Powerful Images for Abstract Elementary Classes

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Two approaches to extending Model Theory

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Two approaches to generalising:

Makkai-Paré: Accessible categories

Shelah: Abstract Elementary Classes

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3. For M and N in $\mathbf{K}$, if $M \leq N$ then $M \subseteq_L N$.

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1. $\mathbf{K}$ is a class of L-structures for a fixed language $L = L(\mathbf{K})$.
2. $\leq$ is a reflexive and transitive binary relation on $\mathbf{K}$.
3. For M and N in $\mathbf{K}$, if $M \leq N$ then $M \subseteq_L N$.
4. Both $\mathbf{K}$ and $\leq$ are closed under isomorphism: if $M, N \in \mathbf{K}$, $M \leq N$ and $f : N \cong N'$, then $f``M$ and $N'$ are in $\mathbf{K}$, and $f``M \leq N$.

. . . continued

Abstract Elementary Classes (AECs)

5. (*Coherence axiom*) If $M_0, M_1, M_2 \in \mathbf{K}$, $M_0 \subseteq_L M_1 \leq M_2$, and $M_0 \leq M_2$, then $M_0 \leq M_1$.
6. (*Tarki-Vaught chain axioms*) If δ is a limit ordinal and $\langle M_i \mid i < \delta \rangle$ is a $\leq$-increasing chain of members of $\mathbf{K}$, then
    1. $M_\delta := \bigcup_{i<\delta} M_i \in \mathbf{K}$,
    2. $M_i \leq M_\delta$ for all $i < \delta$, and
    3. if $N \in \mathbf{K}$ and $M_i \leq N$ for all $i < \delta$ then $M_\delta \leq N$.

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   3. if $N \in \mathbf{K}$ and $M_i \leq N$ for all $i < \delta$ then $M_\delta \leq N$.
7. (*Löwenheim-Skolem-Tarski axiom*) There is a cardinal $\mu \geq \|L(\mathbf{K})\| + \aleph_0$ such that for every $M \leq \mathbf{K}$ and every $A \subseteq |M|$, there exists $M_0 \leq M$ in $\mathbf{K}$ such that $A \subseteq |M|$ and $\|M_0\| \leq \mu + \|A\|$. We define the Löwenheim-Skolem-Tarski number of $\mathbf{K}$, $LS(\mathbf{K})$, to be the least such μ.

Examples

- The class of models of a sentence of first order logic, or of any countable fragment of the infinitary logic $L_{\omega_1,\omega}$, with the associated notion of elementary submodel as $\leq$.

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- $\omega_1 + 1$ as **K**, usual ordinal ordering as $\leq$.

Morley's Theorem

A theorem of classical model theory:

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Conjecture (Shelah)

An analogous statement should be true for AECs.

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Theorem (Boney 2014)

If there are a proper class of strongly compact cardinals, then every AEC is tame.

Generalising from AECS to accessible categories

Lieberman and Rosický characterisation

Up to equivalence of categories, an AEC is an accessible category $\mathcal{K}$ with directed colimits, endowed with a faithful functor U to **Set** satisfying

- every morphism of $\mathcal{K}$ is mono,
- coherence,
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Powerful subcategories and images

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Motivating example

The category **FrAb** of free abelian groups is a powerful subcategory of the category **Ab** of abelian groups.

Note that **FrAb** is also the image of the free abelian group functor $F : \mathbf{Set} \to \mathbf{Ab}$.

For a functor $F : \mathcal{A} \to \mathcal{C}$, the *powerful image* of F is the least powerful subcategory of $\mathcal{C}$ containing the image of F, that is, the full subcategory with objects given by the closure of $\mathrm{Im}(F)$ under subobjects.

Accessible functors

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Lieberman and Rosický showed that Boney's Theorem follows from this result.

Weakening the large cardinals

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γ-strong compactness is consistently a strictly weaker notion than strong compactness.

Embedding reformulation

Theorem (Bagaria & Magidor, 2014)

A cardinal κ is γ-strongly compact if and only if for every $\alpha \geq \kappa$ there is an elementary embedding $j : V \to M$ definable in V, where V is the universe of all sets and M is an inner model of ZFC, such that

- $\kappa \geq \mathrm{crit}(j) \geq \gamma$ *(where* $\mathrm{crit}(j)$ *is the least cardinal δ such that $j(\delta) \neq \delta$),*
- *there is a set $A \supseteq j``\alpha$ such that $A \in M$ and $M \vDash |A| < j(\kappa)$.*

Improving on Makkai & Paré

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Suppose for every γ there is a γ-strongly compact cardinal. Then the powerful image of any accessible functor is accessible.

Corollary

Suppose for every γ there is a γ-strongly compact cardinal. Then every AEC is tame.

Sketch of the proof of the Theorem

Suppose $F\colon \mathcal{K} \to \mathcal{L}$ is a λ accessible functor, and take a cardinal $\mu \rhd \lambda$ that is greater than the number of λ-presentable objects in $\mathcal{K}$.

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For the rest, see the blackboard.