Prying HOD and V apart with HOD-supercompactness

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It is a folklore result that an equivalent characterization of a *Vopěnka cardinal* is that it is a Woodinized supercompact cardinal.

Definition

A cardinal δ is *Woodinized supercompact* if for every $X \subseteq V_\delta$, there is a $\kappa < \delta$ such that for every $\lambda < \delta$, there is a λ-supercompactness embedding $j : V \to M$ with $\text{crit}(j) = \kappa$ such that $X \cap V_\lambda = j(X) \cap V_\lambda$.

We care particularly when $X = \text{HOD} \cap V_\delta$, the set of hereditarily ordinal definable sets in V_δ.

Definition of L

Now some background.

Let's talk about canonical inner models.

- The universe of constructible sets, L was introduced by Gödel in his proof of consistency of the Axiom of Choice and of the Generalized Continuum Hypothesis.
- L is the minimal standard inner model of ZFC.
- All models of set theory with the same ordinals have exactly the same version of L.

More about L

- L can be thought of as being built in "stages" resembling the von Neumann universe, V. In von Neumann's universe, at a successor stage, one takes $V_{\alpha+1}$ to be the set of all subsets of the previous stage, V_α.
- By contrast, In Gödel's constructible universe L, one uses only those subsets of the previous stage that are:
  - definable by a formula in the formal language of set theory with parameters from the previous stage
  - with the quantifiers interpreted to range over the previous stage

More about L

- L has a well understood fine structure.
- Many natural questions in mathematics, such as the Continuum Hypothesis, can be decided by working in the theory $\mathsf{ZFC} + V = L$.
- However, L can only contain very weak large cardinals.
- The holy grail of inner model theory is to build inner models containing suitable large cardinals, particularly a supercompact cardinal, but with as many of the fine structural properties of L as possible.

"L-like" models

- These inner models, called extender models, are constructed in the same way as L but incorporating certain approximations to the relevant elementary embeddings while weakening some of the fine structural framework.
- Woodin uses the term Ultimate-L to refer to the hypothetical inner model that includes supercompact cardinals.
- This model would be robust enough with respect to forcing that one would be able to answer essentially all natural questions by working in $V =$ Ultimate-L.
- If $V =$ Ultimate-L is definable, then it is contained in HOD, the class of hereditarily ordinal definable sets, and then HOD must be "close to V" in a certain sense.

Definition of HOD

- A set is said to be ordinal definable (OD) if it can be defined in terms of a finite number of ordinals by a first order formula.
- HOD is the class of sets that are hereditarily ordinal definable.
- $x \in$ HOD can be formally expressed by a formula that is Σ_2.
- HOD is a canonical inner model and is consistent with all known large cardinal axioms and many independence results.

Definition of the L Dichotomy

The following dichotomy follows from Jensen's Covering Lemma for L and says either L is "close to V" or L is "far from V".

Definition of L Dichotomy

Exactly one of the following holds:

- L computes the singular cardinals and their successors correctly.
- All uncountable cardinals are inaccessible in L.

Canonical inner models other than L have been defined and shown to satisfy corresponding dichotomies, all of these inner models are contained in HOD.

Definition of the HOD Dichotomy

The following result of Woodin expresses a similar idea for HOD; that either HOD is "close to V" or else HOD is "far from V".

Theorem: (Woodin, '12)

Assume that δ is an extendible cardinal. Then exactly one of the following holds.

1. For every singular cardinal $\gamma > \delta$, γ is singular in HOD and $(\gamma^+)^{HOD} = \gamma^+$.
2. Every regular cardinal greater than δ is measurable in HOD.

This result of Woodin extends the dichotomy of core models to HOD itself and is called the HOD Dichotomy.

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Definition of the HOD Conjecture

Key Idea

Woodin's HOD Conjecture proposes that HOD is close to V in a particular way; namely that (1) of the HOD dichotomy holds.

More formally,

HOD Hypothesis:

There exists a proper class of regular cardinals which are not ω-strongly measurable in HOD.

ω-strongly measurable in HOD

An uncountable regular cardinal λ is defined to be ω-strongly measurable in HOD iff there is $\kappa < \lambda$ such that:

1. $(2^\kappa)^{HOD} < \lambda$ and
2. there is no partition $\langle S_\alpha | \alpha < \kappa \rangle$ of $\text{cof}(\omega) \cap \lambda$ into stationary sets such that $\langle S_\alpha | \alpha < \kappa \rangle \in HOD$.

Definition of the HOD Conjecture

Then, the HOD Conjecture is just one sentence;

The HOD Conjecture

ZFC + "There is a supercompact cardinal" proves the HOD Hypothesis.

If there is an extendible cardinal δ, and (1) of the HOD Dichotomy holds, then the HOD Hypothesis holds.

- Woodin posits that we probably cannot force (2) of the HOD Dichotomy with any of the known large cardinal axioms.
- There seem to be no current methods to produce a model of set theory even with a supercompact cardinal in which the HOD Hypothesis fails.
- However, the exploration of to what extent HOD and V can be forced apart
  and to what extent large cardinal and other properties are exhibited in HOD, has been quite fruitful.

Separating HOD and V

Some recent results:

1. Large cardinals in V that are not large in HOD. (Cheng, Friedman, Hamkins, 2015)
2. Different cardinals: a model is produced where the α^+ of HOD is strictly less than α^+ for every infinite cardinal α. (Cummings, Friedman, Golshani, 2015)
3. The continuum function: Building models where 2^κ is large in V but small in HOD. (Golshani, 2017)
4. Using Cofinality and Covering properties: Building models with singular cardinals in V that are measurable or regular and not measurable in HOD. (Ben-Neria, Unger, 2017)
5. Forcing get that any model of ZFC can be made into the Mantle, generic Mantle, HOD and generic HOD of another model V. The HOD of this model V is not a ground model. (Fuchs, Hamkins, Reitz, 2011)

None of these results produce a model of set theory with a supercompact cardinal in which the HOD Hypothesis fails.

Definition of a supercompact cardinal

Let us recall the definition of a supercompact cardinal.

Suppose that κ is a regular cardinal and that $\kappa < \lambda$.

Definition of a supercompact cardinal

A cardinal κ is λ-supercompact if for every $\lambda > \kappa$ there is an elementary embedding $j : V \to M$ with critical point $\kappa, j(\kappa) > \lambda, M^\lambda \subseteq M$.

Definition of a supercompact cardinal

Equivalently,

Definition of a supercompact cardinal

- Let $P_\kappa(\lambda) = \{\sigma \subseteq \lambda \mid |\sigma| < \kappa\}$.
- A measure $\mu \subseteq P(P_\kappa(\lambda))$ on is normal if every function f such that $\{\sigma \in P_\kappa(\lambda) \mid f(\sigma) \in \sigma\} \in \mu$ is constant on a set in μ.
  An measure μ is fine if $\{\sigma \in P_\kappa(\lambda) \mid \alpha \in \sigma\} \in \mu$ for every $\alpha < \lambda$.

A cardinal κ is λ-supercompact if there exists a normal, fine measure μ on $P_\kappa(\lambda)$ which is κ-complete.

A cardinal κ is supercompact if it is λ-supercompact for every λ.

Downward Transference-motivating HOD supercompactness

When are large cardinal properties are transferred into HOD?

In his work on the HOD Conjecture, Woodin isolates the concept of of an inner model N being a weak extender model for δ is supercompact:

Weak extender models for δ is supercompact

N is an inner model of ZFC, and for every $\gamma > \delta$, there is a δ-complete normal fine measure U on $\mathcal{P}_\delta(\gamma)$ such that

- $N \cap \mathcal{P}_\delta(\gamma) \in U$ and
- $U \cap N \in N$

Let us look at a key idea of when (sufficiently) large cardinals relativize down to HOD.

Universality Theorem

Suppose the HOD Hypothesis holds and δ is HOD-supercompact. If $j : \mathrm{HOD} \cap V_{\gamma+1} \longrightarrow M = \mathrm{HOD} \cap V_{j(\gamma)+1}$ is an elementary embedding with $cp(j) \geq \delta$. Then $j \in \mathrm{HOD}$

The following theorem is a consequence of the Universality Theorem:

Theorem (Woodin, 2017)

Suppose N is a weak extender model for δ is supercompact, and $\kappa \geq \delta$ is supercompact. Then κ is supercompact in N.

Downward Transference-motivating HOD supercompactness

Let HOD be an inner model, and let $j : V \to M$ be an elementary embedding, where M is transitive. Then by $j(\mathrm{HOD})$ we mean:

$$\bigcup_{\lambda \in \mathrm{ORD}} j(V_\lambda \cap \mathrm{HOD}) = \mathrm{HOD}^M$$

.

Definition of HOD-supercompactness

A cardinal κ is HOD-supercompact if for every $\lambda > \kappa$, there is an elementary embedding $j : V \to M$, where M is transitive, such that $\mathrm{crit}(j) = \kappa$, $\lambda < j(\kappa)$, $M^{V_\lambda} \subseteq M$ and

$$j(\mathrm{HOD}) \cap V_\lambda = \mathrm{HOD} \cap V_\lambda$$

.

Connection between HOD-supercompactness and HOD Hypothesis

Some key connections between HOD-supercompactness and the HOD Hypothesis:

- By a theorem of Woodin's if there is an extendible cardinal δ, then δ is HOD-supercompact.
- Then under the assumption of the HOD Hypothesis, HOD is a weak extender model for δ is supercompact.
- From the Universality theorem and its consequences, it follows that supercompact cardinals greater than δ are supercompact in HOD.

This gives us the following fact:

Fact

Under the HOD Hypothesis, any HOD-supercompact cardinal is supercompact in HOD.

Let's now separate the notions of:

- supercompactness
- HOD-supercompactness
- supercompactness in HOD

A reasonable place to begin is to ask, if κ is supercompact, is it HOD-supercompact?

Sargsyan answered this question in the negative in the following result.

Sargsyan (2009)

Suppose κ is a supercompact cardinal. Then there is a forcing extension of V in which κ is supercompact, but not HOD-supercompact.

This theorem does not speak to whether κ is supercompact in HOD.

Then, does HOD-supercompactness imply supercompactness in HOD?

- In general, for arbitrary inner models $N \subseteq V$, N-supercompactness does not imply supercompactness in N.
- For example, every supercompact cardinal κ is (trivially) L-supercompact while not being supercompact in L.
- Can we get the analogous version for HOD? That is, a model where there is a cardinal κ that is HOD-supercompact but not supercompact in HOD?
- This turns out to be equivalent to forcing the failure of the HOD Hypothesis.
- That is because by the fact mentioned on the previous slide, if the HOD Hypothesis holds, then HOD-supercompactness implies supercompactness in HOD. So a model with a HOD-supercompact cardinal that's not supercompact in HOD would not satisfy the HOD Hypothesis.

Separation Lemma

Separations Lemma

Suppose there is a supercompact cardinal. Then there exist class forcing notions $\mathbb{P}$ (possibly trivial) to force each of the following to hold:

1. ZFC+ there is a cardinal that is supercompact, HOD-supercompact, and supercompact in HOD.
2. ZFC+ there is a cardinal that is not supercompact, (hence not HOD-supercompact), and supercompact in HOD.
3. ZFC+ there is a cardinal that is supercompact, not HOD-supercompact and supercompact in HOD.
4. ZFC+ there is a cardinal that is supercompact, not HOD-supercompact and not supercompact in HOD.
5. ZFC+ there is a cardinal that is not supercompact, not HOD-supercompact and not supercompact in HOD. (trivial)

Before beginning the proof, we mention some useful results:

The following is a folklore result used extensively in proving things (relating to forcing) about HOD:

Folklore

Let $\mathbb{P}$ be an almost homogeneous notion of forcing which is ordinal definable, and let g be $\mathbb{P}$-generic. Then

$$\mathsf{HOD}^{V[g]} \subseteq \mathsf{HOD}^{V}.$$

Coding sets into HOD

Let's talk about "coding a set A into HOD" via forcing:

Again, a set is OD if it can be defined in terms of a finite number of ordinals by a first order formula.

For example, one can use the continuum function in the following way:

- Suppose you wanted to code $A \subseteq \kappa$ into HOD (and assume GCH holds.)
- Consider $\lambda > \kappa$ and the block of κ many immediate successors of λ.
- So there is a λ_α that corresponds to every $\alpha < \kappa$.
- Force to kill GCH at λ_α according to whether $\alpha \in A$. That is, in your forcing extension $\alpha \in A \iff 2^{\lambda_\alpha} = \lambda_\alpha^{++}$.

Proof of Lemma

To keep the following proofs relatively free from technicalities, let us introduce the following terminology.

Definition: Securely Coded

V is securely coded if whenever g is generic over V for some set-sized forcing notion $\mathbb{P}$, then $V \subseteq \mathrm{HOD}^{V[g]}$.

If V is securely coded:

- We have $V = \mathrm{HOD}$, because V is a forcing extension of itself by trivial forcing.
- So we have that $\mathrm{HOD} \subseteq V \subseteq \mathrm{HOD}$,
- But moreover, every set in V is ordinal definable in any set-forcing extension of V.

Proof of Lemma

- It is well-known that if κ is supercompact, then there is a set forcing extension $V[g]$ in which κ is indestructible under $<\kappa$-directed closed forcing. (Laver, 1978).
- If κ is an indestructible supercompact cardinal, then $V_\kappa \subseteq \text{HOD}$. (Because for any set $a \in V_\kappa$, there is a $<\kappa$-directed closed forcing that codes it into the continuum function beyond κ, and this forcing preserves the supercompactness of κ, so that the fact that a is coded is reflected below κ.)
- One can then perform a class forcing that codes itself unboundedly often into the continuum function above κ to get a model which that is securely coded.
- We thus obtain a model that is securely coded and in which κ is indestructible.

We will call such a model $\bar{V}$, that is a model that is securely coded and where κ is indestructibly supercompact.

Proof of Lemma

This brings us to the following observation:

Observation

Suppose κ is supercompact and indestructible under $<\kappa$-directed closed forcing (or strong and indestructible under $\leq\kappa$-strategically closed forcing). Then the following are equivalent:

1. κ is HOD-supercompact
2. $V = \text{HOD}$.

Proof of Observation:

(2) $\rightarrow$ (1) is obvious.

Proof of Lemma

To show (1) $\to$ (2):

- Since κ is indestructibly supercompact, $V_\kappa \subseteq \mathrm{HOD}$.
- Fix any $A \subseteq \gamma$, $\gamma > \kappa$.
- There exists $\lambda > \gamma$ such there is a λ-supercompactness embedding, where $j(\kappa) > \lambda$ and $M^\lambda \subseteq M$.
- By elementarity, $M_{j(\kappa)} \subseteq \mathrm{HOD}$, and since $j(\kappa) > \lambda$, $A \in \mathrm{HOD}^M = j(\mathrm{HOD})$.
- By HOD-supercompactness, $j(\mathrm{HOD}) \cap V_\lambda = \mathrm{HOD} \cap V_\lambda$, therefore $A \in \mathrm{HOD}$.

Proof of Lemma

Proof of (1):

- If $V = \text{HOD}$, any supercompact cardinal is HOD-supercompact.
- Also obviously supercompact in HOD.
- $\bar{V}$ will work.

Proof of (2):

- Force to get $\bar{V}$ where κ is indestructibly supercompact and securely coded.
- Let $c \subseteq \text{Coll}(\omega, \kappa)$ be $\bar{V}$-generic.
- $\bar{V}[c] \models \kappa$ is no longer supercompact and therefore not HOD-supercompact.
- Collapse forcing is homogeneous therefore $\text{HOD}^{\bar{V}[c]} \subseteq \text{HOD}^{\bar{V}} = \bar{V}$.
- Therefore κ is supercompact in $\text{HOD}^{\bar{V}[c]}$.

Proof of Lemma

Proof of (3):

- Force to get $\bar{V}$ where V is securely coded and κ is indestructibly supercompact by $< \kappa$-directed closed forcing.
- Let $g \subseteq \mathrm{Add}(\kappa, 1)$ be $\bar{V}$-generic.
- Since $\mathrm{Add}(\kappa, 1)$ is $< \kappa$-directed closed, κ remains indestructibly supercompact in $V[g]$.

  To show κ is supercompact in HOD:

- Cohen forcing is homogeneous therefore $\mathrm{HOD}^{\bar{V}[g]} \subseteq \mathrm{HOD}^{\bar{V}} = \bar{V}$.
- Therefore κ is supercompact in $\mathrm{HOD}^{\bar{V}[g]}$.

Proof of Lemma

To show κ is not HOD-supercompact:

- $g \notin \mathrm{HOD}^{V[g]}$, since $\mathrm{Add}(\kappa, 1)$ is weakly homogenous.
- Then, $\mathrm{HOD}^{V[g]} \neq V[g]$
- Since κ is indestructibly supercompact, by the Observation, κ is not HOD-supercompact.

Supercompact, not HOD-supercompact, not supercompact in HOD

Proof of (4):

In (Cheng, Friedman and Hamkins, 2015) a model is produced where a proper class of supercompact cardinals are not even weakly compact in HOD.

To show that this class of cardinals are not HOD-supercompact we use the following definitions and lemma:

Definition of (N,X)-measurability

Let N be an inner model. Let κ be a cardinal and X a set. Then κ is (N, X)-*measurable* if there is a $j : V \to M$ with $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \mathrm{rnk}(X)$ and

$$j(N) \cap X = N \cap X.$$

In particular we are interested in $(\mathrm{HOD}, P(\kappa))$-measurability which gives us,

$$j(\mathrm{HOD}) \cap P(\kappa) = \mathrm{HOD} \cap P(\kappa).$$

Supercompact, not HOD-supercompact, not supercompact in HOD

(Recall) the concept of ineffability:

Ineffable cardinal

An inaccessible cardinal κ is *ineffable* if for every sequence $\langle a_\alpha \mid \alpha < \kappa \rangle$ such that $a_\alpha \subseteq \alpha$, for every $\alpha < \kappa$, it follows that there is a set $A \subseteq \kappa$ such that

$$\{\alpha < \kappa \mid A \cap \alpha = a_\alpha\}$$

is stationary in κ.

This brings us to the following general lemma which we are applying to HOD:

Lemma: Downward transference of Ineffability

Suppose that N is an inner model such that κ is $(N, \mathcal{P}(\kappa))$-measurable. Then κ is ineffable in N.

Supercompact, not HOD-supercompact, not supercompact in HOD

Sketch of proof of lemma:

- Choose $\langle a_\alpha \mid \alpha < \kappa \rangle \in \text{HOD}$.
- Let $j : V \to M$ witness $(\text{HOD}, \mathcal{P}(\kappa))$-measurability.
- Take $j(\langle a_\alpha \mid \alpha < \kappa \rangle) = \langle a'_\alpha \mid \alpha < j(\kappa) \rangle$
- Let $A = a'_\kappa$. Let $B = \{\alpha < \kappa \mid A \cap \alpha = a_\alpha\}$.
- Claim: B is stationary in κ.
- Take $C \subseteq \kappa$, C club.$\kappa \in j(C)$.
- M witnesses $\exists \alpha \in j(C)$ such that $j(A) \cap \alpha = a'_\alpha$.
- (κ witnesses it in M).
- By elementarity, there is a witness in V. So B is stationary in V.
- Since stationarity goes down to HOD, κ is ineffable in HOD.

Supercompact, not HOD-supercompact, not supercompact in HOD

This proper class of supercompact cardinals are not HOD-supercompact. This is because otherwise, these cardinals would be ineffable in the HOD of the extension and they are not even weakly compact there.

Proof of (5):
A model of $V = L$ will trivially satisfy this. Not technically the proof, but ....

Questions relating to $C^{(n)}$ cardinals

- Can the concepts of $C^{(2)}$ or $C^{(n)}$-supercompactness be separated from HOD-supercompactness?
- What about Σ_n-supercompactness?
- Can they be related to some sort of weakening of *Vopěnka's principle*?
- If so, what implication will that have for the failure of the HOD Hypothesis?

Thank you

for listening to my talk on rainbows and unicorns. Also the Loch Ness monster.