Generalized logics, Vopenka principle and accessible categories

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outline

Generalized Logics

Vopenka's Principle

Generalized Logics and Accessible categories

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All of these Properties can be expressed in Second Order Logic

Definition of Abstract logic

Definition

A logic $\mathcal{L}$ consists of

1. A set $\mathcal{F}$ of the "formulas" of $\mathcal{L}$. For every $\Phi \in \mathcal{F}$ is associated with a vocabulary (or a "signature") $\tau(\Phi)$ which is a set of relation symbols and function symbols. ("The relation and function symbols appearing in Φ"). Also every $\Phi \in \mathcal{F}$ has a set of "free variables", $Fr(\Phi)$ associated with it. In most of our applications $Fr(\Phi)$ will be finite but in may be infinite in some cases.

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2. A relation ("The satisfaction relation")

$$Sat(\mathcal{A}, \Phi, \vec{s})$$

between a formula $\Phi \in \mathcal{F}$, A structure $\mathcal{A}$ whose vocabulary contains $\tau(\Phi)$ and $\vec{s}$ which an assignment of values $\mathcal{A}$ to every variable in $Fr(\Phi)$. This relation is also denoted by $\mathcal{A} \models \Phi(\vec{s})$.

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Definition of abstract Logic continued

We assume that the satisfaction relation of the logic $\mathcal{L}$ satisfies the following properties:

Invariance under isomorphisms Assume that $\pi : \mathcal{A} \to B$ is an isomorphism then $\mathcal{A} \models \Phi(\vec{s})$ iff $\mathcal{B} \models \Phi(\pi \circ \vec{s})$.

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Dependence only on the vocabulary of the formula Assume that the vocabulary of two structures $\mathcal{A}, \mathcal{A}^*$ contain $\tau(\phi)$ and that $\mathcal{A} \restriction \tau(\Phi) = \mathcal{A}^* \restriction \tau(\Phi)$ (In particular $\mathcal{A}$ and $\mathcal{A}^*$ have the same domain) then $Sat(\mathcal{A}, \Phi, \vec{s})$ iff $Sat(\mathcal{A}^*, \Phi, \vec{s})$.

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Note that since the class of formulas of the logic $\mathcal{L}$ is a set, the union of the vocabularies of the formulas of $\mathcal{L}$ is a a fixed set vocabulary, which we shall call the vocabulary or the signature of the logic , denoted by $\tau(\mathcal{L})$.

$\mathcal{L}$ -elementarity

Definition

Let $\mathcal{L}$ be a logic . Let $\mathcal{A}$ be a structure in the vocabulary τ,$\mathcal{B}$ a substructure of $\mathcal{A}$. $\mathcal{B}$ is an $\mathcal{L}$ substructure of $\mathcal{A}$ if for every formula Φ of $\mathcal{L}$ such that $\tau(\Phi) \subseteq \tau$ and an assignment of values in $\mathcal{B}$ to $Fr(\Phi), \vec{s}$

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LST property

Definition

The logic $\mathcal{L}$ has the Lowenheim-Skolem-Tarski property (**LST** property) if there is a cardinal κ such that every structure $\mathcal{A}$ of a vocabulary $\tau(\mathcal{L})$, has a $\mathcal{L}$-substructure of cardinality less than κ. Such a κ is called an LST cardinal for the logic $\mathcal{L}$.

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The logic $\mathcal{L}$ has the strong LST property if there is a cardinal κ such that for every structure $\mathcal{A}$ and every subset of $B \subseteq \mathcal{A}$ there is a $\mathcal{L}$ substructure of $\mathcal{A}, \mathcal{D}$ of cardinal less than κ containing B.

LST for second order logic

Theorem

The following are equivalent:

1. *There exists a supercompact cardinal*
2. *Second Order Logic has the LST property.*
3. *Second order logic has the strong LST property.*

The minimal LST cardinal for second order logic is the smallest supercompact cardinal.

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Let κ be a supercompact cardinal, then it is a LST cardinal for every bounded logic whose vocabulary has cardinality less than κ.

Vopenka Principle

Vopenka principle is the axiom schemata that claims that class sequence $\langle \mathcal{A}_\alpha | \alpha \in \mathit{On} \rangle$ of structures of the same vocabulary , there are $\alpha \neq \beta$ such that $\mathcal{A}_\alpha$ can be embedded into $\mathcal{A}_\beta$.

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1. *Vopenka principle*
2. *Every abstract logic has a LST cardinal*
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Examples: free groups, Free* groups, Metric spaces and more.

The definition of supercompact

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Let B be a class. A cardinal κ is B-supercompact if for very δ there is a transitive class M, an elementary embedding $j : V \to M$ with critical point κ such that $M^\delta \subseteq M$ and $j(B) \cap V_\delta = B \cap V_\delta$.

Vopenka's principle is a supercompact version of Woodin

Theorem

The following are equivalent :

1. *Vopenks's principle*
2. *For every class $B \subseteq On$ there is a B-supercompct cardinal.*
3. *For every class function $F : On \to On$ there is a cardinal κ , a transitive class M and an elementary embedding $j : V \to M$ with critical point κ such that $M^{j(F)(\kappa)} \subseteq M$.*

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We shall sketch a proof that (2) implies that every logic has an LST property.

Proof.

Let $\mathcal{L}$ be a logic . We can code all the relevant information about $\mathcal{L}$ as a class *B* (coding the vocabulary , the set of formulas and the satisfaction relation). Let κ be a *B*-supercompact cardinal. We show that κ is a LST cardinal for $\mathcal{L}$.

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Let $\mathcal{A}$ be a structure in the vocabulary of $\mathcal{L}$. Let λ be large enough so that $\mathcal{A} \in V_\lambda$. and $|V_\lambda| = \lambda$. Let $j : V \to M$ be an elementary embedding with critical point κ such that $j(B) \cap V_\lambda = B \cap V_\lambda$ M is closed under λ sequences. Without loss of generality we can assume $j(\kappa) > \lambda$.

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$j(\mathcal{L}$ is a logic in M coded by $j(B)$, but since $j(B) \cap V_lambda = B \cap V_\lambda$, $\mathcal{L}$ and$j(\mathcal{L}$ agree on satisfaction in any structure belonging to V_λ , in particular on $\mathcal{A}$. □

Continued.

For every formula of $\mathcal{L}$ ϕ an an assignment of elements of $\mathcal{A}$ to the free variables of π, we have

$$\mathcal{A} \models_{\mathcal{L}} \phi(\vec{a}) \Leftrightarrow j(\mathcal{A}) \models_{j(\mathcal{L})} \phi(j(\vec{a})$$

By *j* being elementary .

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.
Combining these two equivalences and the fact that $j \restriction \mathcal{A} \in M$ we get that: M satisfies that $j \restriction \mathcal{A}$ is a $j(\mathcal{L}$ embedding of $\mathcal{A}$ into $j(\mathcal{A})$. So M believes that $j(\mathcal{A}$ has an $j(\mathcal{L}$ elementary substructure of cardinality less than $j(\kappa)$. By elementarity of j V believes that $j(\mathcal{A}$ has a $\mathcal{L}$ elementary substructure of cardinality less than κ. □

Goal

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Given a category $\mathcal{C}$. Set a general framework for expressing properties of the objects in $\mathcal{C}$. Naturally we are interested only in properties which are invariant under the isomorphisms of $\mathcal{C}$.

We also have to define an analogue of assignment of elements of the domain to variables. We shall use a distinguished class of morphisms For this we shall use a distinguished class of morphisms from another category to the objects of $\mathcal{C}$.

Category of assignment for a category $\mathcal{C}$

Definition

Given a category $\mathcal{C}$. An category of assignments for $\mathcal{C}$ is a category $\mathcal{A}$, a functor $\mathcal{F} : \mathcal{A} \to \mathcal{C}$ such that for every $X \in \mathcal{A}$ and $Y \in \mathcal{C}$ we have a set of distinguished assignments $As(X, Y)$, which is a subset of $Hom(\mathcal{F}(X), Y)$ such that

- $X, X^* \in \mathcal{A}$, $Y \in \mathcal{C}$,$X \xrightarrow{\eta} X^*$ is a morphism of $\mathcal{A}$, $\pi \in As(X^*, Y)$ then $\pi \circ \mathcal{F}(\eta) \in As(X, Y)$.

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- $Y \xrightarrow{\eta} Y^*$ is an isomorphism of $\mathcal{C}$, $\sigma \in As(X, Y)$ then $\eta \circ \sigma \in As(X, Y^*)$.

We shall say that the pair (X, σ) is an assignment if for Y the codomain of σ we have $\sigma \in As(X, Y)$.

Examples of Categories of assignments

Example

The category $\mathcal{C}$ is the category of first order structures of a given countable signature τ. The category of assignments $\mathcal{A}$ for first order logic will be all countable structures of signature τ with additional constants $\langle c_i | i \in \omega \rangle$. (An object of $\mathcal{A}$ can be considered to be a structure in $\mathcal{C}$ together with an assignment to the variables $\langle x_i | i \in \omega \rangle$. The functor $\mathcal{F}$ will be the forgetful functor. $As(X, Y)$ will be all the morphisms from $\mathcal{F}(X)$ into Y.

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Example

Suppose we want to have a logic that will be able to express properties of objects in **Top** , the topology of topological spaces. The assignment category will be the category of topological spaces, where the morphisms are restricted to subspace embeddings. The Functor $\mathcal{F}$ will be the identity map (both on the objects and the morphisms.) $As(X, Y)$ will be all subspace embeddings of X into Y .

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3. The satisfaction relation $\mathcal{S}(\phi, X, \sigma, Y)$ where ϕ is a formula, $X \in \mathcal{A}$, $Y \in \mathcal{C}$, $\sigma \in As(X, Y)$,$\sigma \in \mathcal{A}(\phi)$. Intuitively when the satisfaction relation holds between a formula ϕ ,an assignment σ and an object of $\mathcal{C}$, Y it means that σ is an assignment which is meaningful for ϕ , and σ is assignment into the object Y which makes the formula ϕ true.

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We also write $Y \models \phi(X, \sigma)$ if $\mathcal{S}(\phi, X, \sigma, Y)$ holds .

Invariance of satisfaction under isomorphism

We assume that the satisfaction relation $\mathcal{S}$ satisfies the following two properties:

1. Let $Y \xrightarrow{\eta} Y^*$ be an isomorphism of the category $\mathcal{C}$. ϕ a formula, $X \in \mathcal{A}$, $\sigma \in As(X, Y)$. Then $Y \models \phi(X, \sigma)$ iff $Y^* \models \phi(X, \eta \circ \sigma)$.

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2. Let $X* \xrightarrow{\eta} X$ be an isomorphism of the category $\mathcal{A}$, ϕ a formula,$\sigma \in As(X, Y)$ then $Y \models \phi(X, \sigma)$ iff $Y \models \phi(X^*, \sigma \circ \eta)$.

$\mathcal{L}$-elementary morphisms

Definition

Let $\mathcal{L}$ be an abstract logic for the category $\mathcal{C}$. Let $Y \xrightarrow{\eta} Y^*$ be a morphism of the category $\mathcal{C}$. Then η is $\mathcal{L}$-elementary if for every formula of $\mathcal{L}$, ϕ, $X \in \mathcal{A}$ $\sigma \in As(X, Y)$ we have σ is relevant for ϕ iff $\eta \circ \sigma$ is relevant for ϕ and $Y \models \phi(X, \sigma)$ iff $Y^* \models \phi(X, \eta \circ \sigma)$.

$\mathcal{L}$-elementary morphisms

Definition

Let $\mathcal{L}$ be an abstract logic for the category $\mathcal{C}$. Let $Y \xrightarrow{\eta} Y^*$ be a morphism of the category $\mathcal{C}$. Then η is $\mathcal{L}$-elementary if for every formula of $\mathcal{L}$, ϕ, $X \in \mathcal{A}$ $\sigma \in As(X, Y)$ we have σ is relevant for ϕ iff $\eta \circ \sigma$ is relevant for ϕ and $Y \models \phi(X, \sigma)$ iff $Y^* \models \phi(X, \eta \circ \sigma)$.
$Y \xrightarrow{\eta}_{\mathcal{L}} Y^*$ is a notation for "η is a $\mathcal{L}$-elementary morphism of Y to Y^*.

Definition

The logic $\mathcal{L}$ for the category $\mathcal{C}$ has the LST property if there is a cardinal λ such that for every $Y, Z \in \mathcal{C}$. $Z \xrightarrow{\eta} Y$ a morphism of $\mathcal{C}$ such that Z is λ presentable , there is a λ presentable Z^* , morphisms $Z \xrightarrow{\sigma} Z^* \xrightarrow{\rho} Y$ such that ρ is $\mathcal{L}$-elementary .

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A cardinal λ as above is an LST cardinal for the logic $\mathcal{L}$.

Vopenka's principle and LST property for logics for accessible categories

Theorem (Vopenka's principle)

Let $\mathcal{L}$ be a logic for the category $\mathcal{C}$. Assume that the category of assignments of $\mathcal{L}$ is a small category and that $\mathcal{C}$ is accessible , then $\mathcal{L}$ has the LST property.

Sketch.

On the board , time permiting. $\square$

Key Lemma

Lemma

Let $\mathcal{L}$ be a logic for the accessible category $\mathcal{C}$, let B be a predicate "coding" $\mathcal{L}$ and $\mathcal{C}$. Let λ be large enough so that

1. *$\mathcal{C}$ is λ accessible.*
2. *$|\mathcal{D}| < \lambda$ where $\mathcal{D}$ is a set of λ-presentable objects such that every object in $\mathcal{C}$ is a λ-directed colimit of members of it.*
3. *The category of $\mathcal{A}$ of assignments of $\mathcal{L}$ has cardinality less than λ.*
4. *The set of formulas of $\mathcal{L}$ has cardinality less than λ*

. Then for every $Y \in \mathcal{C}$ there is a cardinal $\delta(Y)$ such that if $j : V \to M$ is an elementary embedding with critical point $\geq \lambda$ and $j(B) \cap \delta(Y) = B \cap \delta(Y)$. Then in M there is an object Y^* isomorphic to Y and a $\mathcal{L}$ elementary morphism of Y^* to $j(Y)$.

Corollary

Let $\mathcal{C}, \mathcal{L}, B, \lambda$ be as the above lemma. Then every $\kappa \geq \lambda$ which is B-supercompact is a LST cardinal for the logic $\mathcal{L}$.

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Let $\mathcal{C}, \mathcal{L}, B, \lambda$ be as the above lemma. Then every $\kappa \geq \lambda$ which is B-supercompact is a LST cardinal for the logic $\mathcal{L}$.

Vopenka's principle implies that for every $B \subseteq On$ there are unboundedly many *B*-supercompact cardinals, hence the theorem follows.

Thank you for your attention!