Measurable cardinals in category theory

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Reflections on Set Theoretic Reflection

Joan Bagaria's 60th birthday
Catalonia, November 2018

Joint work in progress with Joan Bagaria and Jiří Rosický, and I've been talking about closely related things with Will Boney.

Category theory preliminaries

Recall

A category $\mathcal{C}$ consists of

- a class of *objects*, and
- for every pair of objects A and B of $\mathcal{C}$, a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ of *morphisms* from A to B,

with identity morphisms and a composition (partial) function of morphisms, satisfying suitable axioms.

E.g.s

- **Set** is the category with sets as objects and functions as morphisms.
- **Gp** is the category with groups as objects and group homomorphisms as morphisms.

Limits and colimits

We think of a diagram as being a set of objects and morphisms between them.

- The limit of a diagram $\mathcal{D}$ is an object L along with a *cone* δ of projection maps to the objects of $\mathcal{D}$ (such that the triangles formed with the morphisms of $\mathcal{D}$ commute) such that any other such cone from an object of $\mathcal{C}$ factors uniquely through δ.
- The colimit of a diagram is the same in reverse.

E.g.

In **Set**, every diagram $\mathcal{D}$ has a limit and a colimit:

- The limit is the subset of the product of the sets in $\mathcal{D}$ consisting of all element whose coordinates "cohere" under the functions of the diagram.
- The colimit is the disjoint union of the sets in $\mathcal{D}$, modulo identifying elements with their images under the functions in $\mathcal{D}$.

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Gp has all limits & colimits too: limits are the same as in **Set**, and colimits are free products modulo identifications.

Given a set $\mathcal{A}$ of objects in a category $\mathcal{C}$ and an object C of $\mathcal{C}$, the canonical diagram of C with respect to $\mathcal{A}$ is the diagram with

- for every object A in $\mathcal{A}$ and every morphism $f: A \to C$, a copy of A, which we shall denote by A_f,
- as morphisms, all morphisms $h: A_f \to B_g$ such that $g \circ h = f$.

Note that the morphisms $f\colon A_f \to C$ form a cocone to C. If this cocone makes C the colimit of its canonical diagram with respect to $\mathcal{A}$, we say that C is a canonical colimit of objects from $\mathcal{A}$.

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E.g.s

- ω is dense in **Set**: every set is the colimit of the diagram of all of its finite subsets, which are the images of functions from finite sets.
- Any set of representatives of all the isomorphism classes of finite groups is dense in **Gp**: every group is the colimit of the diagram of all of its finite subgroups.

Note that being a canonical colimit of objects from $\mathcal{A}$ is stronger in general than just being a colimit of *some* diagram of objects from $\mathcal{A}$.

E.g.

Let $\mathbf{Vect}_{\mathbb{R}}$ be the category of real vector spaces, with linear transformations as the morphisms. Consider the set $\mathcal{A} = \{\mathbb{R}\}$. Then every object of $\mathbf{Vect}_{\mathbb{R}}$ is a colimit of objects from $\mathcal{A}$, but $\mathcal{A}$ is *not* dense. Indeed, consider a function $\varphi \colon \mathbb{R}^2 \to \mathbb{R}^2$ respecting scalar multiplication but not addition. Then there is a cocone mapping each $\mathbb{R}_f$ to $\mathbb{R}^2$ by $\varphi \circ f$, but it doesn't factor through the canonical cocone by any linear map.

Given a category $\mathcal{C}$, $\mathcal{C}^{op}$ is the category with the same objects as $\mathcal{C}$, and the same morphisms but in the opposite direction. Identity functions remain identity functions, and compositions of morphisms remain compositions of morphisms, just in the opposite order.

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$\mathbf{Set}^{op}$ is the category with sets as objects, and functions as morphisms, with any $f: X \to Y$ in the usual sense being considered as going from Y to X.

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Question

Is there a dense set in $\mathbf{Set}^{op}$?

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We can think about such functions $f : X \to \alpha$ in terms of the partitions $\{f^{-1}\{\gamma\} \mid \gamma \in \alpha\}$ that they define. In this context, the functions in the diagram represent coarsening maps.

The elements of the limit are elements $\mathbf{u} = (u_f)_{\alpha_f \in \mathcal{D}}$ of the product of the ordinals α_f in $\mathcal{D}$ — in the α_f coordinate, the element u_f of α_f is chosen.

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This corresponds to the choice of a piece from each of the partitions ($f^{-1}\{u_f\}$ in the partition corresponding to $f\colon X \to \alpha$), in a way that the coarsening maps respect — we can think of this as choosing a "big" piece from each partition.

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$\mathcal{U}$ is a κ-complete ultrafilter

Let $\chi_Y \colon X \to 2$ be the characteristic function of Y, $\chi_Y(x) = 1 \leftrightarrow x \in Y$. Then

$$\begin{aligned}\mathcal{U} &= \{Y \subseteq X \mid \exists \alpha < \kappa \exists f \colon X \to \alpha (Y = f^{-1}\{u_f\})\} \\ &= \{Y \subseteq X \mid u_{\chi_Y} = 1\}.\end{aligned}$$

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- If $Y \in \mathcal{U}$ and $\{Z_\gamma \mid \gamma < \alpha\}$ is a partition of Y into fewer than κ many pieces, then since $\{X \smallsetminus Y\} \cup \{Z_\gamma \mid \gamma < \alpha\}$ coarsens to $\{Y, X \smallsetminus Y\}$, one of the Z_γ is in $\mathcal{U}$, so $\mathcal{U}$ is κ-complete.

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- For any $Y \subseteq X$, $Y \in \mathcal{U}$ if $u_{\chi_Y} = 1$ and $X \smallsetminus Y \in \mathcal{U}$ if $u_{\chi_Y} = 0$, so $\mathcal{U}$ is ultra.

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Note that by definition, there is a dense set in $\mathbf{Set}^{op}$ if and only if for some κ, every X is the limit of its canonical diagram with respect to κ, if and only if there are no non-principal κ-complete ultrafilters on any set.

So we have shown

Theorem (Isbell, 1960)

There is a dense set in $\mathbf{Set}^{op}$ *if and only if there are only boundedly many measurable cardinals.*

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Question (Joan)

Can we characterise *normal* ultrafilters on κ?

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However, normality is *not* permutation-invariant: given an ultrafilter $\mathcal{U}$ on κ and a permutation π of κ, we have an ultrafilter $\pi\mathcal{U}$ on κ defined by $Y \in \pi\mathcal{U}$ iff $\pi^{-1}Y \in \mathcal{U}$.

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Consider the permutation π swapping every limit ordinal with its successor. Then if $\mathcal{U}$ is normal, the ultrafilter $\pi\mathcal{U}$ is *not* normal — it contains the set of successor ordinals.

Changing the question

A more suitable notion in this context is selectivity:

Definition

An ultrafilter $\mathcal{U}$ on κ is *selective* if for every partition of κ into κ many sets Y_α none of which is in $\mathcal{U}$, there is some set $X \in \mathcal{U}$ such that $|X \cap Y_\alpha| \leq 1$ for every α.

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Clearly selectivity is permutation-invariant.

Note that any normal ultrafilter is selective: suppose $\{Y_\alpha \mid \alpha < \kappa\}$ is a partition of κ into pieces not in $\mathcal{U}$, and assume WLOG that $\min(Y_\alpha) \geq \alpha$ for all α. Then

$$\mathop{\triangle}_{\alpha<\alpha} \kappa \setminus Y_\alpha$$

is a set in $\mathcal{U}$ as required.

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Proposition

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Proof

Let $j_{\mathcal{U}}\colon V \to M$ be the ultrapower embedding from $\mathcal{U}$. Let $f\colon \kappa \to \kappa$ be such that $[f]_{\mathcal{U}} = \kappa$. By κ-completeness, $f^{-1}\{\alpha\} \notin \mathcal{U}$ for each $\alpha \in \kappa$ (otherwise we would have $[f]_{\mathcal{U}} = j(\alpha) = \alpha$), so by selectivity there is some $X \in \mathcal{U}$ such that $f \restriction X$ is injective.

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Shrinking X if necessary, we may assume that $|\kappa \smallsetminus X| = \kappa$ and $|\kappa \smallsetminus f``X| = \kappa$. But then we may extend $f \restriction X$ to a permutation π of κ with $\pi \restriction X = f \restriction X$, so $[\pi]_{\mathcal{U}} = [f]_{\mathcal{U}} = \kappa$.

Now consider the normal ultrafilter $\mathcal{V}$ defined from $j_{\mathcal{U}}$:

$$\begin{aligned} X \in \mathcal{V} &\leftrightarrow \kappa \in j_{\mathcal{U}}(X) \\ &\leftrightarrow [\pi]_{\mathcal{U}} \in [c_X]_{\mathcal{U}} \text{ where } c_X \text{ is the constant function taking value } X \\ &\leftrightarrow \{\alpha \in \kappa \mid \pi(\alpha) \in X\} \in \mathcal{U} \\ &\leftrightarrow X \in \pi\mathcal{U}. \end{aligned}$$

So $\pi\mathcal{U}$ is normal. □

So a nonprincipal κ-complete ultrafilter on κ is selective if and only if some permutation of it is normal.

Characterising selectivity in category theory

The precise formulation of selectivity we translate is the following:

Proposition

An ultrafilter on κ is selective if and only if for every $f : \kappa \to \kappa$, there is a set X in the ultrafilter on which X is either constant or injective.

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We also need a little more category theory.

Terminal objects and elements

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Note that an element x of a set X can be identified with the function *from* 1 to X taking the unique element of 1 to x. Thus "elements" *can* be translated into category-theoretic language, as "morphisms from the terminal object."

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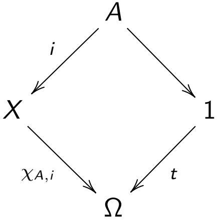
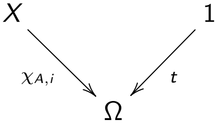
Thus, a subobject is just considered to mean a monomorphism.

Definition

In a category $\mathcal{C}$ with a terminal object 1, a subobject classifier is an object Ω equipped with a distinguished morphism $t\colon 1 \to \Omega$ such that for every monomorphism $i\colon A \to X$ of $\mathcal{C}$, there is a unique morphism $\chi_{A,i}\colon X \to \Omega$ such that



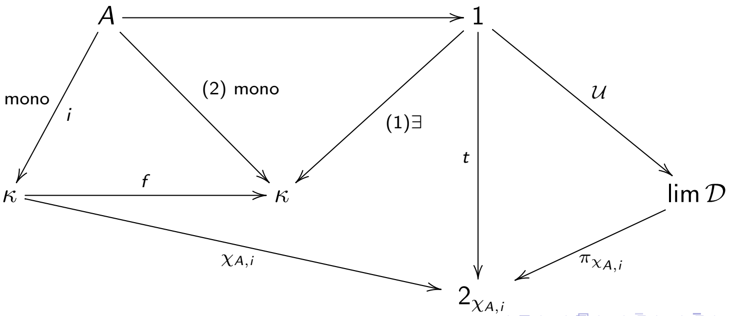
is the pullback (that is, A is the limit) of the following diagram. (E.g. in **Set**, $\Omega = 2$.)



Theorem

Let $\mathcal{U}\colon 1 \to \lim \mathcal{D}$ be (the morphism that picks out) a non-principal κ-complete ultrafilter on κ. Then $\mathcal{U}$ is selective if and only if for every morphism $f\colon \kappa \to \kappa$, there is a subobject A of κ, with inclusion morphism $i\colon A \hookrightarrow \kappa$, such that:

- *the subobject classifier map $t\colon 1 \to 2_{\chi_{A,i}}$ factors as $\pi_{\chi_{A,i}} \circ \mathcal{U}$, and*
- *either*
  1. *$f \circ i$ factors through 1, or*
  2. *$f \circ i$ is a monomorphism.*



Back to normality?

If we work with the category of sets with a preferred partial order (with no requirement that functions respect it), the hom-sets inherit a preferred partial order. In this setting, we can express $f < \mathrm{id}$, and translate the Fodor formulation of normality to get a category-theoretic version.

Congratulations Joan!