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- $\text{cof}(\mathcal{N})$ = the least cardinality of a set of Lebesgue null sets such that every Lebesgue null set is a subset of one of them.

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$$\begin{array}{ccccccc}
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 & & \uparrow{\scriptstyle \leq} & & \uparrow{\scriptstyle \leq} & & \\
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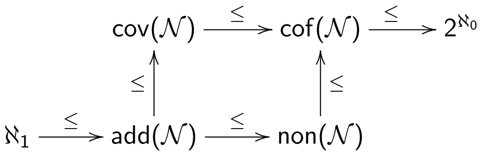
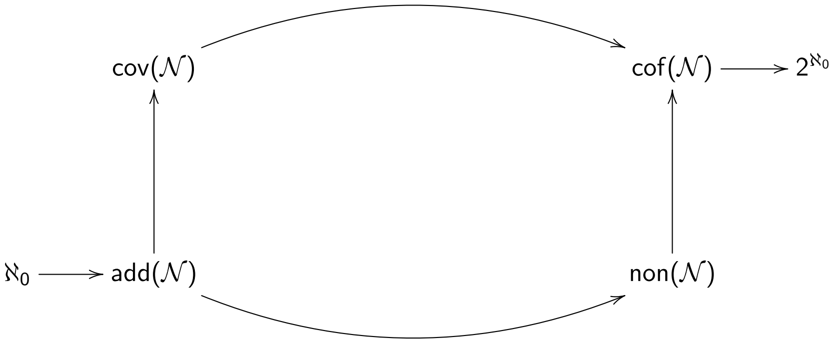


$\mathrm{non}(\mathcal{N}) \leq \mathrm{cof}(\mathcal{N})$:

Suppose $\mathcal{F}$ is as in the definition of $\mathrm{cof}(\mathcal{N})$. For each $F \in \mathcal{F}$, choose $x_F \notin F$. Then $\{x_F : F \in \mathcal{F}\}$ is not covered by any $F \in \mathcal{F}$, and so is not in $\mathcal{N}$. □

Cichoń's diagram



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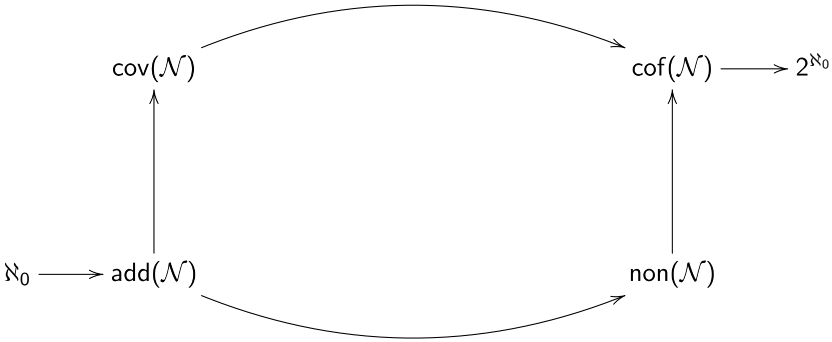

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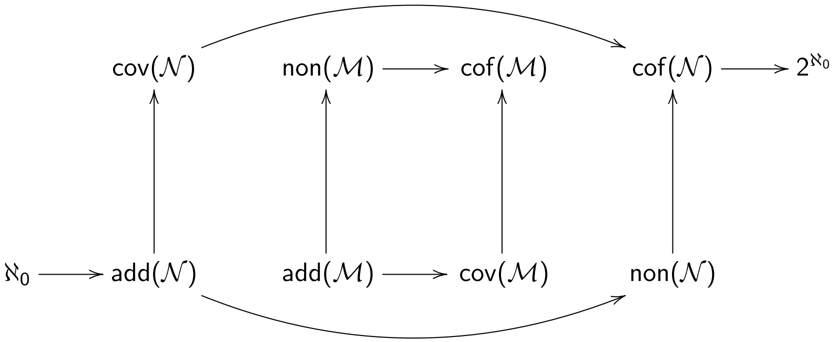

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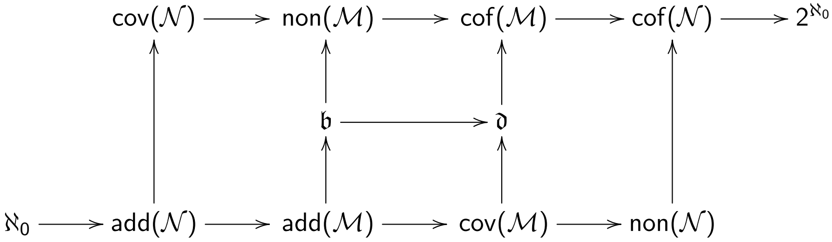

Unbounding and dominating numbers

$\exists^\infty n$ means "there exist infinitely many n",
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Other relations

Let R be a relation on $X \times Y$.

$$\mathfrak{b}(R) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq X \wedge \forall g \in Y \exists f \in \mathcal{F} \neg(f\, R\, g)\}$$
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A *slalom* is a function $\varphi : \omega \to [\omega]^{<\omega}$ such that for every n, $|\varphi(n)| \leq n$. Let Φ denote the set of slaloms.

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- $\neq^*$ on $\omega^\omega \times \omega^\omega$: $f \neq^* g \Leftrightarrow \forall^\infty n(f(n) \neq g(n))$

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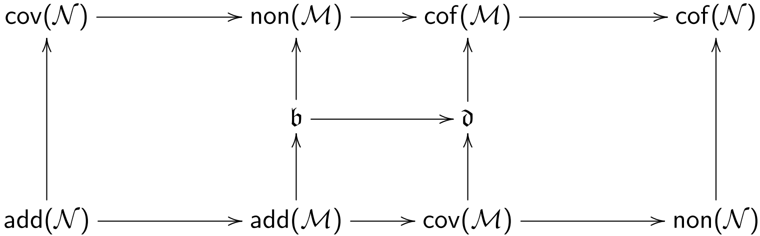

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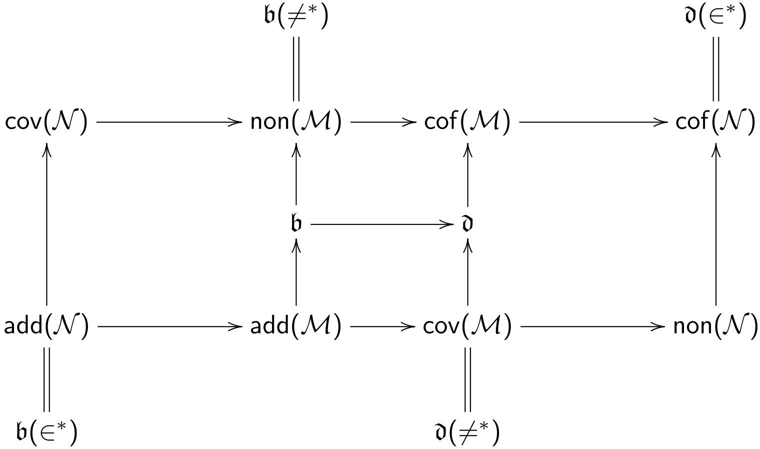

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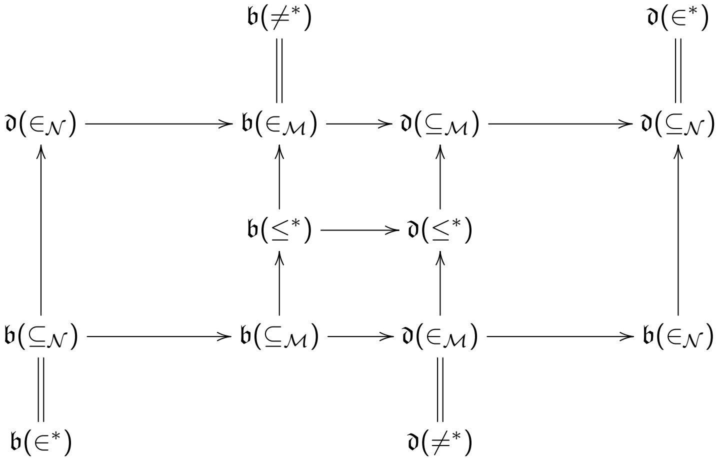

Computability theory

Given $A \in 2^\omega$, computability relative to A, or computability with A as an oracle, means computability where the Turing machine is permitted the additional operation of querying for any n what the nth bit of A is.
$A \leq_T B$ means that A is computable from B.

We consider highness properties of such Turing oracles.

The analogy

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Given a set-theoretic relation R and a computability-theoretic analogue R', we say that

$$\mathfrak{b}(R) \text{ corresponds to } \mathcal{B}(R')(A) := \exists y \leq_T A \, \forall \text{ computable } x (x \, R' \, y)$$
$$\mathfrak{d}(R) \text{ corresponds to } \mathcal{D}(R')(A) := \exists x \leq_T A \, \forall \text{ computable } y \neg (x \, R' \, y)$$

Eg

For $R = \leq^*$ we take $R' = \leq^*$ too, and so corresponding to $\mathfrak{b}(\leq^*)$ we have the property $\mathcal{B}(\leq^*)(A)$, which is that A computes a function that eventually dominates all computable functions.

Relation analogues

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Slaloms

The analogue of a slalom is a *trace*, that is, a computable slalom.

Relation analogues

Lebesgue null sets

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For this purpose our set of reals is 2^ω with the usual product measure λ: if $\mathcal{O}(s)$ is the set of all strings $x \in 2^\omega$ starting with s, then

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A *Schnorr test* is an effective sequence $(G_m)_{m\in\omega}$ of Σ^0_1 sets such that for each m, $\lambda(G_m) < 2^{-m}$, and the function $m \mapsto \lambda(G_m)$ is computable.

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A real $x \in 2^\omega$ is *Schnorr random* if x does not lie in any Schnorr null set.

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All of these notions relativize to any Turing oracle A.

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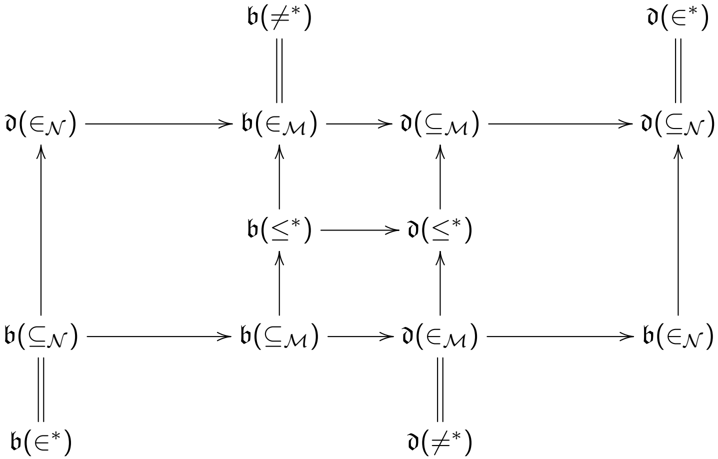

The computability-theoretic diagram

	$\mathcal{B}(\neq^*)$		$\mathcal{D}(\in^*)$
$\mathcal{D}(\in_\mathcal{N})$	$\mathcal{B}(\in_\mathcal{M})$	$\mathcal{D}(\subseteq_\mathcal{M})$	$\mathcal{D}(\subseteq_\mathcal{N})$
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The computability-theoretic diagram

high or DNR

not computably traceable

$A \geq_T$ a Schnorr random

weakly meager engulfing

not low for weak 1-genericity

not low for Schnorr tests

high

hyperimmune

Schnorr engulfing

meager engulfing

weakly 1-generic

weakly Schnorr engulfing

$A \geq_T \varphi$ tracing all computable functions

$A \geq_T f$ agreeing infinitely often with all computable functions

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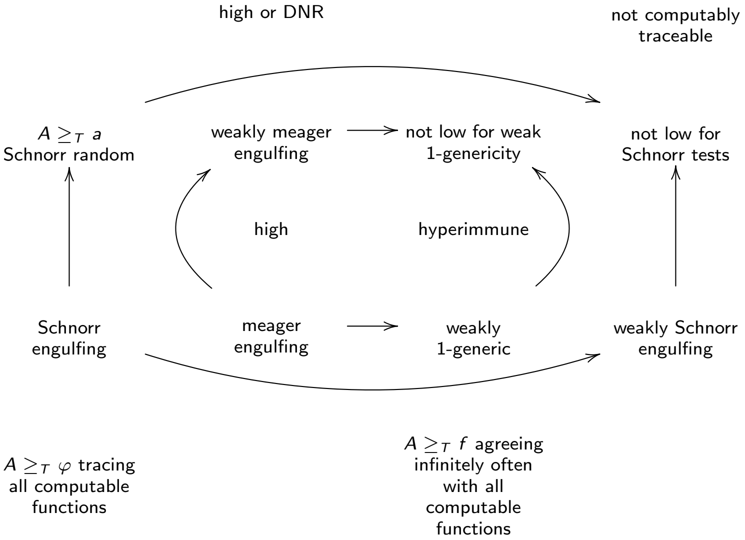

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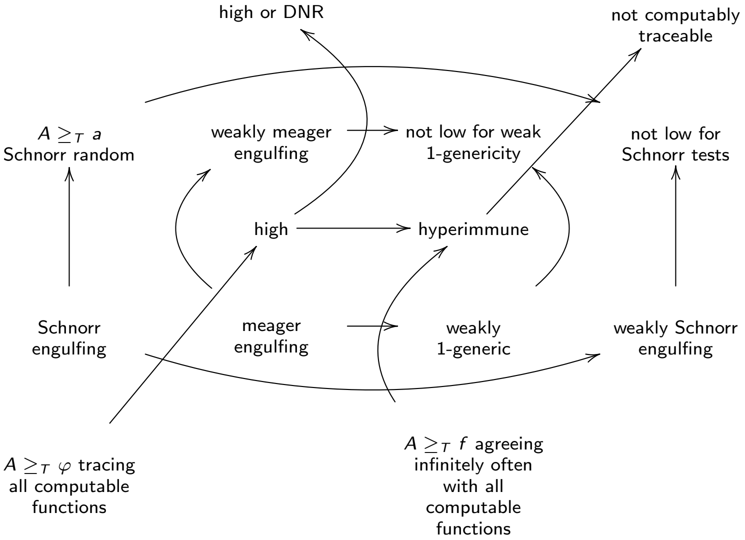

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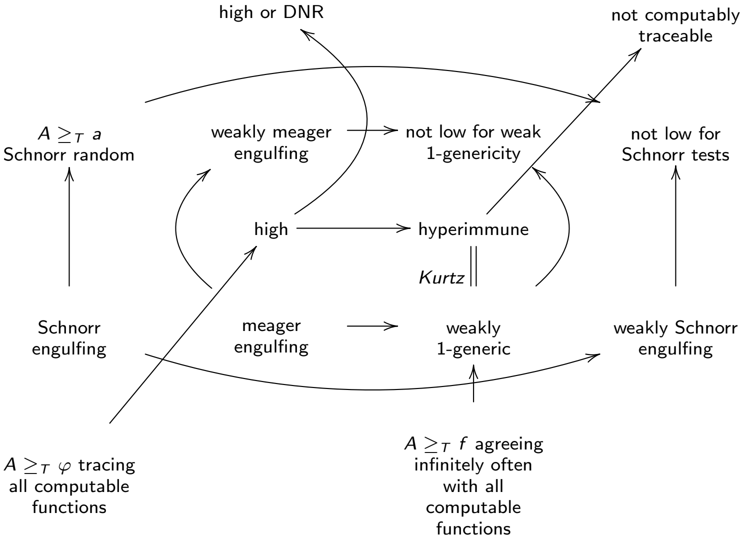

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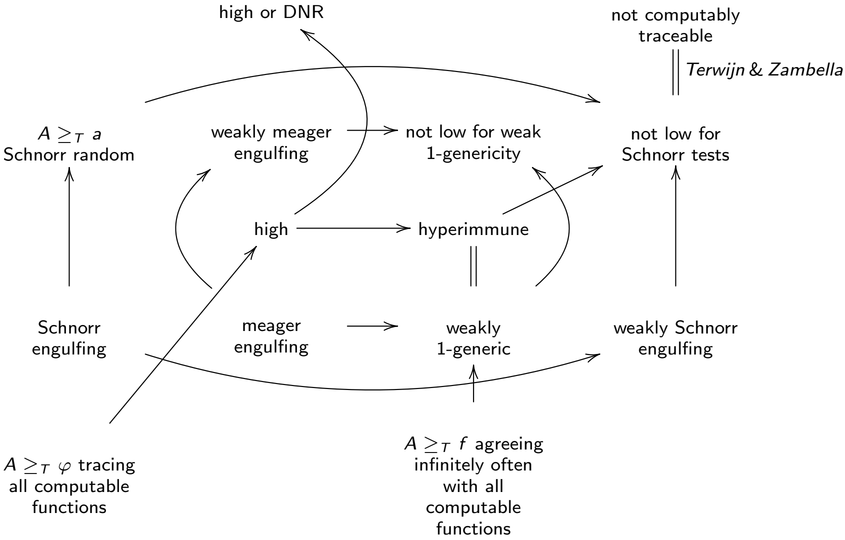

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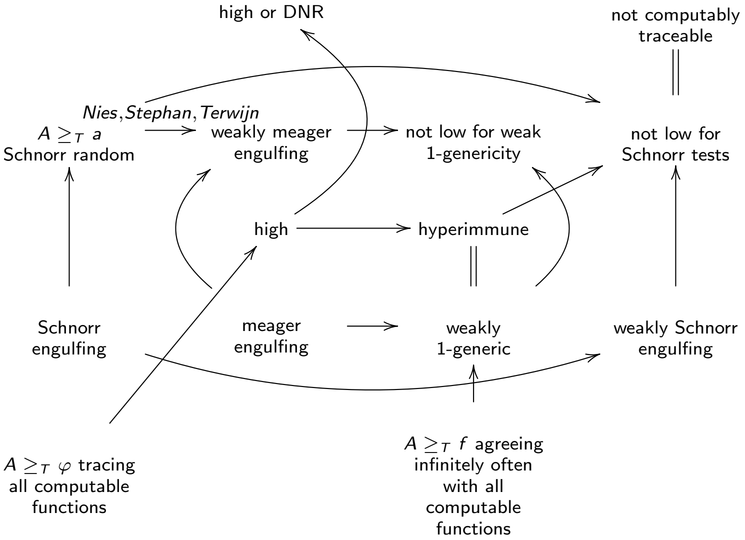

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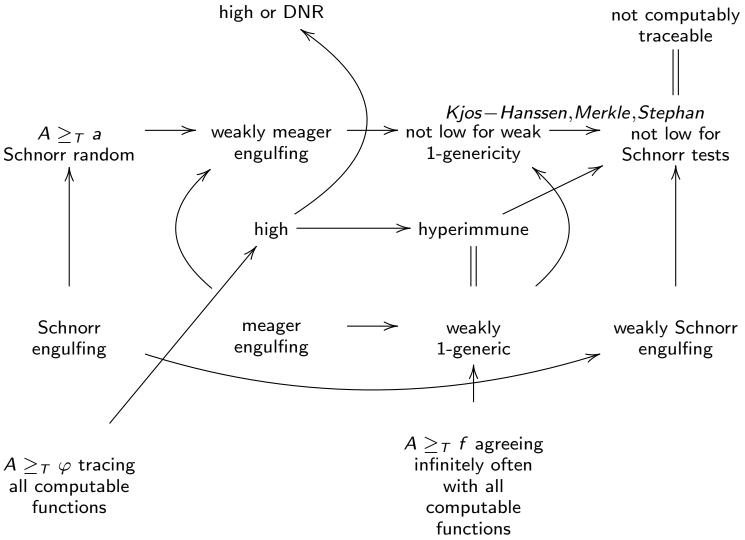

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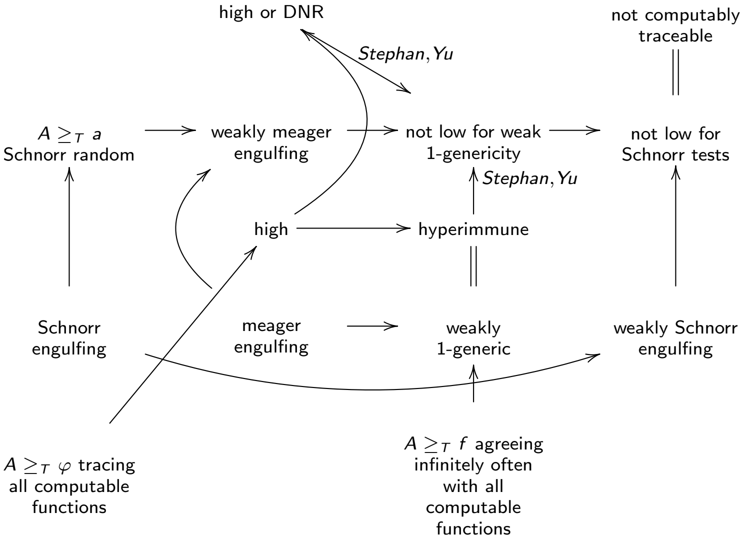

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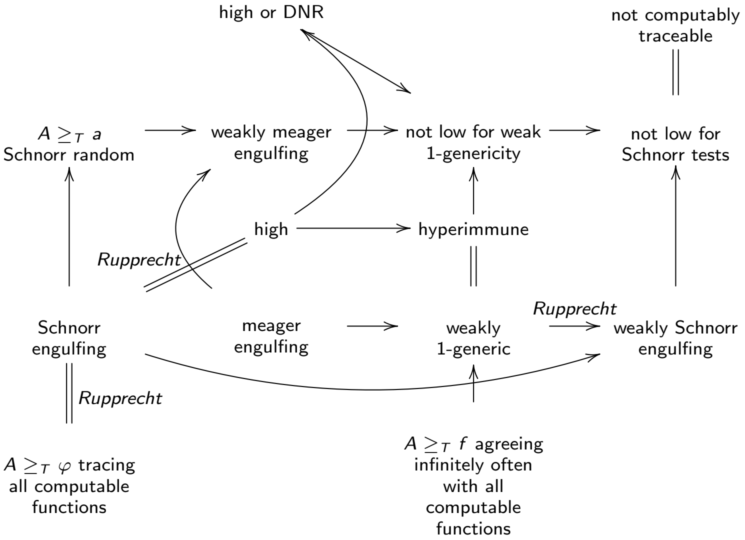

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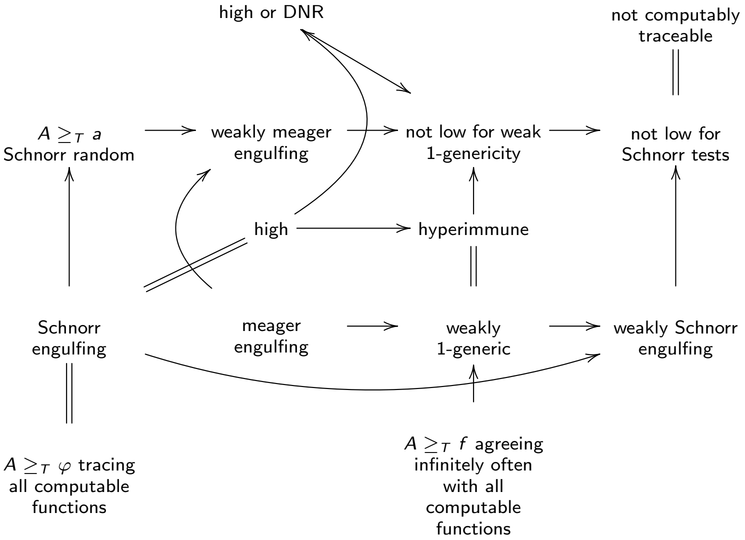

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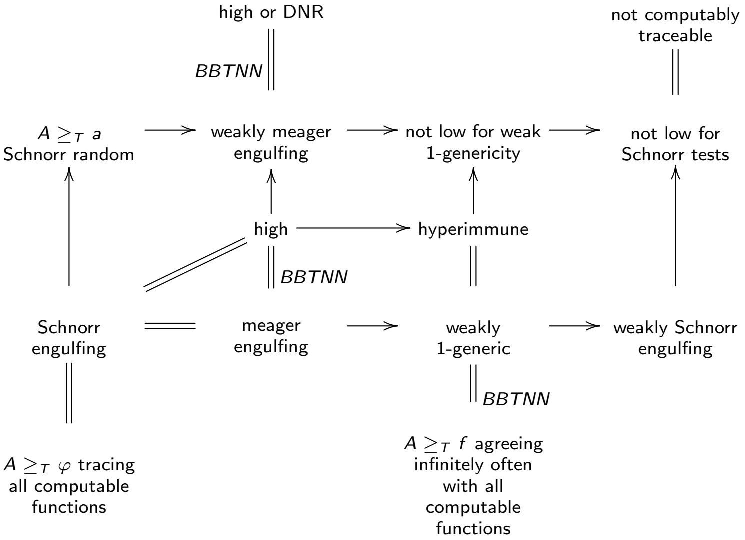

A proof

Theorem (Brendle, B-T, Ng, Nies)

The following are equivalent for a Turing oracle A.

1. *A is high, i.e., A computes a function evenutally dominating each computable function* $\omega \to \omega$*.*
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Proof.

(3) $\Rightarrow$ (2) : trivial — the union of the list is the desired effectively meagre in A set.

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$(2) \Rightarrow (1)$: See our paper! $\dashv$

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Open questions

Is there a Turing degree A that is:

1. weakly meagre engulfing and hyperimmune but does not compute a Schnorr random?
2. weakly meagre engulfing and weakly Schnorr engulfing but not hyperimmune and does not compute a Schnorr random?
3. weakly meagre engulfing but not weakly Schnorr engulfing and does not compute a Schnorr random?
4. not low for Schnorr tests but low for weak 1-genericity and not weakly Schnorr engulfing?