

# Categorifying Borel Reducibility

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Joint work in progress with  
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In this setting, *Borel* (the inverse image of any Borel set is Borel) is a suitably liberal interpretation of “reasonably definable.”



# Borel reducibility

## Definition (Friedman and Stanley, 1989)

Let  $X$  and  $Y$  be standard Borel spaces, let  $E$  be an equivalence relation on  $X$ , and let  $F$  be an equivalence relation on  $Y$ . We say that  $E$  is *Borel reducible* to  $F$ , and write  $E \leq_B F$ , if there is a Borel function  $f : X \rightarrow Y$  such that

$$x_1 E x_2 \quad \text{iff} \quad f(x_1) F f(x_2).$$

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If you can show that there is *no* such Borel reduction, the classification programme cannot be completed. This has led to important unclassifiability results in, e.g.,

- ergodic theory (Hjorth, and Foreman & Weiss)
- $C^*$ -algebras (Farah, Toms & Törnquist).

## Example

Let  $\mathcal{V}$  be a countable vocabulary, with relation symbols  $R_m$  of arity  $a_m$  and function symbols  $f_n$  of arity  $b_n$ . Each  $\mathcal{V}$ -structure  $\mathcal{M}$  with underlying set  $\mathbb{N}$  can be encoded by a single characteristic function  $\chi_{\mathcal{M}}$  in

$$\prod_{R_m \in \mathcal{V}} 2^{\mathbb{N}^{a_m}} \times \prod_{f_n \in \mathcal{V}} \mathbb{N}^{\mathbb{N}^{b_n}}$$

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Similarly, for any first order (or  $\mathcal{L}_{\omega_1, \omega}$ ) sentence  $\varphi$  over  $\mathcal{V}$ , we can consider the subspace  $\text{Mod}(\varphi)$  of  $\text{Str}(\mathcal{V})$  of models of  $\varphi$ .

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## Definition (Friedman and Stanley)

The isomorphism relation of a class of structures  $\text{Mod}(\varphi)$  is *Borel complete* if every isomorphism relation on a class  $\text{Mod}(\psi)$  of structures Borel reduces to it.

# Examples of Borel complete classes

- Graphs
- Groups (Mekler)
- Rooted trees (Friedman & Stanley)
- Linear orders (Friedman & Stanley)
- Fields (Friedman & Stanley)
- Boolean algebras (Camerlo & Gao)
- Quandles (B.-T. & Miller)



# Categories

## Definition

A *category*  $\mathcal{C}$  consists of

- a class  $\text{Obj } \mathcal{C}$  of *objects*, and
- a class  $\text{Mor } \mathcal{C}$  of *morphisms*,

along with

- functions  $\text{dom}, \text{cod}: \text{Mor } \mathcal{C} \rightarrow \text{Obj } \mathcal{C}$ ,
- an “identity morphism” function  $\text{id}: \text{Obj } \mathcal{C} \rightarrow \text{Mor } \mathcal{C}$ ,
- a composition partial binary function  $\circ$  on morphisms, taking pairs  $(g, f)$  of morphisms such that  $\text{dom}(g) = \text{cod}(f)$  to morphisms,

satisfying the following axioms:<sup>a</sup>

- When they are defined,  $(h \circ g) \circ f = h \circ (g \circ f)$ .
- $\forall a \in \text{Obj } \mathcal{C}, \text{dom}(\text{id}(a)) = \text{cod}(\text{id}(a)) = a$ .
- $\forall f \in \text{Mor } \mathcal{C}, \text{id}(\text{cod}(f)) \circ f = f \circ \text{id}(\text{dom}(f)) = f$ .

---

<sup>a</sup>Also,  $\text{dom}^{-1}(A) \cap \text{cod}^{-1}(B) = \text{Hom}(A, B)$  is usually required to be a *set* for all  $A, B \in \text{Obj } \mathcal{C}$ .

E.g.

For any vocabulary  $\mathcal{V}$  and any  $\mathcal{V}$ -sentence  $\varphi$ , we have the category whose objects are all models of  $\varphi$ , and whose morphisms are the  $\mathcal{V}$ -homomorphisms. Such categories are called *algebraic*.

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A *functor* is a homomorphism of categories: it takes objects to objects and morphisms to morphisms in a way that respects composition, identities, domains and codomains.

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- in a way that is functorial, and more.

Idea: extend the Borel reducibility framework to accommodate this.

# Borel categories

## Definition

A *Borel category*  $\mathcal{C}$  is a pair  $(\text{Obj}(\mathcal{C}), \text{Mor}(\mathcal{C}))$  of standard Borel spaces endowed with

- Borel functions  $\text{dom}$  and  $\text{cod}$  from  $\text{Mor}(\mathcal{C})$  to  $\text{Obj}(\mathcal{C})$ ,
  - a Borel function  $\text{id}$  from  $\text{Obj}(\mathcal{C})$  to  $\text{Mor}(\mathcal{C})$ , and
  - a Borel function  $\circ$  from  $\{(g, f) \in \text{Mor}(\mathcal{C})^2 \mid \text{dom}(g) = \text{cod}(f)\}$  to  $\text{Mor}(\mathcal{C})$ ,
- such that  $(\text{Obj}(\mathcal{C}), \text{Mor}(\mathcal{C}))$  with these functions is a category.

## Examples

Let  $\text{Mor } \mathcal{V}$  be the subset of  $\text{Str } \mathcal{V} \times \text{Str } \mathcal{V} \times \mathbb{N}^{\mathbb{N}}$  consisting of those  $(\mathcal{M}, \mathcal{N}, f)$  such that  $f$  is a  $\mathcal{V}$ -homomorphism from  $\mathcal{M} \rightarrow \mathcal{N}$ . Then  $\mathbf{Str } \mathcal{V} = (\text{Str } \mathcal{V}, \text{Mor } \mathcal{V})$  is a Borel category.



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For any sentence  $\varphi$  over  $\mathcal{V}$ ,

$$\mathbf{Mod}(\varphi) = (\text{Mod } \varphi, \text{Mor } \mathcal{V} \cap (\text{Mod } \varphi \times \text{Mod } \varphi \times \mathbb{N}^{\mathbb{N}}))$$

is a Borel category.

# Borel functors

## Definition

A *Borel functor* from a Borel category  $\mathcal{C}$  to a Borel category  $\mathcal{D}$  is a pair of Borel functions  $F_{\text{Obj}}: \text{Obj}(\mathcal{C}) \rightarrow \text{Obj}(\mathcal{D})$  and  $F_{\text{Mor}}: \text{Mor}(\mathcal{C}) \rightarrow \text{Mor}(\mathcal{D})$  that respect the functions  $\text{dom}$ ,  $\text{cod}$ ,  $\text{id}$  and  $\circ$ . For  $A \in \text{Obj}(\mathcal{C})$  we write  $F(A)$  for  $F_{\text{Obj}}(A)$  and for  $f \in \text{Mor}(\mathcal{C})$  we write  $F(f)$  for  $F_{\text{Mor}}(f)$ .

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## Note

Borel functors must respect isomorphisms: if  $g \circ f = \text{id}$  and  $f \circ g = \text{id}$  then

$$F(g) \circ F(f) = F(\text{id}) = \text{id} \text{ and } F(f) \circ F(g) = F(\text{id}) = \text{id}.$$

But they need not respect non-isomorphism.

A functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is *full* if for all objects  $C_0$  and  $C_1$  of  $\mathcal{C}$ ,  $F$  restricted to  $\text{Hom}_{\mathcal{C}}(C_0, C_1)$  is surjective:

$$\forall g: F(C_0) \rightarrow F(C_1) \exists f: C_0 \rightarrow C_1 (F(f) = g).$$

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A functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is *faithful* if for all objects  $C_0$  and  $C_1$  of  $\mathcal{C}$ ,  $F$  restricted to  $\text{Hom}_{\mathcal{C}}(C_0, C_1)$  is injective:

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Note that a fully faithful  $F$  can only fail to be an embedding by identifying isomorphic objects.



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## Proposition

The object map  $F_{\text{Obj}}$  of a functorial reduction  $F: \mathcal{C} \rightarrow \mathcal{D}$  is a Borel reduction from the isomorphism relation on  $\text{Obj}(\mathcal{C})$  to the isomorphism relation on  $\text{Obj}(\mathcal{D})$ .

## Proof

It suffices to show that if  $F(C_0) \cong_{\mathcal{D}} F(C_1)$ , then  $C_0 \cong_{\mathcal{C}} C_1$ .

## Functorial reductions are Borel reductions

If  $F(C_0) \cong_{\mathcal{D}} F(C_1)$ , that means there are morphisms  $h: F(C_0) \rightarrow F(C_1)$  and  $j: F(C_1) \rightarrow F(C_0)$  such that

$$j \circ h = \text{id}(F(C_0)) \quad \text{and} \quad h \circ j = \text{id}(F(C_1)).$$

By fullness there are  $f: C_0 \rightarrow C_1$  and  $g: C_1 \rightarrow C_0$  such that  $F(f) = h$  and  $F(g) = j$ . By functoriality,

$$\begin{aligned} F(g \circ f) &= F(g) \circ F(f) = j \circ h = \text{id}(F(C_0)) = F(\text{id}(C_0)), & \text{and} \\ F(f \circ g) &= F(f) \circ F(g) = h \circ j = \text{id}(F(C_1)) = F(\text{id}(C_1)), \end{aligned}$$

and then by faithfulness,

$$g \circ f = \text{id}(C_0) \quad \text{and} \quad f \circ g = \text{id}(C_1).$$

So  $C_0 \cong C_1$ . □

# Just one?

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Even accepting that some classifications should be done with functors, it would seem reasonable to allow a classification to use multiple different functors to distinguish different objects. This is fine:

### Proposition

*The product of any countable collection of Borel categories is a Borel category. Moreover, for any countable collection of Borel functors  $F_i: \mathcal{C} \rightarrow \mathcal{D}_i$ , the product functor  $\prod_i F_i: \mathcal{C} \rightarrow \prod_i \mathcal{D}_i$  is a Borel functor.*

So countably many Borel functors can be incorporated into a single one.

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## How does this compare with the category theory literature?

There is a substantial body of work that has been done on the analogous category theory notions (without “Borel”).

- “Hands-on”  $\longrightarrow$  probably translate to Borel setting.
- Care required to keep the constructions countable.



# Categorifying Borel completeness

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Which Borel complete classes of structures become functorially universal Borel categories?

# Graphs

Let **Gra** be the Borel category of directed graphs on  $\mathbb{N}$ , with graph homomorphisms as morphisms. Equivalently, this is **Str** $\{E\}$  where  $E$  is a binary relation.

## Proposition

**Gra** is functorially universal.

There are various proofs of similar statements in the literature, with proofs that don't quite carry across. E.g.:

- The proof in Hodges (Theorem 5.5.1) in the context of interpretability is functorial, but not full.
- A proof from the category theory literature (Pultr and Trnkova Section II.5) doesn't preserve countability.

But adjustments can be found to make them work.

# Towards graphs

A step along the way to showing that **Gra** is functorially universal is the following:

## Proposition

*For any countable relational vocabulary  $\mathcal{V}$ , there is a countable binary relational vocabulary  $\mathcal{W}$  such that **Str**  $\mathcal{V}$  functorially reduces to **Str**  $\mathcal{W}$  (in fact, there is a Borel full embedding from **Str**  $\mathcal{V}$  into **Str**  $\mathcal{W}$ ).*

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Using the construction in Pultr and Trnkova’s book,  $\mathcal{N}$  has underlying set  $\mathbb{N}^{\sup(a_i)}$  which could be uncountable. But with a little more care and an extra binary relation to keep track of tuple length, we can build an encoding structure on  $\mathbb{N}^{<\omega}$ .

# Details

Specifically, if  $\mathcal{V} = \{R_i \mid i \in \omega\}$  where  $R_i$  has arity  $a_i$ , let  $\mathcal{W}$  be the binary relational vocabulary with symbols  $\bar{R}_i$  for  $i$  in  $\omega$ ,  $S_n$  for  $n$  in  $\omega$ , and  $T$ .

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Given a  $\mathcal{V}$ -structure  $\mathcal{M}$  with underlying set  $M$ , define a  $\mathcal{W}$ -structure  $\mathcal{N}$  with underlying set  $M^{<\omega}$  by

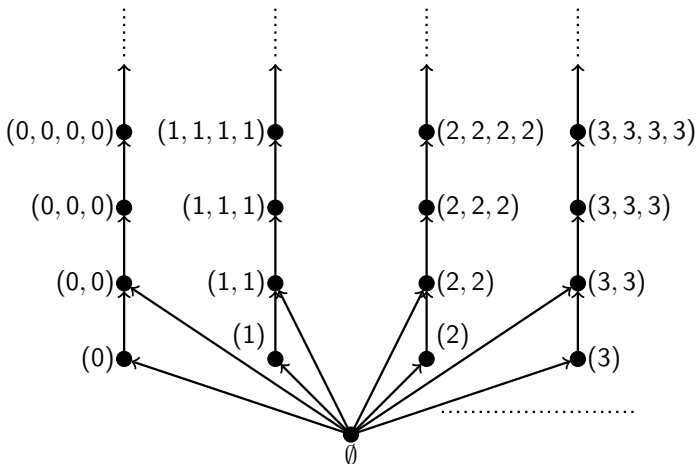
$$\mathbf{x} T^{\mathcal{N}} \mathbf{y} \text{ iff } \exists m \in M (\forall i < |\mathbf{x}| (x_i = m) \wedge \forall j < |\mathbf{y}| (y_j = m)) \\ \wedge (|\mathbf{y}| = |\mathbf{x}| + 1 \vee (|\mathbf{y}| = 2 \wedge |\mathbf{x}| = 0))$$

$$\mathbf{x} S_n^{\mathcal{N}} \mathbf{y} \text{ iff } |\mathbf{x}| = |\mathbf{y}| > n \wedge \forall j < |\mathbf{y}| (y_j = x_n)$$

$$\mathbf{x} \bar{R}_i^{\mathcal{N}} \mathbf{y} \text{ iff } \mathcal{M} \models R_i(\mathbf{x}), \text{ that is, } \mathbf{x} \in R_i^{\mathcal{M}}.$$



T:



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# Mono-unary algebras

Let  $f$  be a unary function symbol. Structures in  $\mathbf{Str}\{f\}$  are called *mono-unary algebras*. Friedman and Stanley showed that  $\mathbf{Str}\{f\}$  is Borel complete.

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## Proposition

$\mathbf{Str}\{f\}$  is not functorially universal.

## Proof sketch

Hedrlín and Pultr showed that if the endomorphism monoid of a mono-ary algebra is a group, then it is a direct product of cyclic groups.

# Mono-unary algebras

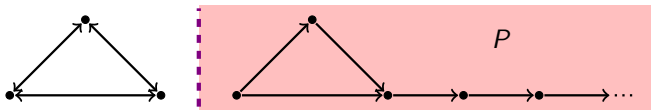
Let  $f$  be a unary function symbol. Structures in  $\mathbf{Str}\{f\}$  are called *mono-unary algebras*. Friedman and Stanley showed that  $\mathbf{Str}\{f\}$  is Borel complete. But:

## Proposition

$\mathbf{Str}\{f\}$  is not functorially universal.

## Proof sketch

Hedrlín and Pultr showed that if the endomorphism monoid of a mono-unary algebra is a group, then it is a direct product of cyclic groups. So for example,  $\mathbf{Str}\{f\}$  does not admit a functorial reduction from  $\mathbf{Str}\{E, P\}$  where  $E$  is a binary relation and  $P$  is a unary relation, since the following structure has endomorphism monoid  $S_3$ :



# Linear Orders

Let **LinOrd** be the Borel category of linear orders with underlying set  $\mathbb{N}$ , with order-preserving functions as homomorphisms. Friedman and Stanley showed that the isomorphism relation of countable linear orders is Borel complete. But:

## Proposition

**LinOrd** *is not functorially universal*.

# Linear Orders

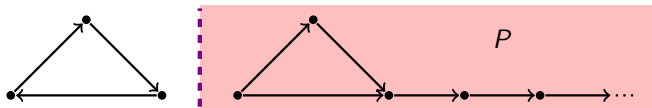
Let **LinOrd** be the Borel category of linear orders with underlying set  $\mathbb{N}$ , with order-preserving functions as homomorphisms. Friedman and Stanley showed that the isomorphism relation of countable linear orders is Borel complete. But:

## Proposition

**LinOrd** is not functorially universal.

## Proof sketch

Every non-trivial automorphism of a linear order has infinite order in the automorphism group. So for example, **Str** $\{f\}$  does not admit a functorial reduction from **Str** $\{E, P\}$  where  $E$  is a binary relation and  $P$  is a unary relation, since this structure has automorphism group  $\mathbb{Z}/3\mathbb{Z}$ :



# Boolean algebras

Let  $\mathbf{BA}$  be the Borel category of Boolean algebras with underlying set  $\mathbb{N}$ , with Boolean algebra homomorphisms as the morphisms. Camerlo and Gao showed that the isomorphism relation of countable Boolean algebras is Borel complete. But:

## Proposition

$\mathbf{BA}$  is not functorially universal.



# Boolean algebras

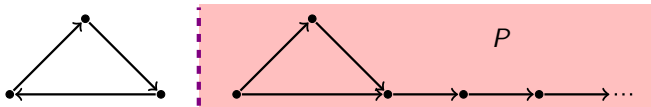
Let  $\mathbf{BA}$  be the Borel category of Boolean algebras with underlying set  $\mathbb{N}$ , with Boolean algebra homomorphisms as the morphisms. Camerlo and Gao showed that the isomorphism relation of countable Boolean algebras is Borel complete. But:

## Proposition

$\mathbf{BA}$  is not functorially universal.

## Proof sketch

McKenzie and Monk showed that every element of the centre of the automorphism group of any Boolean algebra has order at most 2. So for example,  $\mathbf{Str}\{f\}$  does not admit a functorial reduction from  $\mathbf{Str}\{E, P\}$  where  $E$  is a binary relation and  $P$  is a unary relation, since this structure has automorphism group  $\mathbb{Z}/3\mathbb{Z}$ :



# Questions

Hjorth's notion of *turbulence* is a central tool for the impossibility of classification results mentioned at the start — can we find a categorified version of that?

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Hjorth's notion of *turbulence* is a central tool for the impossibility of classification results mentioned at the start — can we find a categorified version of that?

Can we use this framework to show some classification programme is impossible, where standard Borel reducibility was too blunt a tool?

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