Elementary embeddings and category theory

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Background

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Such strong *large cardinal axioms* in set theory are usually expressed in terms of elementary embeddings (and indeed Vopěnka's Principle can be expressed this way too). So it is important to be able to understand and use elementary embeddings in a category theory context.

Accessible categories

For λ a regular cardinal, a poset is λ-directed if every subset of cardinality less than λ has an upper bound.

Eg: the usual notion of directed is $\aleph_0$-directed in this notation.

A λ-directed diagram is one whose index category is a λ-directed poset.

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Eg

In the category of groups and group homomorphisms, $\aleph_0$ presentability agrees with the usual notion of finite presentability.

Accessible categories

A category $\mathcal{K}$ is λ-accessible if

- $\mathcal{K}$ has λ-directed colimits, and
- there is a *set* $\mathcal{A}$ of λ-presentable objects such that every object is a λ-directed colimit of objects from $\mathcal{A}$.

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See the book *Locally presentable and accessible categories* by Adámek and Rosický for (many) more details.

A theorem

Theorem (Rosický, Trnková & Adámek, 1990)

Assuming Vopěnka's Principle, for each full embedding $F : \mathcal{A} \to \mathcal{K}$ *with* $\mathcal{K}$ *an accessible category, there is a regular cardinal* λ *such that* F *preserves* λ*-directed colimits.*

Vopěnka's Principle (VP)

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For any proper class $\mathcal{C}$ of graphs, there are distinct graphs G_0 and G_1 in $\mathcal{C}$ such that there exists a graph homomorphism from G_0 to G_1.

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Bagaria, Casacuberta, Mathias and Rosický: VP for classes defined by formulae of a given quantifier complexity is strictly weaker than full VP, so many specific applications of VP can be obtained from weaker assumptions.

Stratifying the theorem

Theorem (Bagaria & B-T)

Suppose that $\mathcal{K}$ is a full subcategory of $\mathbf{Str}\,\mathbf{\Sigma}$ for some signature $\mathbf{\Sigma}$. Let $F : \mathcal{A} \to \mathcal{K}$ be any Σ_n-definable full embedding with Σ_n-definable domain category $\mathcal{A}$, for some $n > 0$. If there exists a $C^{(n)}$-extendible cardinal greater than

- *the rank of $\mathbf{\Sigma}$,*
- *the arity of each function or relation symbol in $\mathbf{\Sigma}$, and*
- *the ranks of the parameters used in some Σ_n definitions of F and $\mathcal{A}$ and in some definition of $\mathcal{K}$,*

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Vopěnka's Principle, elementary embeddings formulation

For any signature $\mathbf{\Sigma}$, and any proper class $\mathcal{C}$ of $\mathbf{\Sigma}$-structures, there are distinct structures A and B in $\mathcal{C}$ such that there exists an elementary embedding from A to B.

Most large cardinal axioms from set theory can be expressed in terms of elementary embeddings, but often with the universe of sets itself as the domain.

Using elementary embeddings of the universe in category theory

Basic idea

If all the categories, functors, etc that you care about are definable, and j is an elementary embedding of the whole universe to itself fixing the parameters in the definitions, then j commutes with everything.

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Kunen proved that the only such elementary embedding is the identity map.

But non-identity elementary embeddings j with domain and codomain that *approximate* the universe seem to be consistent, with better approximations giving stronger axioms.

Most such axioms people have considered are actually weaker than Vopěnka's Principle.

The set-theoretic framework

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Before you ask:

Open Problem

Recast these results in topos theory.

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The von Neumann hierarchy

$$\begin{aligned} V_0 &= \emptyset \\ V_{\alpha+1} &= \mathcal{P}(V_\alpha) \\ V_\lambda &= \bigcup_{\alpha<\lambda} V_\alpha \quad \text{for limit ordinals } \lambda \\ V &= \bigcup_{\alpha\in\mathrm{Ord}} V_\alpha, \quad \text{the full set-theoretic universe.} \end{aligned}$$

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Categories and functors are taken to be classes.

Formula complexity

Levy hierarchy

In the language of set theory, $\boldsymbol{\Sigma} = \{\in\}$, a formula is

- Σ_0 and Π_0 if all of its quantifiers are bounded (i.e., of the form $\forall x \in y$ or $\exists x \in y$).
- Σ_{n+1} if it is of the form $\exists x(\varphi(x))$ for some Π_n formula φ.
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For a structure $\mathcal{M}$, we write $\mathcal{M} \vDash \varphi(m)$ for "$\mathcal{M}$ satisfies formula φ with parameter m".

Example

$$\langle \mathbb{Z}, + \rangle \vDash \forall x \exists y (x + y = 3)$$

$C^{(n)}$ cardinals

We denote by $C^{(n)}$ the class of cardinals κ such that $V_\kappa \prec_{\Sigma_n} V$, that is, for every Σ_n formula $\varphi(x)$ and set $x_0 \in V_\kappa$,

$$\langle V_\kappa, \in\rangle \models \varphi(x_0) \quad \text{if and only if} \quad \langle V, \in\rangle \models \varphi(x_0).$$

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For every n, $C^{(n)}$ is unbounded: given any cardinal γ, one can find a cardinal κ greater than γ in $C^{(n)}$.

Proof sketch

By induction on n. Iteratively take larger and larger κ in $C^{(n-1)}$ so that V_κ contains sets x witnessing statements of the form $\exists x(\varphi(x))$ with φ a Π_{n-1} formula. This process "closes off" at a limit point κ in $C^{(n)}$. $\square$

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Note however that trying this for all formulae (i.e., all n) at once raises Gödelian, definability of definability problems.

$C^{(n)}$-extendible cardinals

Definition

A cardinal κ is $C^{(n)}$-extendible if for every $\lambda > \kappa$ there is a cardinal $\mu > \lambda$ and an elementary embedding $j : V_\lambda \to V_\mu$ such that

1. $\kappa = \mathrm{crit}(j)$, *i.e., κ is the least ordinal such that $j(\kappa) \neq \kappa$,*
2. $j(\kappa) > \lambda$, *and*
3. $j(\kappa) \in C^{(n)}$.

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κ is $C^{(n)+}$-extendible if moreover, for every $\lambda > \kappa$ in $C^{(n)}$, there is a $\mu > \lambda$ in $C^{(n)}$ and an elementary embedding $j : V_\lambda \to V_\mu$ such that (1), (2) and (3) hold.

Relationships of large cardinals

Theorem (Bagaria & B-T)

For all α,

$$\exists\kappa > \alpha(\kappa \textit{ is } C^{(n)}\textit{-extendible})$$

is equivalent to

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Theorem (Bagaria, Casacuberta, Mathias & Rosický)

Vopěnka's Principle is equivalent to the existence of a proper class of $C^{(n)+}$*-extendible cardinals for every* n*. Morever, the existence of a* $C^{(n)+}$*-extendible* κ *corresponds exactly to Vopěnka's Principle for classes that are* Σ_{n+2}*-definable with parameters from* V_κ*.*

The main theorem again

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i. *the rank of $\boldsymbol{\Sigma}$,*
ii. *the arity of each function or relation symbol in $\boldsymbol{\Sigma}$, and*
iii. *the ranks of the parameters used in some Σ_n definitions of F and $\mathcal{A}$ and in some definition of $\mathcal{K}$,*

then there exists a regular cardinal λ such that F preserves λ-directed colimits.

Proof

Want to show $F : \mathcal{A} \to \mathcal{K}$ preserves λ-directed colimits.

Sufficient:

$$i \circ F : \mathcal{A} \to \mathbf{Str}\,\mathbf{\Sigma} \text{ preserves } \lambda\text{-directed colimits,}$$

where $i : \mathcal{K} \to \mathbf{Str}\,\mathbf{\Sigma}$ is the inclusion functor (and this notional inclusion doesn't change the quantifier complexity). So WLOG assume $F : \mathcal{A} \to \mathbf{Str}\,\mathbf{\Sigma}$.

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Note: $\mathbf{Str}\,\boldsymbol{\Sigma}$ has all λ-directed colimits, for λ greater than the arities of the symbols in $\boldsymbol{\Sigma}$ (i.e. cardinals as per (ii)).

Let β be sufficiently large as per (i), (ii) and (iii).

Proof

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Consider the following category $\mathcal{C}$:

Objects: $\mathbf{Str}\,\mathbf{\Sigma}$ morphisms $a : \bar{A} \to F(A)$ such that for some $\lambda > \beta$ and some λ-directed diagram $\mathcal{D}$ in $\mathcal{A}$,

- A is the colimit of $\mathcal{D}$ in $\mathcal{A}$,
- $\bar{A}$ is the colimit of $F\mathcal{D}$ in $\mathbf{Str}\,\mathbf{\Sigma}$, and
- a is the morphism induced by the image under F of the $\mathcal{A}$-colimit cocone from $\mathcal{D}$ to A.

Morphisms: From a to b: pairs $\langle g, h\rangle$ of $\mathbf{Str}\,\mathbf{\Sigma}$ morphisms such that

$$\begin{array}{ccc} \bar{A} & \xrightarrow{\;a\;} & F(A) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ \bar{B} & \xrightarrow[\;b\;]{} & F(B). \end{array}$$

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If the theorem fails, then $\mathcal{C}^*$ is not essentially small.

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Obj($\mathcal{C}^*$) is Σ_{n+2}-definable over the language of set theory (extended with $\mathcal{P}_\beta$): $a \in \text{Obj}(\mathcal{C}^*)$ iff

$$\begin{aligned}\exists\lambda\exists\mathcal{D}\exists\langle\bar{A},\bar{\eta}\rangle\exists\langle A,\eta\rangle(&\lambda \text{ is a regular cardinal} \wedge \mathcal{D} \text{ is a diagram in } \mathcal{A} \wedge \\ &\mathcal{D} \text{ is } \lambda\text{-directed} \wedge \\ &\langle\bar{A},\bar{\eta}\rangle = \text{Colim}_{\mathbf{Str}\,\mathbf{\Sigma}}(F\mathcal{D}) \wedge \langle A,\eta\rangle = \text{Colim}_{\mathcal{A}}(\mathcal{D}) \wedge \\ &a : \bar{A} \to F(A) \text{ is the induced homomorphism} \wedge \\ &a \text{ is not an isomorphism}).\end{aligned}$$

The universal property of colimits makes the middle line Π_{n+1}.

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Let κ be a $C^{(n)+}$-extendible cardinal greater than β.

Let a be an object of $\mathcal{C}^*$ of rank $> \kappa$, arising from a λ_a-directed diagram $\mathcal{D}_a$ for some $\lambda_a > \kappa$.

Let $\lambda \in C^{(n)}$ be greater than the ranks of $a, \mathcal{D}_a, F\mathcal{D}_a$, and the corresponding cocones $\langle \bar{A}, \bar{\eta} \rangle_a$ and $\langle A, \eta \rangle_a$.

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If not take κ a $C^{(n)+}$-extendible, $a \in \mathcal{C}^*$ "bigger", $\lambda \in C^{(n)}$ yet bigger

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If not take κ a $C^{(n)+}$-extendible, $a \in \mathcal{C}^*$ "bigger", $\lambda \in C^{(n)}$ yet bigger

Proof

Want to show $F : \mathcal{A} \to \mathbf{Str}\,\boldsymbol{\Sigma}$ preserves λ-directed colimits.
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Note that because $\kappa > \beta$, the definition of F is unaffected by j, so j commutes with F.

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Since j is elementary, we have a morphism in ${\mathcal{C}^*}^{V_\mu}$

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Now, $\mathcal{D}_a$ is λ_a-directed, so $j(\mathcal{D}_a)$ is $j(\lambda_a)$-directed, and $j(F\mathcal{D}_a) = Fj(\mathcal{D}_a)$ is $j(\lambda_a)$-directed.

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