(\mathcal{H}\). Moreover, one says that \(\Omega\) has the Reeh-Schlieder property if \(\Omega\) is cyclic for \(A(O)\) for each \(O \subset M\) which is open, non-void, and relatively compact. By the locality assumption, this then implies that \(\Omega\) is also separating for all local algebras \(A(O)\) (as long as \(O^\perp\) contains a non-void open set) and this means that \(A\Omega = 0 \Rightarrow A = 0\) for all \(A \in A(O)\). (A vector \(\Omega \in \mathcal{H}\) which is cyclic and separating for all local algebras \(A(O)\) is sometimes also called a standard vector for the family \(\{A(O)\}_{O \subset M}\).)

The generic occurrence of the Reeh-Schlieder property for large sets of physical states in quantum field theory — as so far known in quantum field theory on manifolds possessing suitable groups of isometries [40, 47, 1, 17, 31, 49, 42, 2] — is a mathematically precise way of expressing that long-range correlations are

\(^1\)i.e. \(M\) is a smooth, \(n\)-dimensional manifold, and \(g\) is a Lorentzian metric on \(M\)
a fundamental feature of quantum field theory. Furthermore, the Reeh-Schlieder property plays a very important role in analyzing the mathematical structure of quantum field theory. For instance, it is being used at some stages in the development of charge superselection theory (see [1 41] and references cited there). Another very important aspect of the Reeh-Schlieder property is that one may naturally associate with each von Neumann algebra together with a cyclic and separating vector the so-called Tomita-Takesaki modular objects [48]. In the seminal work of Bisognano and Wichmann [3, 4] it has been shown that the Tomita-Takesaki modular objects associated with the vacuum-vector and the von Neumann algebra $A(W)$ of a “wedge-region” (which is actually infinitely extended) in a Wightman-type quantum field theory on Minkowski spacetime have a specific geometric significance. This insight has initiated considerable progress in the mathematical development of general quantum field theory on which the recent review by Borchers [5] reports exhaustively; therefore we refer the reader to that reference for further discussion. We limit ourselves to mentioning that quite promising generalized forms of such a “geometric modular action” that are applicable to quantum field theories on curved spacetimes have been suggested and investigated more recently [16 15]. The Reeh-Schlieder property is also responsible for (maximal) violations of Bell’s inequalities in quantum field theory [46], and more recently, Reeh-Schlieder properties have been found to imply various forms of long-range entanglement of states in relativistic quantum field theory [23 32 51]; see also [43 13] for related discussions. A possible significance of Reeh-Schlieder properties for questions related to cosmology has been proposed in [54].

As indicated above, Reeh-Schlieder properties have, either in the model-independent approach or for concrete quantum field models, so far only been established under the assumption that the spacetime in which the quantum system is situated possesses a sufficient amount of spacetime symmetries. This constitutes a considerable limitation, and the question is if Reeh-Schlieder like properties can also be established for quantum field theories on spacetimes not admitting any isometries. This is feasible since the main mathematical argument leading to the Reeh-Schlieder theorems is the case where there are sufficiently many space-time symmetries is an analytical argument of the type of the edge-of-the-wedge-theorem [47] or Schwartz’ reflection principle in order to derive a certain global
property of a quantum state from local information, and these arguments don’t use spacetime-symmetries (in particular, timelike isometry groups) directly. On the other hand, commonly the analytic properties of correlation functions in quantum field theory are consequences of the relativistic spectrum condition whose formulation requires a form of time-translation symmetry. Time-translation symmetry is also required in order to formulate conditions of thermal equilibrium in relativistic quantum field theory from which Reeh-Schlieder properties may also be deduced [31].

At any rate, certain analytical properties of correlation functions are prerequisite in order to establish Reeh-Schlieder theorems in quantum field theory and the question arises how to generalize the analyticity properties known to hold e.g. for ground states or thermal equilibrium states with respect to a time-symmetry group to more general situations. A way how to proceed in more general situations may be to follow, and to refine, the approach pioneered by Radzikowski [38, 39] who proved that, for a free scalar quantum field on generic curved space-times, the condition that a quasifree state be a Hadamard state can equivalently be expressed as a condition on the wavefront set of the two-point function of that state (see also [36, 25, 45] for related work). Since it can convincingly be argued that Hadamard states are most likely candidates for physical states of quantum fields obeying linear wave-equations [53], it appears natural to sharpen the condition on the two-point function of a physical state by demanding that it applies to the analytic wavefront set and not only, as in most previous considerations, to the $C^\infty$-wavefront set. In order that this makes sense independently of particular coordinates, the underlying space-time manifold ought to be real analytic.

Thus, we will propose a stricter form of the wavefront set spectrum condition formulated in [38], or of the microlocal spectrum condition [12], for the $n$-point functions of a generic scalar quantum field on a real analytic spacetime in terms of their analytic wavefront sets, and we will show that such states possess the Reeh-Schlieder property. In doing so, we will present a transcription of the (analytic) microlocal spectrum condition for a two-point function as a condition on the wavefront set of a certain Hilbert-space valued distribution. It turns out that working with wavefront sets of Hilbert-space valued distributions has several advantages. One of those is that in terms of Hilbert-space valued distributions the (analytic) microlocal spectrum condition (for two-point functions of free fields)
assumes a very simple and elegant form which is, in fact, most reminiscent of the
usual spectrum condition in quantum field theory in Minkowski spacetime. Part
of this analysis appears already in [11] for the case of the $C^\infty$-wavefront set where
the calculus of Hilbert space valued distributions is used for the definition of Wick
products of free fields. This appeared motivation enough to discuss several aspects
of microlocal analysis of Hilbert-space valued distributions more systematically,
and that discussion thus forms the first part of the present article in Sec. 2.

In Sec. 3 we summarize some basics on the description of general scalar quan-
tum fields on curved spacetimes, together with the example of the free scalar
Klein-Gordon field.

We recapitulate the definition of the microlocal spectrum condition (“µSC”),
referring to the $C^\infty$-wavefront sets of $n$-point functions of quantum fields on man-
ifolds, according to [12] in Sec. 4. In the same section, we introduce our analytic
microlocal spectrum condition (“aµSC”) which will be defined similarly to µSC
but using now analytic wavefront sets of $n$-point functions of quantum fields on
real analytic manifolds. (We should note that Hollands and Wald [29] have re-
cently also introduced a similar concept of analytic microlocal spectrum condition
which refers to a whole class of states, and is used in a different context).

In Sec. 5 we will present our main result, which is a Reeh-Schlieder theorem
for quantum field states fulfilling the aµSC. Here, we draw on results of Sec. 2,
and also on a microlocal version of the edge-of-the-wedge theorem, which appears
as Thm. 8.5.6’ in [27].

Finally, in Sec. 6 we present the characterization of aµSC and of µSC for
two-point functions of a free scalar field on a manifold in terms of a simple conic
form of the wavefront set of certain Hilbert-space valued distributions. Moreover,
we prove that ground- and KMS-states of the free scalar field on a real analytic
stationary spacetime obey the aµSC. This shows in particular that the Hartle-
Hawking state on the Schwarzschild spacetime satisfies the aµSC.
2 Microlocal analysis for Hilbert space valued distributions

Assume we are given a Hilbert space $\mathcal{H}$ and a smooth manifold $M$ which is second countable and Hausdorff. The space $\mathcal{D}'(M, \mathcal{H})$ of $\mathcal{H}$ valued distributions is defined to be the set of all weakly continuous linear maps $C^\infty_0(M) \to \mathcal{H}$.

Note that due to the nuclearity of $C^\infty_0(M)$ these maps are automatically strongly continuous. If $X \subset \mathbb{R}^n$ is an open subset a linear map $\psi : C^\infty_0(X) \to \mathcal{H}$ is in $\mathcal{D}'(X, \mathcal{H})$ if and only if for each compact subset $K \subset X$ there are constants $C > 0$ and $\alpha \in \mathbb{N}_0$ such that

$$\|\psi(f)\| < C \sum_{|k| \leq \alpha} \sup_x |(\partial^k f)(x)|,$$

for all $f \in C^\infty_0(K)$. If $\alpha$ can be chosen independently of $K$, then we say that $\psi$ is of order $\alpha$ and write $\psi \in \mathcal{D}'^\alpha(X, \mathcal{H})$. The set of $\psi \in \mathcal{D}'(X, \mathcal{H})$ with compact support in $X$ will be denoted by $\mathcal{E}'_0(X, \mathcal{H})$. If $\psi$ has compact support then one can easily extend $\psi$ to a linear map $C^\infty(X) \to \mathcal{H}$ and we have for all compact sets $K$ which contain $\text{supp}(\psi)$

$$\|\psi(f)\| < C \sum_{|k| \leq \alpha} \sup_{x \in K} |(\partial^k f)(x)|, \quad \forall f \in C^\infty(X),$$

for some $C > 0$, $\alpha \in \mathbb{N}_0$. Conversely, if there exists a compact set $K$ such that for a given linear map $\psi : C^\infty(X) \to \mathcal{H}$ the inequality (2) holds for some $C$ and $\alpha$, then $\psi$ is a distribution with support in $K$. Therefore, $\mathcal{E}'(X, \mathcal{H})$ can be identified with the set of strongly continuous maps $C^\infty(X) \to \mathcal{H}$. If a subset $L$ of $\mathcal{E}'(X, \mathcal{H})$ is bounded then (2) holds for all $\psi \in L$ and $f \in C^\infty(X)$ with constants $C$ and $\alpha$ independent of $\psi$. We define $\mathcal{E}'^\alpha(X, \mathcal{H}) := \mathcal{E}'(X, \mathcal{H}) \cap \mathcal{D}'^\alpha(X, \mathcal{H})$ and obviously $\mathcal{E}'(X, \mathcal{H}) = \bigcup_\alpha \mathcal{E}'^\alpha(X, \mathcal{H})$.

If $\psi \in \mathcal{E}'(\mathbb{R}^n, \mathcal{H})$ one may define the Fourier transform $\hat{\psi}$ in the same way as this is done for ordinary distributions. Namely, the Fourier transform $\hat{\psi}$ is the $\mathcal{H}$ valued function on $\mathbb{R}^n$ given by $\hat{\psi}(k) := \psi(e^{-ik\cdot})$.  

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Definition 2.1. Let $X$ be open in $\mathbb{R}^n$ and let $\psi$ be in $\mathcal{D}'(X, \mathcal{H})$. Then a point $(x, k) \in X \times (\mathbb{R}^n \setminus \{0\})$ is called regular directed for $\psi$ if the following holds: There exists a function $f \in C^\infty_0(X)$ with $f(x) = 1$ and an open conic neighbourhood $\Gamma$ of $k$ such that for each $N \in \mathbb{N}$ there exists a constant $C_N$ with

$$\sup_{\lambda \in \Gamma} (1 + |\lambda|)^N \|\hat{f}\psi(\lambda)\| < C_N. \quad (3)$$

The set of regular directed points is open. Its complement in $X \times (\mathbb{R}^n \setminus \{0\})$ is called the wavefront set $\text{WF}(\psi)$ of $\psi$.

The following proposition shows that microlocal analysis of Hilbert space valued distributions is analogous to the case of ordinary distributions. See also [11] for related discussion.

Proposition 2.2. Let $X$ be open in $\mathbb{R}^n$ and let $\psi \in \mathcal{D}'(X, \mathcal{H})$ be a Hilbert space valued distribution.

1) If $\psi$ holds for $f \in C^\infty_0(X)$ then it also holds with $f$ replaced by $gf$ for any $g \in C^\infty(X)$.

2) If $\psi$ has compact support then $\hat{\psi}$ is polynomially bounded in the norm, i.e. there is a constant $C$ and an integer $M$, such that

$$\|\hat{\psi}(k)\| < C(1 + |k|)^M. \quad (4)$$

If moreover a subset $L \in \mathcal{E}'(X, \mathcal{H})$ is bounded then $\psi$ holds for all $\psi \in L$ with $C$ and $M$ independent of $\psi$.

3) If $\text{WF}(\psi)$ is empty then $\psi$ is smooth in the norm.

4) If we define the distribution $w \in \mathcal{D}'(X \times X)$ by $w(f, g) = \langle \psi(\overline{f}), \psi(g) \rangle$, then

$$(x, k) \in \text{WF}(\psi) \iff ((x, -k), (x, k)) \in \text{WF}(w), \quad (5)$$

and moreover, if $(x, k) \notin \text{WF}(\psi)$ with $k \neq 0$, then

$$((x, -k), (x_1, k_1)) \notin \text{WF}(w) \text{ and } ((x_1, k_1), (x, k)) \notin \text{WF}(w), \quad (6)$$

for arbitrary $(x_1, k_1) \in X \times \mathbb{R}^n$. 

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5) Under change of coordinates $\text{WF}(\psi)$ transforms as a subset of the cotangent bundle. Hence, $\text{WF}(\phi)$ may be defined for distributions $\phi \in \mathcal{D}'(M, \mathcal{H})$, for a smooth manifold $M$ and $\text{WF}(\phi) \subset T^*M \setminus 0$. Here $T^*M \setminus 0$ is the cotangent bundle with zero section removed.

**Proof.** The first statement is proved in the same manner as for ordinary distributions (cf. [27], Lemma 8.1.1). The inequality (2) immediately gives the second statement if $f(x) = e^{-ikx}$. To see that 3) holds we first note that $\text{WF}(\psi) = \emptyset$ implies that for each point $x \in X$ there is a positive function with $f(x) = 1$ such that $\|\hat{f}\psi\| \in \mathcal{S}(\mathbb{R}^n)$. The same arguments as in the complex valued case show that $f\psi$ can be represented by the inverse Fourier transform of $\hat{f}\psi$, which is a smooth function, i.e. all derivatives in the norm sense exist and are given by $D^\alpha(f\psi)(x) = 2\pi(\hat{\alpha}f\psi)(k)(-x)$. 5) is a simple consequence of 4) and it remains to show 4). Assume first that $(x, -k, x, k)$ is regular directed for $w$. Then there is a function $f_1 \in C^\infty_0(X \times X)$ such that $\hat{f_1}\psi$ decays rapidly in a conic neighbourhood of $(x, -k, x, k)$. Because of 1) we may choose $f_1$ to be of the form $f \otimes f$, where $f \in C^\infty_0(X)$ is a positive function. Since

$$\|\hat{f}\psi(k)\|^2 = ((f \otimes f)\psi)(-k, k),$$

(7) $(x, k)$ is regular directed for $\psi$. Suppose conversely we knew that $(x, k)$ is regular directed for $\psi$. By the Cauchy inequality we have

$$|((f \otimes g)\psi)(k_1, k_2)| \leq \|\psi(\hat{f}(\cdot)e^{ik_1\cdot})\| \cdot \|\psi(g(\cdot)e^{-ik_2\cdot})\|.$$  

(8) By assumption there is a $g \in C^\infty_0(X)$ with $g(x) = 1$ such that in a conic neighbourhood of $k$ the second factor is rapidly decreasing. Since the other is polynomially bounded for any $f \in C^\infty_0(X)$, $(x_1, k_1, x, k)$ is a regular directed point for $w$ for any $(x_1, k_1)$. In the same way one shows that $(x, -k, x_1, k_1)$ is regular directed. This concludes the proof. 

The fourth statement in the above proposition allows one to take over many results known for ordinary distributions.

**Proposition 2.3.** If $P : C^\infty(M) \to C^\infty(M)$ is a differential operator and $\psi \in \mathcal{D}'(M, \mathcal{H})$ such that $\psi \circ P^* = 0$, where $P^*$ is the formal adjoint of $P$, then

$$\text{WF}(\psi) \subset \text{char}(P).$$
Here \( \text{char}(P) \) is the characteristic set of \( P \), i.e. the set of points \((x, k)\) in \( T^*M \setminus 0 \) on which the principal symbol \( \sigma_P \) of \( P \) vanishes.

**Proof.** We define \( w \) as in Proposition 2.2. Note that \((x, -k), (x, k)\) is in the characteristic set of the operator \( L := P \otimes P \) if and only if \((x, k)\) is in the characteristic set of \( P \). Moreover, \( Lw = 0 \). The result follows now from the fourth statement in Proposition 2.2.

One may also define the analytic wavefront set of a Hilbert space valued distribution. We follow the definition in [27] (Def. 8.4.3).

**Definition 2.4.** Let \( X \) be an open subset of \( \mathbb{R}^n \) and \( \psi \in \mathcal{D}'(X, \mathcal{H}) \). We denote by \( \text{WF}_A(\psi) \) the complement in \( X \times (\mathbb{R}^n \setminus \{0\}) \) of the set of points \((x_0, k_0)\) such that there is a neighbourhood \( U \subset X \) of \( x_0 \), a conic neighbourhood \( \Gamma \) of \( k_0 \) and a bounded sequence \( \psi_N \) of distributions with compact support which is equal to \( \psi \) in \( U \), such that there exists a constant \( C \) with

\[
|||k|^N \hat{\psi}_N(k)|| \leq C(C(N + 1))^N, \tag{9}
\]

for all \( k \in \Gamma \).

The bounded sequence \( \psi_N \) can always be chosen to be the product \( f_N \psi \), where \( f_N \) is a sequence of smooth functions. One has

**Lemma 2.5.** Let \( \psi \in \mathcal{D}'(X, \mathcal{H}) \), \( K \) a compact subset of \( X \), and let \( F \) be a closed cone in \( \mathbb{R}^n \) such that \( \text{WF}_A(\psi) \cap (K \times F) = \emptyset \). If \( f_N \in C^\infty_0(K) \) and for all \( \alpha \)

\[
|D^{\alpha + \beta} f_N| \leq C_\alpha(C_\alpha(N + 1))^{|\beta|}, \quad |\beta| \leq N = 1, 2, \ldots, \tag{10}
\]

then \( f_N \psi \) is a bounded sequence and we have

\[
|||k|^N \hat{\psi}_N(k)|| \leq C(C(N + 1))^N, \tag{11}
\]

for all \( k \in F \). Moreover, if \( x \) is a point in the interior of \( K \), there always exists a neighbourhood \( \mathcal{U} \) of \( x \) and a sequence of functions \( f_N \) such that (10) is satisfied and \( f_N = 1 \) on \( \mathcal{U} \).

**Proof.** The proof of this statement is the same as for ordinary distributions (see [27], Lemma 8.4.4). \( \square \)
Proposition 2.6. Let $X$ be an open set in $\mathbb{R}^n$ and let $\psi \in \mathcal{D}'(X, \mathcal{H})$ be a Hilbert space valued distribution.

1) If $\text{WF}_A(\psi)$ is empty then $\psi$ is strongly real analytic.

2) If we define the distribution $w \in \mathcal{D}'(X \times X)$ by $w(f, g) = \langle \psi(f), \psi(g) \rangle$, then

$$ (x, k) \in \text{WF}_A(\psi) \iff ((x, -k), (x, k)) \in \text{WF}_A(w), $$

and moreover, if $(x, k) \notin \text{WF}_A(\psi)$ with $k \neq 0$, then

$$ ((x, -k), (x_1, k_1)) \notin \text{WF}_A(w) \text{ and } ((x_1, k_1), (x, k)) \notin \text{WF}_A(w), $$

for arbitrary $(x_1, k_1) \in X \times \mathbb{R}^n$.

3) Under analytic change of coordinates $\text{WF}_A(\psi)$ transforms as a subset of the cotangent bundle. Hence, $\text{WF}_A(\phi)$ may be defined for distributions $\phi \in \mathcal{D}'(M, \mathcal{H})$, for a smooth real analytic manifold $M$ and $\text{WF}_A(\phi) \subset T^* M \setminus 0$.

Proof. We start with 2). Assume that $(x, -k, x, k) \notin \text{WF}_A(w)$. We choose a sequence of functions $f_N$ in $C_0^\infty(X)$ which satisfies the inequality (10) and which is equal to 1 in a neighbourhood of $x$. Then the sequence $g_N := f_N \otimes f_N \in C_0^\infty(X \times X)$ also satisfies an inequality of the form (10), and hence

$$ |(k_1, k)|^N |(\widehat{g_N w})(k_1, k)| \leq C'(N + 1)^N, $$

for some constant $C > 0$ and all $k_1$ in a conic neighbourhood of $k$. We have

$$ \|\widehat{f_N \psi}(k_1)\|^2 = \langle \widehat{g_N w}, (k_1, k_1) \rangle, $$

and a quick estimate shows that for all $k_1$ in a conic neighbourhood of $k$ we have

$$ \|\widehat{f_N \psi}(k_1)\|^2 \leq C'(N + 1)^N, $$

for some constant $\tilde{C}$. Therefore, $(x, k) \notin \text{WF}_A(\psi)$. Suppose conversely that $(x_1, k_1) \notin \text{WF}_A(\psi)$. Hence, there is a sequence $\psi_N$ bounded in $\mathcal{E}'(X, \mathcal{H})$ and equal to $\psi$ in a neighbourhood of $x_1$ such that the inequality (9) holds in a conic neighbourhood of $k_1$. Choose another function $g$ which is equal to 1 in a neighbourhood of a point $x_2$. Then the distribution $w_N$ defined by $w_N(h_1, h_2) := \langle \psi_N(h_1), \psi_N(h_2) \rangle$
\( \langle \psi(g \cdot \hbar_1), \psi_N(h_2) \rangle \) is bounded in \( \mathcal{E}'(X \times X) \). Moreover, an application of the Cauchy inequality shows that
\[
|\hat{w}_N(k_2, k)| \leq \|\psi(ge^{ik_2})\| \cdot \|\psi_N(e^{-ik})\|. \tag{17}
\]
The first factor is bounded by \( C_M(1 + |k_2|)^M \) for some \( M \in \mathbb{N} \) and a simple estimate shows that for all \( k_2 \) there is a conic neighbourhood \( \Gamma \) of \((k_2, k_1)\) and a \( C > 0 \) with
\[
|(k_2', k)|^N \cdot |\hat{w}_{N,M}(k_2', k)| \leq C(C(N + 1))^N, \tag{18}
\]
for all \((k_2', k) \in \Gamma\). Since \( w_N \) is equal to \( w \) in a neighbourhood of \((x_2, x_1)\), we get
\[
((x_2, k_2), (x_1, k_1)) \notin WF_A(w).
\]
In the same way one shows that
\[
((x_1, -k_1), (x_2, k_2)) \notin WF_A(w).
\]
Statement 3) is an immediate consequence of 2) since the analytic wavefront set of an ordinary distribution transforms as a subset of the cotangent bundle. The first statement can be shown in the same way as for ordinary distributions (see [27], Thm. 8.4.5). This concludes the proof. \( \square \)

**Proposition 2.7.** If \( P : C^\infty(M) \rightarrow C^\infty(M) \) is a differential operator with real analytic coefficients on a real analytic manifold \( M \) and \( \psi \in \mathcal{D}'(M, \mathcal{H}) \) such that \( \psi \circ P^* = 0 \), where \( P^* \) is the formal adjoint of \( P \), then
\[
WF_A(\psi) \subset char(P),
\]
where \( char(P) \) is the characteristic set of \( P \).

**Proof.** Analogous to the proof of Proposition 2.3. \( \square \)

**Theorem 2.8.** Let \( X \subset \mathbb{R}^n \) be an open subset and assume \( \psi \in \mathcal{D}'(X, \mathcal{H}) \). Assume that there is a smooth \( \mathcal{H} \)-valued function \( G : I \times X \rightarrow \mathcal{H} \), with \( I = (0, \epsilon) \), such that
\[
\begin{align*}
&\lim_{t \to 0} G(t, \cdot) = \psi \text{ in the sense of distributions,} \\
&(\partial_t G)(t, x_1, \ldots, x_n) = i(\partial_{x_1} G)(t, x_1, \ldots, x_n).
\end{align*}
\]
Then $WF_A(\psi) \subset \{(x_1, \ldots, x_n) \times (y_1, \ldots, y_n) \in X \times \mathbb{R}^n \setminus \{0\}; y_1 \geq 0\}$.

**Proof.** The proof is a variation of the proof of Thm. 8.4.8 in [27]. Since the statement is local we can assume without loss of generality that $X = X_1 \times \ldots \times X_n$ and that $\psi$. First note that for each given $g \in C^\infty_0(X_2 \times \ldots \times X_n)$ the function

$$H(x + iy) := \int_{X_2 \times \ldots \times X_n} G(y, x, y_2, \ldots, y_n) g(y_2, \ldots, y_n) dy_2 \cdots dy_n$$

is defined on the strip $X_1 + iI$ and is holomorphic. Moreover, it has a distributional boundary value $\psi(\cdot \otimes g)$. We will slightly vary the proof of Thm. 3.1.14 in [27] to show that the following bound holds:

$$\|H(x + iy)\| \leq C' y^{-m-1}$$

(20)

if $(x, y) \in \bar{X}_1 \times I/2$ and $\text{clo}(\bar{X}_1) \subset X_1$ for some $m > 0$. Let $f \in C^\infty_0(X_1 \times I)$ be a function with support in $K \times I$, where $K$ is a compact subset of $X_1$, such that $f$ is equal to one in a neighbourhood of $\bar{Z}$, where $Z := \bar{X}_1 \times I/2$. Cauchy’s integral formula applied to $fH$ in the set $\Im z > \Im(\zeta/2)$ shows that if $\zeta = \xi + i\eta \in Z$

$$H(\zeta) = - \pi^{-1} \int_{y > \eta/2} H(x + iy) \partial f(x, y) / \partial \bar{z}(z - \zeta)^{-1} dx dy$$

$$+ (2\pi i)^{-1} \int f(x, \eta/2)(x - \xi - i\eta/2)^{-1} H(x + i\eta/2) dx.$$

(21)

An application of the uniform boundedness principle (Banach-Steinhaus theorem) shows that $\|\int_{X_1} H(x + iy) h(x) dx\| \leq C \sum_{\alpha \leq m} \sup |\partial^\alpha h|$ for all $h \in C^\infty_0(K)$ with constants $C$ and $m$ independent of $y$ (c.f. [27], Thm. 2.1.8). Therefore, the last integral in (21) can be estimated in the norm by

$$C_1 \sum_{\alpha \leq m} \sup |\partial^\alpha_\xi f(x, \eta/2)(x - \xi - i\eta/2)^{-1}| \leq C_2 |\eta|^{-m-1}.$$

(22)

The first integral in (21) is bounded and this proves the inequality (20).

Since the statement of the theorem is local we can always replace $X_1$ by $\bar{X}_1$ and $I$ by $I/2$ and we can therefore assume without loss of generality that the bound (20) holds in $X_1 \times I$. From Stokes integral formula one gets for $y, Y \in \mathbb{R}_+$
and \( N \in \mathbb{N} \) such that \( y + Y < \epsilon / 2 \) the following formula (compare 3.1.19 in [27]) for any \( \tilde{\phi} \in C_0^\infty(X) \) with \( \tilde{\phi} = \phi \otimes g \):

\[
\int_X \tilde{\phi}(x)G(y, x)dx = \int_{X_1} \Theta(x, Y)H(x + iy + iY)dx
\]

\[
+ (N + 1) \int_{X_1} dx \int_{0<t<1} dt H(x + itY + iy)(\partial^N \phi)(x)\frac{(iY)^N}{N!}t^N,
\]

where

\[
\Theta(x, y) := \sum_{j=0}^{N} \partial^j \phi(x)(iy)^j / j!.
\]

Because of the bound (20) the integrand under the double integral in (23) is uniformly bounded by an integrable function if \( N > (m + 1) \) and the first term even converges uniformly as \( y \to 0 \). Therefore, we have for \( N > (m + 1) \)

\[
\psi(\tilde{\phi}) = \int_{X_1} \Theta(x, Y)H(x + iY)dx
\]

\[
+ (N + 1) \int_{X_1} dx \int_{0<t<1} dt H(x + itY)(\partial^N \phi)(x)\frac{(iY)^N}{N!}t^N.
\]

Now let \( x' = (x'_1, \ldots, x'_n) \) be a point in \( X \) and let \( \phi_\nu \) be a sequence of functions on \( X_1 \) which are all equal to 1 in a common neighbourhood of \( x'_1 \) such that

\[
|\partial^\alpha \phi_\nu| \leq (C_1(\nu + 1))\alpha, \quad \alpha \leq \nu + 1.
\]

Assume that \( g \) is equal to 1 in a neighbourhood of \( (x'_2, \ldots, x'_n) \). With

\[
\Theta_\nu(x, y) := \sum_{j=0}^{\nu} \partial^j \phi_\nu(x)(iy)^j / j!
\]

we get from (25) for \( \nu > (m + 1) \)

\[
(\phi_\nu \otimes g)\psi(k) = \int_X G(Y, x)\Theta_\nu(x_1, Y)g(x_2, \ldots, x_n)e^{-i(x+iy,k)}dx
\]

\[
+ (\nu + 1) \int_X dx \int_{0<t<1} dt G(tY, x)e^{-i(x+itY,k)}
\]

\[
\cdot (\partial^\nu \phi_\nu)(x_1)g(x_2, \ldots, x_n)\frac{(iY)^\nu}{\nu!}t^\nu.
\]
Here we used the notations $x = (x_1, \ldots, x_n)$ and $Y = (Y, 0, \ldots, 0)$. With $C_2 = 2e^{C_1Y}$ we have $|\Theta_\nu(x, Y)| \leq C_2^{\nu+1}$ and because of the bound \(20\) we get
\[
\|(\hat{\phi}_\nu \otimes g)\psi(k)\| \leq C_3^{\nu+1}(e^{(Y,k)} + (\nu - m - 1)!(-Y, k)^m - \nu), \quad (Y, k) < 0.
\] (29)
We define $\psi_\nu := (\phi_{m+\nu} \otimes g)\psi$ and if $(Y, k) < -c|k|$ for a fixed $c$, we obtain for some $C_4$
\[
\|\hat{\psi}_\nu(k)\| \leq C_4^{\nu+1}\nu!|k|^{-\nu},
\] (30)
since $e^{-c|k|} \leq \nu!(c|k|)^{-\nu}$. If we chose $\phi_\nu$ bounded in $C_0^\infty$, then $\psi_\nu$ is bounded in $\mathcal{E}'(X, \mathcal{H})$. We have shown that
\[
\text{WF}_A(\psi) \subset X \times \{k, (Y, k) \geq 0\}.
\]
\[\square\]

An immediate corollary of this theorem is

**Corollary 2.9.** Let $X \subset \mathbb{R}^n$ be an open subset. Suppose that $V$ is an open cone in $\mathbb{R}^n \setminus \{0\}$ and $Z$ is an open neighbourhood of $0$ in $\mathbb{R}^n$. Denote by $V^\circ$ the dual cone.\[\footnote{The dual cone $V^\circ$ of an open cone $V$ is defined as the set $V^\circ = \{\xi \in \mathbb{R}^n : \langle v, \xi \rangle \geq 0 \ \forall v \in V\}$, where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product on $\mathbb{R}^n$.} If $\psi \in \mathcal{D}'(X, \mathcal{H})$ is the boundary value in the sense of distributions of a function $G$ which is analytic in $X \times (V \cap Z)$, then $\text{WF}_A(\psi) \subset X \times V^\circ$.

### 3 Quantum field theory on curved spacetimes

By a spacetime we mean in the following a connected smooth manifold of dimension $n \geq 2$ which is second countable and Hausdorff and which is endowed with a Lorentzian metric $g$ such that $M$ is both oriented and time-oriented. A spacelike hypersurface $C$ in a spacetime $M$ is called Cauchy surface if each inextendible causal curve intersects with $C$ exactly once. In case there exists a Cauchy surface the spacetime $M$ is said to be globally hyperbolic (see e.g. \cite{52, 24} for further discussion).
3.1 Scalar fields on curved spacetimes

The Borchers-Uhlmann algebra $B$ of a manifold $M$ is defined to be the topological tensor algebra

$$
B := \mathbb{C} \oplus \bigoplus_{m=1}^{\infty} \bigotimes_{m} C^\infty_0(M)
$$

(31)
endowed with a star defined by $(f_1 \otimes \ldots \otimes f_k)^* = \overline{f_k} \otimes \ldots \otimes \overline{f_1}$. A state $\omega$ over $B$ determines a sequence of distributions $\omega_m \in \mathcal{D}'(M^m)$, the so called $m$-point functions, by

$$
\omega_m(f_1, \ldots, f_m) := \omega(f_1 \otimes \cdots \otimes f_m).
$$

(32)

If $\mathcal{H}$ is a Hilbert space and $D$ a dense subset we denote by $\mathcal{L}_D^+$ the set of (possibly unbounded) operators $A$ on $\mathcal{H}$ with the properties

$$
dom(A) = D, \ A D \subset D, \ dom(A^*) \supset D, \ A^* D \subset D.
$$

(33)

The involution $A^+ := A^*|_D$ and the locally convex topology defined by the seminorms $p_{\phi, \psi}(A) := |\langle \phi, A \psi \rangle|$, $\phi, \psi \in D$ turn $\mathcal{L}_D^+$ into a locally convex topological $*$-algebra.

Each state over the Borchers-Uhlmann algebra $B$ determines, by the GNS construction, a Hilbert space $\mathcal{H}$ with a dense domain $D$ and a $*$-representation $\pi : B \to \mathcal{L}_D^+$ with cyclic vector $\Omega \in D$ such that $\pi(B)\Omega = D$. If $M = \mathbb{R}^4$ is the Minkowski space it is well known that Wightman fields can be constructed from states over the Borchers-Uhlmann algebra which satisfy certain requirements like translation invariance or the spectrum condition. The field is, in this case, the operator valued distribution $f \mapsto \Phi(f) := \pi(0 \oplus f \oplus 0 \oplus \cdots)$. We will think of a quantum field on a curved spacetime in the same way, i.e. a quantum field can be defined by a state over the Borchers-Uhlmann algebra of test functions on the underlying spacetime. A state is called quasifree if all the odd $m$-point functions vanish and the even $m$-point functions can be expressed by

$$
\omega_m(f_1, \ldots, f_m) = \sum_P \prod_r \omega_2(f_{(r,1)}, f_{(r,2)}),
$$

(34)

where $P$ denotes a partition of the set $\{1, \ldots, m\}$ into subsets which are pairings of points labelled by $r$. 

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For quantum fields on Minkowski spacetime one usually requires the properties of Poincaré covariance, spectrum condition, existence of an invariant vacuum vector as well as local commutativity (for observable fields) to hold (see e.g. [47, 22]). Due to the lack of an analogue of the Poincaré group, only the last requirement can straightforwardly be generalized to curved spacetimes. However, as will be seen in section 4, there is a microlocal version of the spectrum condition which can be stated independently of the coordinate system and hence can be applied to quantum fields on curved spacetimes.

For a field \( \Phi(\cdot) \) defined by a state \( \omega \) over the Borchers-Uhlmann algebra we can associate a net of von Neumann algebras \( \{ \mathcal{A}(O) \}_{O \subset M} \) in the following way. Let as above be \( D \) the dense domain \( \pi(B)\Omega \) arising by the GNS construction from \( \omega \). For a subset \( S \subset L_+^D \) the weak commutant \( S'_{w} \) of \( S \) is defined to be the set of bounded operators \( A \) on \( H \) such that

\[
\langle B^* \phi, A \psi \rangle = \langle A^* \phi, B \psi \rangle, \quad \forall \ B \in S, \ \phi, \psi \in D.
\]  

(35)

A net of von Neumann algebras \( \{ \mathcal{A}(O) \}_{O \subset M} \) is then defined by

\[
\mathcal{A}(O) := \left( \{ \Phi(f); \ \text{supp}(f) \subset O \} \right)'_{w}'.
\]

(36)

The requirement of local commutativity may now be formulated by demanding that the von Neumann algebras associated with causally separated regions commute, i.e.,

\[
\mathcal{A}(O_1) \subset \mathcal{A}(O)' \quad \text{if} \quad O_1 \subset O^\perp.
\]

This is a strong form of local commutativity which is to be seen as a selective constraint on the Hilbert-space representation \( \pi \) of the Borchers-Uhlmann algebra \( \mathcal{B} \) induced by the state \( \omega \), and hence as a constraint on \( \omega \) itself. It implies in particular spacelike commutativity of field operators, \([\Phi(f), \Phi(h)] = 0\) whenever the supports of \( f \) and \( h \) are causally separated.

### 3.2 The Klein-Gordon field on curved spacetimes

Since the construction of free fields relies heavily on the presence of a Cauchy surface (time-zero formalism) we restrict our considerations of quantum fields on curved spacetimes to the globally hyperbolic case. The evolution of the free
scalar field of mass $m$ and with coupling $\kappa$ on a globally hyperbolic spacetime is described by the Klein-Gordon equation

$$ P\phi := (\Box_g + m^2 + \kappa R)\phi = 0, \quad \phi \in C^\infty(M). $$

(37)

Here $\Box_g$ is the Laplace operator with respect to the metric and $R$ is the scalar curvature of $M$. The operator $P$ is a differential operator of second order acting on the smooth functions on $M$. It has unique advanced and retarded fundamental solutions (see [37,18]) $E^\pm : C^\infty_0(M) \to C^\infty(M)$ satisfying

$$ PE^\pm = E^\pm P = \text{id} \quad \text{on} \quad C^\infty_0(M), $$

$$ \text{supp}(E^\pm f) \subset J^\pm(\text{supp}(f)), $$

where $J^\pm (O)$ denotes the causal future/past of a set $O$, i.e. the set of points which can be reached by future/past directed causal curves emanating from $O$. The difference $E := E^+ - E^-$ is the so called commutator function. It maps $C^\infty_0(M)$ onto the space of smooth solutions to the Klein-Gordon equation which have compactly supported restriction to all Cauchy surfaces. The map $E : C^\infty_0(M) \to C^\infty(M)$ is continuous and hence has a distributional kernel in $\mathcal{D}'(M \times M)$ which we also denote by $E$, i.e. $E(f, h) = \int_M f(Eh)$ where integration is taken with respect to the Lorentzian metric-volume form. The field algebra $\mathcal{F}$ of the Klein-Gordon field is defined to be the unital $\ast$-algebra generated by the symbols $\phi(f)$, $f \in C^\infty_0(M)$ and the relations

1. $f \mapsto \phi(f)$ is complex linear,

2. $\phi(f)* = \phi(\overline{f}),$

3. $\phi(P f) = 0,$

4. $[\phi(f_1), \phi(f_2)] = iE(f_1, f_2).$

Clearly, each state $\omega$ over $\mathcal{F}$ with the further property that the $\omega(\phi(\cdot) \cdots \phi(\cdot))$ are distributions defines a state $\tilde{\omega}$ over the Borchers-Uhlmann algebra by

$$ \tilde{\omega}(f_1 \otimes \ldots \otimes f_m) := \omega(\phi(f_1) \cdots \phi(f_m)). $$

(38)
The corresponding quantum field \( \Phi : C_0^\infty (M) \to \mathcal{L}_D^+ \) satisfies the Klein-Gordon equation

\[ \Phi (P \cdot) = 0, \quad (39) \]

and the canonical commutation relations

\[ \left[ \Phi (f_1), \Phi (f_2) \right] = i E(f_1, f_2) \mathbb{1}_D. \quad (40) \]

Since the commutator function \( E \) vanishes for spacelike separation of the arguments the field satisfies the requirement of local commutativity, i.e. \( \left[ \Phi (f), \Phi (h) \right] = 0 \) if the supports of \( f \) and \( h \) are spacelike separated. For many states, among them the quasifree states, also the stronger requirement of local commutativity at the level of the net of von Neumann algebras \( \{ A(O) \}_{O \subset M} \) in their GNS-representations described above is fulfilled. In the following we call states over the Borchers-Uhlmann algebra arising in this way states for the Klein-Gordon field.

### 4 The microlocal spectrum condition

In the investigation of the Klein-Gordon field a crucial role is played by the so called Hadamard states (see e.g. \([20, 35, 55]\)). They are thought of as the appropriate counterpart of the vacuum in Minkowski space and are characterized by the short distance behaviour of their 2-point function. The investigation of such states is partially motivated by the result of Wald (\([53]\)) that the expectation value of the energy momentum tensor \( T_{\mu \nu} \) with respect to a Hadamard state can be made sense of in a satisfactory way. This is a very important feature of Hadamard states, since it is this expectation value that appears in the Einstein equations in the semi-classical theory of gravity coupled to the Klein-Gordon field. We will not give the original definition of Hadamard states here, since such states can as well be characterized by the wavefront set of their 2-point function. This was shown by Radzikowski (\([38, 39]\)) and relies heavily on the work of Duistermaat and Hörmander (\([26, 28]\)) on Fourier integral operators. We first would like to note that the wavefront set of the commutator distribution \( E \) for the wave operator
on a globally hyperbolic spacetime with metric $g$ is given by

$$\text{WF}(E) = \{((x_1, -k_1), (x_2, k_2)) ; (x_1, k_1) \sim (x_2, k_2) \}$$

and $g^{\mu\nu}(k_1)_\mu(k_1)_\nu = g^{\mu\nu}(k_2)_\mu(k_2)_\nu = 0$,}

where $(x_1, k_1) \sim (x_2, k_2)$ means that there is a lightlike geodesic $\gamma$ connecting $x_1$ and $x_2$ such that $k_1$ is coparallel to the tangent vector of the curve at $x_1$ and $k_2$ is the parallel transport of $k_1$ from $x_1$ to $x_2$. Radzikowski’s result is that a quasifree state $\omega$ for the Klein-Gordon field is a Hadamard state if and only if the wavefront set of its 2-point function $\omega_2$ is given by

$$\text{WF}(\omega_2) = \{((x_1, -k_1), (x_2, k_2)) \in \text{WF}(E); (k_2)_0 > 0 \},$$

where $(k_2)_0 > 0$ is shorthand for “$k_2$ is future-pointing”.

The microlocal characterization of Hadamard states has meanwhile led to a rich theory. In fact, it turned out that quasifree Hadamard states allow for a construction of Wick polynomials of field operators ([12, 29]) and a perturbative construction of interacting fields on curved spacetimes ([11, 30]). It was shown in [50] that such states are locally quasiequivalent and therefore (at least locally) distinguish a single folium of states. A passive state for a free quantum field theory on a stationary spacetime is always a Hadamard state (see [19] and [44] for the statement in its full generality). The Hadamard condition can also be formulated almost without change for arbitrary free quantum fields. Radzikowski’s result is known to hold also in these cases ([45]). Adiabatic vacuum states satisfy a similar condition with the wavefront set replaced by an appropriate Sobolev-wavefront set as shown in [33].

Motivated by the observations just mentioned, a microlocal spectrum condition ($\mu SC$) that applies to general quantum fields on curved spacetimes was introduced by Brunetti, Fredenhagen and Köhler [12]; we shall now summarize its definition. We denote by $G_k$ the set of all finite graphs with vertices $\{1, \ldots, k\}$ such that for every element $G \in G_k$ all edges occur in both admissible directions. We write $s(e)$ and $r(e)$ for the source and the target of an edge respectively. Following [12] we define an immersion of a graph $G \in G_k$ into a spacetime $M$ an assignment of the vertices $\nu$ of $G$ to points $x(\nu)$ in $M$, and of edges $e$ of $G$ to piecewise smooth curves $\gamma(e)$ in $M$ with source $s(\gamma(e)) = x(s(e))$ and range
\[ r(\gamma(e)) = x(r(e)), \] together with a covariantly constant causal covector field \( k_e \) on \( \gamma \) such that

1. If \( e^{-1} \) denotes the edge with opposite direction as \( e \), then the corresponding curve \( \gamma(e^{-1}) \) is the inverse of \( \gamma(e) \).

2. For every edge \( e \) the covector field \( k_e \) is directed towards the future whenever \( s(e) < r(e) \).

3. \( k_{e^{-1}} = -k_e. \)

**Definition 4.1 (\( \mu SC \), [12]).** A state \( \omega \) over the Borchers-Uhlmann algebra \( \mathcal{B} \) is said to satisfy the microlocal spectrum condition iff its \( m \)-point functions \( \omega_m \in D'(M^m) \) satisfy

\[
WF(\omega_m) \subset \{(x_1, k_1; \ldots; x_m, k_m) \in T^*M^m \setminus 0; \ \exists G \in \mathcal{G}_m \text{ and an immersion } (x, \gamma, k) \text{ of } G \text{ in } M \}
\]

such that

\[
x_i = x(i) \quad \forall i = 1, \ldots, m \text{ and }
\]

\[
k_i = -\sum_{e, s(e) = i} k_e(x_i) = \Gamma_m
\]

We note here that quasifree Hadamard states of the Klein-Gordon field satisfy the microlocal spectrum condition. For a motivation of this definition and further properties of states satisfying the \( \mu SC \) we refer the reader to [12]. For later purposes we will need the following property of the sets \( \Gamma_m \) which is Lemma 4.2 in [12].

**Proposition 4.2.** The sets \( \Gamma_m \) are stable under addition, i.e. \( \Gamma_m + \Gamma_m \subset \Gamma_m \). Moreover, if \( (x, k) \in \Gamma_m \) then \( (x, -k) \not\in \Gamma_m \).

**5 The analytic microlocal spectrum condition and the Reeh-Schlieder property**

In the following we restrict our consideration to the case when \( M \) is a real analytic spacetime, i.e. \( M \) is real analytic as a manifold and the metric \( g \) is analytic. Passing from the smooth to the analytic category it seems reasonable to require
that the state satisfies a microlocal spectrum condition with the wavefront set \( \text{WF} \) replaced by the analytic wavefront set \( \text{WF}_A \). (See [29] for a related concept.)

**Definition 5.1 (\(a\mu\text{SC}\)).** A state \( \omega \) over the Borchers-Uhlmann algebra is said to satisfy the Analytic Microlocal Spectrum Condition (\(a\mu\text{SC}\)) iff its \( m \)-point functions satisfy

\[
\text{WF}_A(\omega_m) \subset \Gamma_m,
\]

where the notations of Def. 4.1 are used.

For Wightman fields in Minkowski spacetime the spectrum condition is equivalent to the requirement that the \( m \)-point functions are boundary values of functions which are analytic in the tube

\[
T_m := \{(z_1, \ldots, z_m); \Im(z_{j+1} - z_j) \in V_+; j = 1, \ldots, m - 1\}, \tag{43}
\]

where \( V_+ \) is the forward lightcone.

**Theorem 5.2.** Suppose that \( M \) is the \( n \)-dimensional Minkowski spacetime \((n \geq 2)\) and let \( \omega \) be a state over the Borchers-Uhlmann algebra such that its \( m \)-point functions are boundary values in the sense of distributions of functions that are analytic in \( T_m \cap Z \), where \( Z \) is a complex neighbourhood of \((\mathbb{R}^n)^m \subset (\mathbb{C}^n)^m\). Then \( \omega \) satisfies \( a\mu\text{SC} \).

**Proof.** Clearly, \( T_m \) is of the form \( T_m = (\mathbb{R}^n)^m + i\mathcal{C} \), where the cone \( \mathcal{C} \) is defined by \( \mathcal{C} = \{(k_1, \ldots, k_m); k_{j+1} - k_j \in V_+\} \). The dual cone \( \mathcal{C}^\circ \) can easily be calculated and the result is

\[
\mathcal{C}^\circ = \left\{ (k_1, \ldots, k_m); k_m, k_{m-1} + k_m, \ldots, \sum_{j=2}^m k_j \in V_+, \sum_{j=1}^m k_j = 0 \right\}. \tag{44}
\]

Hence, by Corollary 2.9, the set \( \text{WF}_A(\omega_m) \) is contained in \((\mathbb{R}^n)^m \times \mathcal{C}^\circ\). The set

\[
\left\{ (x_1, k_1; \ldots; x_m, k_m); k_m, k_{m-1} + k_m, \ldots, \sum_{j=2}^m k_j \in V_+, \sum_{j=1}^m k_j = 0 \right\}
\]

is contained in \( \Gamma_m \) (see e.g. the proof of Thm. 4.6 in [12]) which concludes the proof. \( \square \)
This theorem applies to the vacuum state of Wightman fields in Minkowski spacetime and, provided that the invariant domain of all field operators includes e.g. the $C^\infty$-vectors for the energy, it applies also to vector states which are analytic in the energy (in vacuum representation, see e.g. Chp. 12 in [1]); moreover, it applies also to states which satisfy the relativistic KMS condition proposed by Buchholz and Bros ([6, 7]). Quasifree states for the Klein-Gordon field on the de Sitter spacetime that satisfy the weak spectral condition ([14, 8, 9]) can also be shown to satisfy $a\mu SC$.

Let as before $(\pi, \Omega, D, \mathcal{H})$ be the GNS-representation of the state $\omega$ and denote by $\Phi$ the corresponding quantum field. We will show now that a state that satisfies $a\mu SC$ has the Reeh-Schlieder property, i.e. the set 
$$\{ \Phi(f_1) \cdots \Phi(f_n) \Omega; \text{supp}(f_i) \subset O, m \in \mathbb{N} \}$$
is total in $\mathcal{H}$ for each non-void open set $O \subset M$.

The main technical tool for proving this is a microlocal version of the edge of the wedge theorem (Thm. 8.5.6’ in [27]).

**Proposition 5.3.** Let $M$ be a real analytic connected manifold and $u \in \mathcal{D}'(M)$ a distribution with the property that $WF_A(u) \cap -WF_A(u) = \emptyset$. Then the following conclusion holds for each non-void open subset $O \subset M$:

$$u|_O = 0 \implies u = 0.$$

**Proof.** For a closed subset $X \subset M$ the exterior normal set $N_e(X) \subset T^*M \setminus 0$ is defined to be the set of all $(x, k)$ such that $x \in X$ and there is a real valued function $f \in C^2(M)$ with $df(x) = k \neq 0$ and $f(y) \leq f(x)$ for all $y \in X$. The normal set $N(X)$ is the union $N_e(X) \cup -N_e(X)$. Thm. 8.5.6’ in [27] states that $N(\text{supp}(u)) \subset WF_A(u)$ and since $N(\text{supp}(u)) = -N(\text{supp}(u))$ the assumption implies that $N(\text{supp}(u)) = \emptyset$. Consequently, $N_e(\text{supp}(u)) = \emptyset$. Prop. 8.5.8 in [27] states that the projection of $N_e(X)$ in $M$ is dense in $\partial X$. Therefore, $\partial(\text{supp}(u)) = \emptyset$. Since $M$ is connected this implies that either $\text{supp}(u) = \emptyset$ or $\text{supp}(u) = M$. The latter is excluded by $u|_O = 0$.

Our main result is stated in the following theorem.

**Theorem 5.4.** Let $\omega$ be a state over the Borchers-Uhlmann algebra on a real analytic spacetime and suppose furthermore that $\omega$ satisfies $a\mu SC$. Denote by
its GNS-representation and by $\Phi$ the associated quantum field. Then the set
\[
\{\Omega\} \cup \{\Phi(f_1) \cdots \Phi(f_m)\Omega; \ \text{supp}(f_i) \subset O, m \in \mathbb{N}\}
\]
is total in $\mathcal{H}$ for each non-void open set $O \subset M$.

**Proof.** We define a Hilbert space valued distribution $\psi_m \in \mathcal{D}'(M^m, \mathcal{H})$ by
\[
\psi_m(f_1, \ldots, f_m) := \Phi(f_1) \cdots \Phi(f_m)\Omega. \tag{45}
\]
Note that due to Prop. 2.6 a point $(x, k) \in T^*M^m \setminus 0$ is in $WF_A(\psi_m)$ if and only if $(x, -k; x, k) \in WF_A(w^m)$, where the distribution $w_2^m \in \mathcal{D}'(M^{2m})$ is defined by
\[
w_2^m(f_1, \ldots, f_m, g_1, \ldots, g_m) := \omega_2^m(f_m, \ldots, f_1, g_1, \ldots, g_m). \tag{46}
\]
Prop. 4.2 implies that $WF_A(w^m) \cap -WF_A(w^m) = \emptyset$ and therefore $WF_A(\psi_m) \cap -WF_A(\psi_m) = \emptyset$.

Suppose now that $\phi \in \mathcal{H}$ is orthogonal to the set
\[
\{\Omega\} \cup \{\Phi(f_1) \cdots \Phi(f_m)\Omega; \ \text{supp}(f_i) \subset O, m \in \mathbb{N}\}. \tag{47}
\]
Then the distributions $v_m(\cdot) = \langle \phi, \psi_m(\cdot) \rangle \in \mathcal{D}'(M^m)$ vanish on $O^m$ and satisfy $WF_A(v_m) \cap -WF_A(v_m) = \emptyset$. By Prop. 5.3 we conclude that $v_m = 0$ for all $m$. Therefore, $\phi$ is even orthogonal to the set
\[
\{\Omega\} \cup \{\Phi(f_1) \cdots \Phi(f_m)\Omega; \ f_i \in C_0^\infty(M), m \in \mathbb{N}\}, \tag{48}
\]
which is total in $\mathcal{H}$. We conclude that $\phi = 0$ which proves the theorem. $\square$

An immediate corollary is the Reeh-Schlieder property of the associated net of local algebras.

**Corollary 5.5.** Let the assumptions of Theorem 5.4 be fulfilled and denote by $\{A(O)\}_{O \subset M}$ the associated local net of von Neumann algebras. Then $A(O)\Omega$ is dense in $\mathcal{H}$ for each non-void open set $O$.
Remark 5.6. (a) Clearly, the conclusion of Theorem 5.4 also holds if we impose the weaker condition $\text{WF}_\mathcal{A}(\omega_{2m}) \cap -\text{WF}_\mathcal{A}(\omega_{2m}) = \emptyset$ on the state instead of the analytic microlocal spectrum condition. Our result is therefore insensitive to the precise form of analytic microlocal spectrum condition as long as an analogue of Prop. 4.2 holds.

(b) The same method works for fields with values in an analytic vector bundle. Note that local commutativity is not an assumption of Thm. 5.4 and therefore it applies to fermionic fields as well.

(c) The existence of states fulfilling $\alpha_{\mu SC}$ on generic, real analytic, globally hyperbolic spacetimes remains an open problem, even for free field theories. See our comments at the end of this article on this point.

6 Quasifree states and the Klein-Gordon field

As indicated in the introduction the microlocal spectrum condition can be simplified for quasifree states of the Klein-Gordon field using microlocal analysis of Hilbert space valued distributions. This will allow us to give a rather simple proof of the fact that quasifree ground- and KMS-states for the Klein-Gordon field on (analytic) stationary spacetimes are (analytic) Hadamard states.

Proposition 6.1. Let $M$ be a globally hyperbolic spacetime and let $\omega$ be a quasifree state for the Klein-Gordon field on $M$. Let $\Phi$ and $\Omega$ be as in the previous section and denote by $\psi$ the Hilbert space valued distribution $\Phi(\cdot)\Omega$. Then the following statements are equivalent.

1) $\omega$ satisfies $\mu SC$.

2) $\text{WF}(\psi) \subset \overline{V}_+$.

3) $\text{WF}(\psi) = N_+$.

4) $\omega$ is a Hadamard state.

Here $V_+$ denotes the set of future directed causal covectors $(x, k)$ and $N_+$ is the set of future directed non-zero null-covectors.
Proof. We first show that 1) \(\Rightarrow\) 2). Suppose that \((x, k) \in \text{WF}(\psi)\). Then by Prop. 2.2 we get \((x, -k; x, k) \in \text{WF}(\omega_2)\). By \(\mu\text{SC}\) the covector \((x, k)\) must be in \(\overline{V}_+\). 2) \(\Rightarrow\) 3) is a simple consequence of the fact that \(\psi\) solves the Klein-Gordon equation and Prop. 2.3. The implication 4) \(\Rightarrow\) 1) is Prop. 4.3 in [12] and it remains to show 3) \(\Rightarrow\) 4). By Prop. 2.2 we conclude that if \((x_1, k_1; x_2, k_2) \in \text{WF}(\omega_2)\) then \(k_2 \in \overline{V}_+\) and \(k_1 \in V_-\). Therefore, \(\text{WF}(\omega_2) \cap \text{WF}(\omega_2) = \emptyset\) with \(\omega_2(f_1, f_2) := \omega_2(f_2, f_1)\). Note that \(\omega_2 - \omega_2\) is proportional to the commutator function \(E\) and consequently \(\text{WF}(\omega_2) \cup \text{WF}(\omega_2) = \text{WF}(E)\). This implies Eq. (42) and thus concludes the proof.

Remark 6.2. It was shown in [11] that a vector \(\Omega\) inducing a quasifree Hadamard state is contained in the microlocal domain of smoothness (introduced in the same article), which implies the conclusion 4) \(\Rightarrow\) 3). Prop. 6.1 shows that the microlocal domain of smoothness coincides with the set of vectors inducing quasifree Hadamard states.

A completely analogous statement holds in the analytic category.

Proposition 6.3. Let \(M\) be an globally hyperbolic analytic spacetime and let \(\omega\) be a quasifree state for the Klein-Gordon field on \(M\). With the same notation as in Thm. 6.1 the following statements are equivalent.

1) \(\omega\) satisfies \(\alpha\mu\text{SC}\).

2) \(\text{WF}_A(\psi) \subset \overline{V}_+\).

3) \(\text{WF}_A(\psi) = N_+\).

4) \(\omega\) is an analytic Hadamard state, meaning that (42) is satisfied with \(\text{WF}\) replaced by \(\text{WF}_A\).

Proof. Taking into account Prop. 2.6 and Prop. 2.7 the proof is identical to that of Prop. 6.1 with \(\text{WF}\) replaced by \(\text{WF}_A\).
reasonable replacements for the vacuum. For example the Schwarzschild spacetime is stationary and the Hartle-Hawking state for the Klein-Gordon field is a KMS state with respect to the group of time translations. Other examples are the Rindler wedge and wedge-like regions in the de Sitter space. We investigate in the following ground and KMS states for the Klein-Gordon field on a stationary spacetime. Let us first fix some notation. The push-forward \( h_t \) defined by \((h_t f)(x) := f(h_t x)\) acts on the space \( C_0^\infty(M) \) and this action lifts uniquely to an action \( \alpha_t \) on the Borchers-Uhlmann algebra \( B \) by \(*\)-automorphisms. A state \( \omega \) over \( B \) is called ground state if the function \( t \mapsto \omega(A\alpha_t B) \) is bounded for all \( A,B \in B \) and
\[
\int_{-\infty}^{+\infty} \hat{f}(t)\omega(A\alpha_t(B))dt = 0, \quad (49)
\]
holds for all \( f \in C_0^\infty((\infty, 0)) \). A state \( \omega \) is called KMS state at inverse temperature \( \beta > 0 \) if the function \( t \mapsto \omega(A\alpha_t B) \) is bounded for all \( A,B \in B \) and
\[
\int_{-\infty}^{+\infty} \hat{f}(t)\omega(A\alpha_t(B))dt = \int_{-\infty}^{+\infty} \hat{f}(t+i\beta)\omega(\alpha_t(B)A)dt, \quad (50)
\]
for all \( f \in C_0^\infty(\mathbb{R}) \). Note that ground and KMS states are necessarily invariant, i.e. \( \omega(\alpha_t(\cdot)) = \omega(\cdot) \).

**Theorem 6.4.** Let \( M \) be a globally hyperbolic stationary spacetime and suppose that \( \omega \) is a quasifree KMS- or ground-state for the Klein-Gordon field. Then \( \omega \) satisfies the microlocal spectrum condition. If moreover \( M \) is real analytic (as a spacetime) and the flow \( \mathbb{R} \times M \to M \) induced by \( h_t \) is analytic, then \( \omega \) satisfies the analytic microlocal spectrum condition.

**Proof.** Let \((\pi, \Omega, \mathcal{H}, D)\) be the GNS representation of \( \omega \). Let \( \Phi(\cdot) \) be the associated field and define \( \psi(\cdot) := \Phi(\cdot)\Omega \). Since \( \omega \) is invariant there exists a strongly continuous one-parameter group \( U(t) = e^{itH} \) on \( \mathcal{H} \), such that \( U(t)\psi(\cdot) = \psi(h_t \cdot) \) and \( U(t)\Omega = \Omega \). If \( \omega \) is a \( \beta \)-KMS state it follows from the KMS condition that the vectors \( \psi(f) \) are in the domain of \( e^{-\beta/2H} = U(i\beta/2) \) for all \( f \in C_0^\infty(M) \). If \( \omega \) is a ground state this is even true for all \( \beta > 0 \). Therefore, we may define the distribution \( G \in \mathcal{D}'((0, \beta/2) \times M, \mathcal{H}) \) by \( G(t, x) = U(it)\psi(x) \). We use a local
coordinate system \((x_0, \ldots, x_{n-1})\) such that the vector field \(\partial_{x_0}\) generates locally the flow \(h_t\). Then \(G\) satisfies the system of equations

\[
(\partial_t^2 + \partial_{x_0}^2)G = 0 \quad \text{(51)}
\]

\[
(id \otimes P)G = 0. \quad \text{(52)}
\]

This system is elliptic and therefore \(G\) is indeed a smooth function satisfying the conditions of Thm. \(\text{2.8}\). It follows that \(WF_A(\psi)\) is a subset of the set \(\{(x, k) \in T^*M; k(\partial_{x_0}(x)) > 0\}\) in this coordinate system. Since \(\psi\) solves the Klein-Gordon equation \(WF(\psi)\) is confined to the forward light cone. If \(M\) is real analytic and the flow \(\mathbb{R} \times M \to M\) induced by \(h_t\) is analytic we can choose the local coordinate system to be analytic and therefore \(WF_A(\psi) \subset \mathcal{V}_+\).

The part of Theorem \(6.4\) stating that ground- and KMS-states on smooth stationary spacetimes are Hadamard states was shown before in \([44]\), cf. also \([19]\). The method we used in our proof is however rather different from the methods employed there. The advantage of our approach is that it applies to the analytic case without any changes.

As a simple consequence of Theorem \(6.4\) the Hartle-Hawking state on the Schwarzschild spacetime is an analytic Hadamard state. On the de Sitter spacetime the so called Euclidian vacuum state (also called Bunch-Davies vacuum) is known to be a KMS state when restricted to certain stationary wedge-like regions (see e.g. \([21, 10]\) and references therein). As a consequence this state satisfies the \(a\mu SC\). One can also conclude this more directly from the fact that the 2-point function of this state is the boundary value of a function holomorphic in a certain complex tuboid (\([10]\)).

Whereas the existence of Hadamard states for the Klein-Gordon field on an arbitrary globally hyperbolic spacetime is well established a general construction of analytic Hadamard states seems to be a rather difficult task. In the literature there exist two methods of construction of Hadamard states that apply to generic spacetime manifolds. The method used by Fulling et al. (\([19]\)) takes advantage of the fact that any globally hyperbolic spacetime can be deformed to a spacetime that is static in the past of a given Cauchy surface and coincides with the original spacetime in its future. Such a \(C^\infty\)-deformation, however, destroys the analyticity of the metric and thus this method can not be exploited for the construction of analytic Hadamard states. Another construction of Hadamard states is due to Junker
His method relies heavily on the calculus of (smooth) pseudodifferential operators and we are so far not aware of a calculus that would allow it to carry over that method to the analytic case.

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References


