Building κ-complete filters for supercompact κ

Andrew Brooke-Taylor

University of Bristol

Generalised Baire Space workshop
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Two aspects of generalised Baire space:

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- generalised descriptive set theory

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First order of business:

Which proofs lift from the classical ω case to the generalised setting?

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I'll be talking about a case where we assume κ is supercompact.

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Question

How can we preserve κ-completeness at limit stages of the construction?

A simple example

$$\mathfrak{u}(\kappa) = \min\{|\mathcal{F}| : \mathcal{F} \text{ is a filter base for a uniform ultrafilter on } \kappa\}$$

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Mathias forcing $\mathbb{M}_{\mathcal{F}}$

Conditions of the form (s, A) with $s \in 2^{<\omega}$ and $A \in \mathcal{F}$;
$(s, A) \leq (t, B)$ iff $A \subseteq B$, s end-extends t, and $t^{-1}(1) \setminus s^{-1}(1) \subset B$.

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Making $\mathfrak{u}(\aleph_0) < 2^{\aleph_0}$

Start with a model of $2^{\aleph_0} = \aleph_2$.
Do an ω_1-length finite support iteration of Mathias forcing with ultrafilters, where at each stage of the iteration the ultrafilter used contains the Mathias generic subsets from the previous stages.

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Do an ω_1-length finite support iteration of Mathias forcing with ultrafilters, where at each stage of the iteration the ultrafilter used contains the Mathias generic subsets from the previous stages. This is c.c.c., so $2^{\aleph}_0$ remains large, but the set of Mathias reals added forms an ultrafilter filter base of size $\aleph_1$.

Generalising to κ

$\mathbb{M}^{\kappa}_{\mathcal{U}}$ is κ^+-c.c., but to preserve cardinals we also want it to be κ-closed, which is equivalent to $\mathcal{U}$ being κ-complete.

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Garti & Shelah:

> *For the limit stages of the iteration, one has to employ the arguments in [Džamonja & Shelah].*

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Central idea

As usual with a large cardinal κ, we this final ultrafilter $\mathcal{U}$ should be given by

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In this case, the j in quesiton will be the lift of a λ-supercompactness embedding for κ from the ground model. We can carefully control the behaviour of this lift, ensuring that reasonable sets that we want to be in the ultrafilter are in it, and κ-completeness comes for free.

See the whiteboard.

The main claim (1.18 of Džamonja & Shelah's paper)

In V^{S_κ}, there exist sequences

$$\begin{aligned} \bar{\alpha} &= \langle \alpha_i : i < \Upsilon^+ \rangle, \\ \bar{p}^* &= \langle p_i^* : i < \Upsilon^+ \rangle, \text{ and} \\ \bar{q}^* &= \langle q_i^* = ({}^1q_i, {}^2q_i) : i < \Upsilon^+ \rangle, \end{aligned}$$

such that the following hold.

- $\bar{\alpha}$ is a strictly increasing continuous sequence of ordinals less than Υ^+.
- Each p_i^* is purely full in P_{α_i+1}.
- $\bar{p}^*$ is a decreasing sequence of conditions in P_{Υ^+}.
- $\bar{q}^* \in M^{S_\kappa}$, and in M^{S_κ} we have for each $i < \Upsilon^+$ that

$$(p_i^*, {}^1q_i) \in P_{\Upsilon^+} * \dot{S}^*$$

  and

$$(p_i^*, {}^1q_i, {}^2q_i) \in P_{\Upsilon^+} * \dot{S}^* * \dot{P}'_{j(\alpha_i+1)}.$$

- In M^{S_κ}, $\langle (p^*_i, {}^1q_i, {}^2q_i) : i < \Upsilon^+ \rangle$ is a decreasing sequence of conditions in $P_{\Upsilon^+} * \dot{S}^* * \dot{P}'_{\sup_{i<\Upsilon^+}(j(\alpha_i+1))}$.
- In M^{S_κ}, $(p^*_{i+1}, {}^1q_{i+1})$ forces that ${}^2q_{i+1}$ is a common extension of

$$\{j(r) : r \in G_{P_{\alpha_i+1}}\}$$

- If $\dot{B}$ is an S_κ-name for a P_{α_i+1}-name for a subset of κ then there is an $S_\kappa * \dot{P}_{\Upsilon^+}$-name $\tau_{\dot{B}}$ for an element of $\{0,1\}$ such that:
  1. in V, $(\mathbb{1}_{S_\kappa}, \dot{p}^*_{i+1})$ forces $\tau_{\dot{B}}$ to be a $P_{\alpha_{i+1}+1} \downarrow p^*_{i+1}$-name, and
  2. $M \vDash [(\mathbb{1}_{S_\kappa}, \dot{p}^*_{i+1}, q^*_{i+1}) \Vdash \check{\kappa} \in j(\dot{B}) \leftrightarrow \tau_{\dot{B}} = \check{1}]$.
- If $\mathrm{cf}(i) > \kappa$, then in $V^{S_\kappa * \dot{P}_{\alpha_i}}$ we have that $p^*_i(\alpha_i) = \left\{ \dot{B}[G_{P_{\alpha_i}}] : \begin{array}{l} \dot{B} \text{ is a } P_{\alpha_i} \downarrow (p^*_i \restriction \alpha_i)\text{-name for a subset of} \\ \kappa \text{ and } \tau_{\dot{B}}[G_{P_{\alpha_i}}] = 1 \end{array} \right\}$.
  In particular, this is a normal ultrafilter on κ.

Singular cardinals

Džamonja and Shelah originally used the technique to prove the follow result:

Theorem

Suppose there is a supercompact cardinal κ. Then there is a forcing extension in which there is a singular strong limit cardinal μ of cofinality ω with $2^{\mu^+} > \mu^+$, and μ^{++}-many graphs on μ^+ that taken jointly embed every graph on μ^+.

Proof.

Use the technique we've described to get a universal family of graphs at κ^+. Then apply Prikry forcing at κ. $\square$

Downsides

Key constraint:

The ultrafilters used are normal. This is not always desirable.

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E.g.

κ-Mathias forcing with a normal ultrafilter always adds a dominating κ-real. For $\mathfrak{b}(\kappa) < \mathfrak{a}(\kappa)$, we want an iterated forcing that blows up $\mathfrak{a}(\kappa)$ but keeps $\mathfrak{b}(\kappa)$ small, and for this we do *not* want to add dominating κ-reals.

Definition

An ultrafilter is Canjar if Mathias forcing with it does not add a dominating real.

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- More generally, look at κ-complete ultrafilters on κ satisfying any Boolean combination of the κ-analogues of rapidity, being a p-point, and being a q-point.

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- Exhibit a (κ-complete) Canjar ultrafilter on a measurable cardinal κ.
- Characterise Canjar ultrafilters on measurable cardinals.
- More generally, look at κ-complete ultrafilters on κ satisfying any Boolean combination of the κ-analogues of rapidity, being a p-point, and being a q-point.
- Show $\mathrm{Con}(\mathfrak{b}(\kappa) < \mathfrak{a}(\kappa))$ for $\kappa > \omega$, possibly from large cardinals.