Locating Spectra in the Bousfield Lattice

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Spectra and Bousfield equivalence

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For two spectra E and Z, say that Z is *E-acyclic* if $E \wedge Z \simeq *$.

The Bousfield class $\langle E \rangle$ of a spectrum E is the class of E-acyclic spectra.

Two spectra E and F are Bousfield equivalent if $\langle E \rangle = \langle F \rangle$.

Partial order and upper bounds

We order Bousfield classes by reverse inclusion:

$$\langle E\rangle \leq \langle F\rangle \quad \text{iff} \quad \langle E\rangle \supseteq \langle F\rangle \quad \text{iff} \quad F \wedge Z \simeq * \Rightarrow E \wedge Z \simeq *.$$

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Since $\wedge$ distributes over $\bigvee$, $\bigvee$ is well-defined on Bousfield classes by

$$\bigvee_{\alpha\in A} \langle E_\alpha\rangle = \langle \bigvee_{\alpha\in A} E_\alpha\rangle,$$

and $\bigvee_{\alpha\in A}\langle E_\alpha\rangle$ is the least upper bound of the $\langle E_\alpha\rangle$ (lucky notation!).

Lower bounds

$\wedge$ is also well-defined for Bousfield classes by $\langle E \rangle \wedge \langle F \rangle = \langle E \wedge F \rangle$, and $\langle E \rangle \wedge \langle F \rangle$ is a lower bound for $\langle E \rangle$ and $\langle F \rangle$. But it is not in general a greatest lower bound.

Example

I, the Brown-Comenetz dual of the sphere, satisfies $I \wedge I = *$, so $\langle I \rangle \wedge \langle I \rangle = \langle * \rangle$, but of course the greatest lower bound of $\langle I \rangle$ and $\langle I \rangle$ is $\langle I \rangle$.

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Theorem (Ohkawa, 1989)

There are only set many Bousfield classes.

So we can define the greatest lower bound $\langle E \rangle \curlywedge \langle F \rangle$ by

$$\langle E \rangle \curlywedge \langle F \rangle = \bigvee \{ \langle G \rangle \mid \langle G \rangle \leq \langle E \rangle \text{ and } \langle G \rangle \leq \langle F \rangle \}.$$

This makes the **B** of Bousfield classes a lattice, the *Bousfield lattice*.

Suborders: **DL**

Removing the problem with $\wedge$

Bousfield defined

$$\mathbf{DL} = \{\langle E\rangle \mid \langle E\rangle \wedge \langle E\rangle = \langle E\rangle\}.$$

Restricted to **DL**, $\wedge$ *does* give the greatest lower bound!

Lemma

$\wedge$ *respects* $\leq$.

Proof.

By associativity and commutativity of $\wedge$: if $\langle E_1 \rangle \leq \langle E_2 \rangle$ and $\langle F_1 \rangle \leq \langle F_2 \rangle$, and $* \simeq E_2 \wedge F_2 \wedge Z$, then

$$\begin{aligned} * &\simeq E_1 \wedge F_2 \wedge Z \\ &\simeq F_2 \wedge E_1 \wedge Z, \quad \text{so} \\ * &\simeq F_1 \wedge E_1 \wedge Z \\ &\simeq E_1 \wedge F_1 \wedge Z. \end{aligned}$$

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So in **DL**, if $\langle E \rangle \leq \langle F_1 \rangle$ and $\langle E \rangle \leq \langle F_2 \rangle$, then $\langle E \rangle = \langle E \rangle \wedge \langle E \rangle \leq \langle F_1 \rangle \wedge \langle F_2 \rangle$.

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Restricted to **DL**, $\wedge$ *does* give the greatest lower bound. $\vee$ still gives least upper bound, so $(\mathbf{DL}, \vee, \wedge, \langle S\rangle, \langle *\rangle)$ is a lattice; moreover, it is a distributive lattice.

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This is a reasonable suborder to consider from an algebraic topology perspective: for example, every ring spectrum lies in **DL**.

Suborders: BA

Since **B** is a complete lattice, there is is a natural pseudocomplementation operation defined using $\wedge$:

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$$\mathbf{BA} = \{\langle X\rangle \in \mathbf{B} \mid a\langle X\rangle \vee \langle X\rangle = \langle S\rangle\}.$$

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$(\mathbf{BA}, \vee, \wedge, \langle S \rangle, \langle * \rangle, a)$ forms a Boolean algebra.

$\mathbf{BA} \subset \mathbf{DL}$, but they are not equal: for example, the Eilenberg-Maclane space of the integers $H\mathbb{Z}$ has Bousfield class $\langle H\mathbb{Z} \rangle$ in $\mathbf{DL} \setminus \mathbf{BA}$. On the other hand, many reasonable spectra have classes in **BA**, such as the Moore specturm of any abelian group, finite CW spectra, and KO.

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BA is *not* a complete Boolean algebra.

$\mathbf{B} \to \mathbf{DL}$

There is a natural retraction r from **B** onto **DL** (right adjoint to the inclusion):

$$r\langle X\rangle = \bigvee\{\langle Y\rangle \in \mathbf{DL} \mid \langle Y\rangle \leq \langle X\rangle\}.$$

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However, A^2 is *not* the identity map. So Hovey and Palmieri defined

$$\mathbf{cBA} = \{\langle X\rangle \in \mathbf{DL} \mid A^2\langle X\rangle = \langle X\rangle\}.$$

cBA

In **cBA**, $\bigvee$ is no longer the least upper bound, but we can define

$$\bigcurlyvee \mathcal{X} = A^2 \bigvee \mathcal{X}.$$

With this definition, $(\mathbf{cBA}, \curlyvee, \wedge, \langle S \rangle, \langle * \rangle, A)$ is a *complete* Boolean algbera.

$\mathbf{BA} \subset \mathbf{cBA} \subset \mathbf{DL}$, with both inclusions known to be strict.

Problem:

Exhibit a concrete element of $\mathbf{DL} \smallsetminus \mathbf{cBA}$.

Let $K(n)$ denote the nth Morava K-theory spectrum for $n \in \mathbb{N}$, and let $K(\infty)$ denote the mod p Eilenberg-Maclane spectrum $H\mathbb{F}_p$. For $S \subseteq \mathbb{N} \cup \{\infty\}$, let $K(S)$ denote $\bigvee_{n \in S} K(n)$.

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Theorem (B-T, Löwe, & Richter)

For any coinfinite $S \subset \mathbb{N} \cup \{\infty\}$ *with* $\infty \in S$, $K(S) \in \mathbf{DL} \setminus \mathbf{cBA}$.

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Corollary

There are at least $2^{\aleph_0}$ elements of $\mathbf{DL} \setminus \mathbf{cBA}$.

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For every prime p and $n \in \mathbb{N}$, $\langle H\mathbb{Z}_{(p)} \rangle$ and $\langle BP(n) \rangle$ are in $\mathbf{DL} \setminus \mathbf{cBA}$.

$(\langle H\mathbb{Z}_{(p)} \rangle = \langle K(\{0, \infty\}) \rangle$ and $\langle BP(n) \rangle = \langle K(\{0, \dots, n, \infty\}) \rangle)$

The proof rests on the following Dichotomy Lemma

Lemma

Let X and Y be spectra, and let E be a spectrum such that $\langle E\rangle \neq \langle *\rangle$. Suppose that the following conditions hold:

1. $\langle X\rangle \in \mathbf{DL}$,
2. $\langle Y\rangle \in \mathbf{DL}$,
3. $\langle X\rangle \wedge \langle Y\rangle = \langle *\rangle$,
4. $\langle E\rangle \leq \langle X\rangle$, *and*
5. $\langle E\rangle \leq \langle Y\rangle$.

Then $\langle X\rangle$ or $\langle Y\rangle$ is not in **cBA**.

Proof of the Lemma

Suppose for contradiction that $A^2\langle X\rangle = \langle X\rangle$ and $A^2\langle Y\rangle = \langle Y\rangle$.

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Since A converts joins to meets, we have

$$\langle S\rangle = A\langle *\rangle = A(\langle X\rangle \wedge \langle Y\rangle) = A(A^2\langle X\rangle \wedge A^2\langle Y\rangle) = A^2(A\langle X\rangle \vee A\langle Y\rangle)$$

The rightmost expression is $A\langle X\rangle \curlyvee A\langle Y\rangle$ by definition of $\curlyvee$.

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A is order reversing, so

$$\langle S\rangle = A\langle X\rangle \curlyvee A\langle Y\rangle \leq A\langle E\rangle \curlyvee A\langle E\rangle = A\langle E\rangle.$$

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But now if $\langle S\rangle = A\langle E\rangle$, then $\langle E\rangle = \langle S\rangle \wedge \langle E\rangle = A\langle E\rangle \wedge \langle E\rangle = \langle * \rangle$, contradicting the assumption on $\langle E\rangle$. □