Some more cardinals for your repertoire

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UNIVERSITY OF LEEDS

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Plan

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Setting the Scene: Back to Cantor

Definition

Two sets have the same cardinality if there is a bijection (i.e. a one-to-one and onto function) between them.

Notation

- $|X|$ denotes the cardinality of X.
- $\aleph_0$ denotes the cardinality of $\mathbb{N}$.
- $2^{\aleph_0}$ denotes the cardinality of $\mathcal{P}(\mathbb{N})$ (the power set of $\mathbb{N}$, i.e., the set of all subsets of $\mathbb{N}$), which is the same as the cardinality of $\mathbb{R}$.

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Fact

There is a way to take a canonical representative from each cardinality equivalence class.

The order on the cardinals

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$|X| \leq |Y|$ if there is an injection (1-to-1 function) from X to Y.

The Schröder-Bernstein theorem shows that $|X| = |Y|$ if and only if $|X| \leq |Y|$ and $|X| \geq |Y|$, so the notation is reasonable.

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Fact

The cardinals are well-ordered by $\leq$.

So we can give definitions in the form "the least cardinal such that…".

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much as the axioms for groups on their own won't tell you whether your favourite group is abelian.

Analogy: Universes of Sets vs Groups

Universes of Sets

Groups

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The ZFC axioms
E.g. $\exists x \forall y \neg(y \in x)$

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E.g. $\forall x \forall y \forall z (x \cdot (y \cdot z) = (x \cdot y) \cdot z)$

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The group axioms don't imply failure of commutativity.
E.g.: $\mathbb{Z}$ is abelian

What could possibly go strictly between $|\mathbb{N}|$ and $|\mathbb{R}|$?