Borel reducibility and category theory

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Borel reducibility

Definition

Let X and Y be standard Borel spaces, let E be an equivalence relation on X, and let F be an equivalence relation on Y. We say that E is *Borel reducible* to F, and write $E \leq_B F$, if there is a Borel function $f : X \to Y$ such that

$$x_1 \; E \; x_2 \qquad \text{iff} \qquad f(x_1) \; F \; f(x_2).$$

Example

Let $\mathcal{V}$ be a countable relational vocabulary, with relation symbols R_i of arity a_i. Each $\mathcal{V}$-structure $\mathcal{M}$ with underlying set $\mathbb{N}$ can by encoded by a single characteristic function $\chi_{\mathcal{M}}$ in $\prod_{R_i \in \mathcal{V}} 2^{\mathbb{N}^{a_i}}$. We denote this space $\prod_{R_i \in \mathcal{V}} 2^{\mathbb{N}^{a_i}}$ of (codes for) structures by $\operatorname{Str} \mathcal{V}$.

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Definition (H. Friedman and L. Stanley)

The isomorphism relation of a class of structures $\mathbf{Mod}(\varphi)$ is *Borel complete* if every isomorphism relation on a class of structures Borel reduces to it.

Examples of Borel complete classes

- Graphs
- Groups (A. Mekler)
- Rooted trees (H. Friedman & L. Stanley)
- Linear orders (H. Friedman & L. Stanley)
- Fields (H. Friedman & L. Stanley)
- Boolean algebras (R. Camerlo & S. Gao)
- Quandles (A.B.-T. & S. Miller)

Another perspective

Recall

A *category* $\mathcal{C}$ consists of

- a class of *objects*, and
- for every pair of objects A and B of $\mathcal{C}$, a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ of *morphisms* from A to B,

along with identity morphisms and a composition (partial) function of morphisms, satisfying suitable axioms. A *functor* is a homomorphism of categories: it takes objects to objects and morphisms to morphisms in a way that respects composition and identities.

E.g.

For any vocabulary $\mathcal{V}$ and any $\mathcal{V}$-sentence φ, we have the category whose objects are all models of φ, and whose morphisms are the $\mathcal{V}$-homomorphisms. Such categories are called *algebraic*.

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is *full* if for all objects C_0 and C_1 of $\mathcal{C}$, F restricted to $\mathrm{Hom}_{\mathcal{C}}(C_0, C_1)$ is surjective: for all $g\colon F(C_0) \to F(C_1)$ there is an $f\colon C_0 \to C_1$ such that $F(f) = g$.

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A functor $F\colon \mathcal{C} \to \mathcal{D}$ is *faithful* if for all objects C_0 and C_1 of $\mathcal{C}$, F restricted to $\mathrm{Hom}_{\mathcal{C}}(C_0, C_1)$ is injective: for all $g\colon F(C_0) \to F(C_1)$ there is at most one $f\colon C_0 \to C_1$ such that $F(f) = g$.

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A functor $F\colon \mathcal{C} \to \mathcal{D}$ is a *full embedding* if it is full and injective both on objects and morphisms.

Note that a fully faithful F can only fail to be an embedding by identifying isomorphic objects.

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Idea

Let's smush these two notions of maximal complexity together!

Definition

A *Borel category* $\mathcal{C}$ is a pair $(\mathsf{Obj}(\mathcal{C}), \mathsf{Mor}(\mathcal{C}))$ of standard Borel spaces endowed with

- Borel functions dom and cod from $\mathsf{Mor}(\mathcal{C})$ to $\mathsf{Obj}(\mathcal{C})$,
- a Borel function id from $\mathsf{Obj}(\mathcal{C})$ to $\mathsf{Mor}(\mathcal{C})$, and
- a Borel function $\circ$ from $\{(g, f) \in \mathsf{Mor}(\mathcal{C})^2 \mid \mathsf{dom}(g) = \mathsf{cod}(f)\}$ to $\mathsf{Mor}(\mathcal{C})$,

such that $(\mathsf{Obj}(\mathcal{C}), \mathsf{Mor}(\mathcal{C}))$ with these functions is a category, that is,

- for all f, g and h in $\mathsf{Mor}(\mathcal{C})$ with $\mathsf{dom}(g) = \mathsf{cod}(f)$ and $\mathsf{dom}(h) = \mathsf{cod}(g)$, $(h \circ g) \circ f = h \circ (g \circ f)$,
- for all $a \in \mathsf{Obj}(\mathcal{C})$, $\mathsf{dom}(\mathrm{id}(a)) = \mathsf{cod}(\mathrm{id}(a)) = a$,
- for all $f \in \mathsf{Mor}(\mathcal{C})$, $f \circ \mathrm{id}(\mathsf{dom}(f)) = f$ and $\mathrm{id}(\mathsf{cod}(f)) \circ f = f$.

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such that $(\mathsf{Obj}(\mathcal{C}), \mathsf{Mor}(\mathcal{C}))$ with these functions is a category, that is,

- for all f, g and h in $\mathsf{Mor}(\mathcal{C})$ with $\mathsf{dom}(g) = \mathsf{cod}(f)$ and $\mathsf{dom}(h) = \mathsf{cod}(g)$, $(h \circ g) \circ f = h \circ (g \circ f)$,
- for all $a \in \mathsf{Obj}(\mathcal{C})$, $\mathsf{dom}(\mathrm{id}(a)) = \mathsf{cod}(\mathrm{id}(a)) = a$,
- for all $f \in \mathsf{Mor}(\mathcal{C})$, $f \circ \mathrm{id}(\mathsf{dom}(f)) = f$ and $\mathrm{id}(\mathsf{cod}(f)) \circ f = f$.

E.g.

Let $\mathsf{Mor}\,\mathcal{V}$ be the subset of $\mathsf{Str}\,\mathcal{V} \times \mathsf{Str}\,\mathcal{V} \times \mathbb{N}^{\mathbb{N}}$ consisting of those $(\mathcal{M}, \mathcal{N}, f)$ such that f is a $\mathcal{V}$-homomorphism from $\mathcal{M} \to \mathcal{N}$. Then $\mathbf{Str}\,\mathcal{V} = (\mathsf{Str}\,\mathcal{V}, \mathsf{Mor}\,\mathcal{V})$ is a Borel category.

Definition

A *Borel functor* from a Borel category $\mathcal{C}$ to a Borel category $\mathcal{D}$ is a pair of Borel functions $F_{\mathsf{Obj}}\colon \mathsf{Obj}(\mathcal{C}) \to \mathsf{Obj}(\mathcal{D})$ and $F_{\mathsf{Mor}}\colon \mathsf{Mor}(\mathcal{C}) \to \mathsf{Mor}(\mathcal{D})$ that respect the functions dom, cod, id and $\circ$.

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- $F_{\mathsf{Obj}}(\mathsf{dom}(f)) = \mathsf{dom}(F_{\mathsf{Mor}}(f))$,
- $F_{\mathsf{Obj}}(\mathsf{cod}(f)) = \mathsf{cod}(F_{\mathsf{Mor}}(f))$,
- $F_{\mathsf{Mor}}(\mathrm{id}(C)) = \mathrm{id}(F_{\mathsf{Obj}}(C))$,
- $F_{\mathsf{Mor}}(g \circ f) = F_{\mathsf{Mor}}(g) \circ F_{\mathsf{Mor}}(f)$.

For $A \in \mathsf{Obj}(\mathcal{C})$ and $f \in \mathsf{Mor}(\mathcal{C})$ we write $F(A)$ for $F_{\mathsf{Obj}}(A)$ and $F(f)$ for $F_{\mathsf{Mor}}(f)$.

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- $F_{\mathsf{Obj}}(\mathsf{dom}(f)) = \mathsf{dom}(F_{\mathsf{Mor}}(f))$,
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Note

Borel functors must respect isomorphisms: if $g \circ f = \mathrm{id}$ and $f \circ g = \mathrm{id}$ then

$$F(g) \circ F(f) = F(\mathrm{id}) = \mathrm{id} \text{ and } F(f) \circ F(g) = F(\mathrm{id}) = \mathrm{id}.$$

But they need not respect non-isomorphism.

Definition

A *functorial Borel reduction* $F\colon \mathcal{C} \to \mathcal{D}$ is a Borel functor from $\mathcal{C}$ to $\mathcal{D}$ such that F_{Obj} is a Borel reduction from the isomorphism relation on $\mathcal{C}$ to the isomorphism relation on $\mathcal{D}$. That is, $C_0 \cong_{\mathcal{C}} C_1$ if and only if $F(C_0) \cong_{\mathcal{D}} F(C_1)$.

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Note

A fully faithful Borel functor is a functorial Borel reduction.