Preservation of colimits under changes of universe

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Accessible Categories and Their Connections

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Motivation

Zhen Lin Low

28 November 2014

Abstract

The Grothendieck universe axiom asserts that every set is a member of some set-theoretic universe **U** that is itself a set. One can then work with entities like the category of all **U**-sets or even the category of all locally **U**-small categories, where **U** is an "arbitrary but fixed" universe, all without worrying about which set-theoretic operations one may legitimately apply to these entities. Unfortunately, as soon as one allows the possibility of changing **U**, one also has to face the fact that universal constructions such as limits or adjoints or Kan extensions could, in principle, depend on the parameter **U**. We will prove this is *not* the case for adjoints of accessible functors between locally presentable categories (and hence, limits and Kan extensions), making explicit the idea that "bounded" constructions do not depend on the choice of **U**.

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Question: Under which conditions are limits or colimits preserved under a change of universe?

A first difficulty: What does it **mean** to change a category from a universe to another universe?

Even more basic: What **is** a category? (A *set* or a *formula*?)

A second difficulty: Perhaps a category may change from a universe to another universe in nonequivalent ways.

Defining categories

Let κ be a (strongly) innaccessible cardinal. Let V_κ denote the set of all sets of rank less than κ. Hence $V_{\kappa+1} = \mathcal{P}(V_\kappa)$.

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A **κ-category** $\mathcal{C}$ consists of a subset $\mathrm{Ob}(\mathcal{C}) \subseteq V_\kappa$ together with sets $\mathcal{C}(X, Y) \in V_\kappa$ for all $(X, Y) \in \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{C})$ equipped with associative composition functions

$$\mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \longrightarrow \mathcal{C}(X, Z)$$

and a left and right identity $\mathrm{id}_X \in \mathcal{C}(X, X)$ for every $X \in \mathrm{Ob}(\mathcal{C})$.

Thus, a κ-category is a quadruple (O, M, C, I) of sets in $V_{\kappa+1}$ satisfying suitable conditions. It is called **small** if it is in V_κ.

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Example 1: If $\mathcal{C}$ is a λ-category, then

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Example 2: Let Set_κ be the category of sets of rank less than κ with their functions. Then Set_* is a bundle of categories.

Example 3: Let Gr_κ be the category of groups whose underlying set has rank less than κ with their homomorphisms.

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A (definable) **category** is a class

$$\mathcal{C} = \{X \mid \phi(X, P)\}$$

where ϕ is a first-order formula in four variables with a set of parameters P such that, for all inaccessible cardinals κ,

$$V_{\kappa+1} \models \phi(O, M, C, I, P)$$

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If $\mathcal{C}$ is a category defined by a formula $\phi(X, P)$, then

$$\mathcal{C}_\kappa = \{(O, M, C, I) \in V_{\kappa+1} \mid V_{\kappa+1} \models \phi(O, M, C, I, P)\}$$

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Defining categories

Example 4: Let R be a ring with 1. The category of R-modules is defined by a formula $\phi(X, R)$ stating that X is an R-module.

More generally, for every theory T with signature Σ, the category of models of T (i.e., Σ-structures satisfying the sentences in T) is defined by a formula with parameters Σ and T.

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Example 5: If $\mathcal{C}$ is a λ-category, then the formula

$$\exists\lambda \wedge (X \in \mathcal{C})$$

defines $\mathcal{C}$, where λ and $\mathcal{C}$ are parameters.

The associated bundle is $\mathcal{C}_\kappa = \mathcal{C}$ if $\kappa \geq \lambda$ and $\mathcal{C}_\kappa = \emptyset$ otherwise.

Change of universe

Let $\mathcal{C}_*$ be a bundle of categories and let $\kappa < \kappa'$ be inaccessible cardinals. A **change of universe** of $\mathcal{C}_*$ from κ to κ' is a faithful functor $F\colon \mathcal{C}_\kappa \to \mathcal{C}_{\kappa'}$ that is injective on objects.

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If $\mathcal{C}_*$ is definable by an **upward absolute** formula ϕ, then there is a canonical embedding $\mathcal{C}_\kappa \hookrightarrow \mathcal{C}_{\kappa'}$ for all $\kappa < \kappa'$. The κ'-category $\mathcal{C}_{\kappa'}$ is called the **logical enlargement** of $\mathcal{C}_\kappa$.

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A category is called **absolute** if it is definable by an (upward and downward) absolute formula.

Example 6: If Σ is a λ-ary signature and T is a set of sentences in the language of Σ, then the categories of Σ-structures and models of T are absolute between transitive classes closed under sequences of length less than λ and containing Σ and T.

Locally presentable enlargements

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For a small κ-category B, let $\mathbf{Cont}_{\boldsymbol{\lambda}}(\boldsymbol{B}, \mathbf{Set}_{\boldsymbol{\kappa}})$ denote the full subcategory of $\mathrm{Fun}(B, \mathrm{Set}_\kappa)$ of **$\boldsymbol{\lambda}$-continuous** functors, i.e., functors $B \to \mathrm{Set}_\kappa$ that preserve all λ-small limits that exist in B.

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If A is a small κ-category, then the Yoneda embedding factors as

$$A \hookrightarrow \mathrm{Cont}_\lambda(A^{\mathrm{op}}, \mathrm{Set}_\kappa) \hookrightarrow \mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Set}_\kappa).$$

Locally presentable enlargements

While

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is a **λ-free cocompletion** of A, meaning that it preserves all λ-small colimits that exist in A and every functor $A \to \mathcal{C}$ with $\mathcal{C}$ cocomplete that preserves λ-small colimits extends uniquely to a colimit-preserving functor $\mathrm{Cont}_\lambda(A^{\mathrm{op}}, \mathrm{Set}_\kappa) \to \mathcal{C}$.

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The **Representation Theorem** of Adámek–Rosický states that a κ-category is locally λ-presentable if and only if it is equivalent to $\mathrm{Cont}_\lambda(B, \mathrm{Set}_\kappa)$ for some small κ-category *B*.

Locally presentable enlargements

Let $\mathcal{C}$ be a locally λ-presentable κ-category, where $\lambda < \kappa$.

Let A be a skeleton of the full subcategory of λ-presentable objects in $\mathcal{C}$. Then $\mathcal{C} \simeq \mathsf{Cont}_\lambda(A^{\mathsf{op}}, \mathsf{Set}_\kappa)$.

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If $\kappa' > \kappa$, the κ'-category $\mathrm{Cont}_\lambda(A^{\mathrm{op}}, \mathrm{Set}_{\kappa'})$ is called the **locally presentable enlargement** of $\mathcal{C}$ from κ to κ'.

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By the Representation Theorem, $\mathrm{Cont}_\lambda(A^{\mathrm{op}}, \mathrm{Set}_{\kappa'})$ is locally λ-presentable. Furthermore, the embedding

$$\mathrm{Cont}_\lambda(A^{\mathrm{op}}, \mathrm{Set}_\kappa) \hookrightarrow \mathrm{Cont}_\lambda(A^{\mathrm{op}}, \mathrm{Set}_{\kappa'})$$

preserves colimits (and limits), since they are calculated in sets.

Example

The category Gr of groups can be defined in at least two (equivalent) ways:

- By means of a formula $\phi(X)$ stating that X is a group.
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The same happens for all categories of structures or categories of models of theories.

Warnings

- An embedding $\mathrm{Mod}\, T \hookrightarrow \mathrm{Str}\,\Sigma$ need not preserve colimits (Rosický 1994, Bagaria–Brooke-Taylor 2013).

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- There is an ω-sequence of monads on Set_κ whose colimit depends on κ (Bowler 2012).
- The property of being locally presentable need not be preserved under changes of universe.

Warnings

There are inaccessible cardinals $\kappa < \kappa'$ and a definable category $\mathcal{C}$ such that $\mathcal{C}_\kappa$ is locally presentable but $\mathcal{C}_{\kappa'}$ is not.

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For instance, if λ is the smallest **measurable cardinal,** then the bundle $\mathcal{C}_*$ may consist of locally presentable categories or not, depending on set theory.