Erdős-Rado classes; or What categories characterize finitely accessible categories by faithful embeddings into them?

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Accessible Categories and Their Connections
Leeds, England

Motivation (for the category theorists)

Theorem (Morley, as read by Makkai-Paré)

If $\mathbb{K}$ is a large, finitely accessible category, then there is a faithful functor $\Phi : LinOrd \to \mathbb{K}$ preserving directed colimits.

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- What other categories have this property?
  - Makkai-Paré call this **minimality**, but don't elaborate
- Although the statement is categorical, the proof and techniques are really model theoretic
  - Ehrenfuect-Mostowski models

Motivation (for the model theorists)

- Ramsey classes are popular (for certain definitions of popular)
- They are, for a logician, the index classes where you can build indiscernibles in first-order (or compact logics more generally)

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- Ramsey classes are popular (for certain definitions of popular)
- They are, for a logician, the index classes where you can build indiscernibles in first-order (or compact logics more generally)
- How does the non-elementary model theorist cash in on this craze?

Pictoral outline

[See board]

Outline

- Colorings
- Indiscernibles
- Blueprints
- Applications
- Future Directions

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Ramsey's Theorem

Theorem (Ramsey)

For all $n, k, \ell < \omega$, there is $R < \omega$ such that

$$(R) \to (k)^n_\ell$$

Also, for all $n, \ell < \omega$,

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- ('Coloring unordered tuples' is the same as 'coloring tuples in increasing order')
- Statement about colorings of LinOrd at the finite/countable level
- Two possible generalizations:
  - What if we want to color structures other than linear orders?
  - What if we want *uncountable* homogeneous sets?

Ramsey Classes

Let $\mathcal{K}$ be a class of finite structures.

Definition

For $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \mathcal{K}$ *and* $\ell < \omega$, *we write*

$$(\mathfrak{C}) \to (\mathfrak{B})^{\mathfrak{A}}_{\ell}$$

iff whenever we color copies of $\mathfrak{A}$ *in* $\mathfrak{C}$ *with* ℓ *different colors, there is copy of* $\mathfrak{B}$ *in* $\mathfrak{C}$ *that is homogeneous for* $\mathfrak{C}$.

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Definition

$\mathcal{K}$ *is a Ramsey class iff for every* $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}$ *and* $\ell < \omega$, *there is some* $\mathfrak{C} \in \mathcal{K}$ *such that*

$$(\mathfrak{C}) \to (\mathfrak{B})^{\mathfrak{A}}_{\ell}$$

Ramsey Classes

Example

- Linear orders
- Linearly ordered structures in a relational language
- Trees
- Convexly ordered equivalence relations

- Study of Ramsey classes is a nice confluence of combinatorialists and model theorists (for reasons we will see later).

Erdős-Rado Theorem

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Theorem (Erdős-Rado)

For all $n < \omega$, $\alpha < \kappa$,

$$\beth_{n-1}(\kappa)^+ \to (\kappa^+)^n_\kappa$$

- This is a tight bound

Erdős-Rado Classes

- Want to jointly generalize both Ramsey classes and Erdős-Rado Theorem
- Need to formalize a new partition relation. Unlike Ramsey classes, we will want to color all n-tuples at once.
- Strange issues creep in:

Erdős-Rado Classes

Example

Let $\mathcal{K}^{2-or}$ be the class of two disjoint linear orders and let $(I_0, I_1) \in \mathcal{K}^{2-or}$.

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Let $\mathcal{K}^{2-or}$ be the class of two disjoint linear orders and let $(I_0, I_1) \in \mathcal{K}^{2-or}$.Take a coloring of pairs

$$c : [(I_0, I_1)]^2 \to 3$$

given by

$$c(i,j) = \begin{cases} 0 & i \in I_0 \iff i \in I_0 \\ 1 & \text{otherwise} \end{cases}$$

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Want a **big** part of both linear orders $(I_0^*, I_1^*) \subset (I_0, I_1)$ that is **homogeneous**.

- No large subset can be homogeneous because information about how the pairs relate to each other can be used to color them.

Erdős-Rado Classes: Types

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   - In general, can consider Galois type. Key is that same type means tuples can be sent to each other.
2. We always consider types over the empty set, so simplifies notation:

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   - In most examples of index classes, type will be determined by quantifier-free formulas or something nearby.
   - In general, can consider Galois type. Key is that same type means tuples can be sent to each other.
2. We always consider types over the empty set, so simplifies notation:
   1. If $I \in \mathcal{K}$ and $i_1, \ldots, i_n \in I$, the type of these elements is denoted
      $$tp_{\mathcal{K}}(i_1, \ldots, i_n; I)$$
   2. S($\mathcal{K}$) is the collection of all types realized in a model of $\mathcal{K}$
   3. (Formally, we want to really mod out by a permutation of the tuples, but our examples will have a linear order, so it's a wash)

Erdős-Rado Classes: examples

1. $\underline{\mathcal{K}^{or}}$: Linear orders
    - In fact, for each n, there is a unique n-type (up to permutation/increasing ordering)
2. $\underline{\mathcal{K}^{n-mlo}}$: n different linear orders
3. $\underline{\mathcal{K}^{\chi-or}}$: χ disjoint linear orders

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4. $\underline{\mathcal{K}^{n-tr}}$: Trees of height n (in the appropriate language)

Erdős-Rado Classes: bigness notions

- For LinOrd, size is measured by cardinality
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  - $(\omega, \sqsupset_{\omega_1})$ shouldn't be considered big in $\mathcal{K}^{2-or}$ because the first partition is small

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Definition

A bigness notion for $\mathcal{K}$ is a collection $\{\mathcal{K}_\alpha \mid \alpha \in ON\}$ with the following properties:

1. *If $I \subset J$ and $\alpha \leq \beta$, then*
$$I \in \mathcal{K}_\beta \implies J \in \mathcal{K}_\alpha$$
2. *Every I in $\mathcal{K}_\omega$ realizes all types in $S(\mathcal{K})$.*

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- (Not presentability rank, tends to be some kind of saturation...)

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   α-big means of cardinality at least α
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4. $\underline{\mathcal{K}^{n-tr}}$: Trees of height n (in the appropriate language)
   α-big means at least α-splitting at each terminal node

Erdős-Rado Classes: type homogeneity

Our example with $\mathcal{K}^{2-or}$ tells us we need a different notion of homogeneity for coloring that takes the type into account.

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Definition

Let $c : [I]^n \to \mu$ (with $I \in \mathcal{K}$). Then $I_0 \subset I$ is type-homogeneous (for c) iff there is a function $c^* : S(\mathcal{K}) \to \mu$ such that, for all $i_1, \ldots, i_n \in I$,

$$c\,(i_1, \ldots, i_n) = c^*\,(tp_{\mathcal{K}}(i_1, \ldots, i_n; I))$$

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There is no ω-big homogeneous subset for this coloring. However, the whole (I_0, I_1) is type-homogeneous.

- Generalizes our old notion of homogeneity sine LinOrd has exactly one type (up to permutation) of an n-tuple.

Erdős-Rado Classes: structural partition relation

Definition

Let $\mathcal{K}$ be a class of structures (with a notion of type and a bigness notions). For cardinals λ, κ, μ and $n < \omega$, we write

$$(\lambda) \xrightarrow{\mathcal{K}} (\kappa)^n_\mu$$

iff

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iff for every $I \in \mathcal{K}_\lambda$ and coloring $c : [I]^n \to \mu$, there is a type-homogeneous $I_0 \subset I$ with I_0 being κ-big.

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Note: This is much different than the Ramsey partition relation!!! The Ramsey partition relation specified the type of an n-tuple to color, but we want to color all n-tuples at once (polarized partition relations)

Combinatorial Erdős-Rado Classes

Definition

Let $\mathcal{K}$ be a class of structures. We say that $\mathcal{K}$ is a combinatorial Erdős-Rado class iff for every μ, κ and $n < \omega$, there is λ such that

$$(\lambda) \xrightarrow{\mathcal{K}} (\kappa)^n_\mu$$

Combinatorial Erdős-Rado Classes: examples

- (Note: These bounds tend to not be tight! But, not really worried about 'off by omega iterations of $\beth$'-errors)

1. $\underline{\mathcal{K}^{or}}$: Linear orders

$$\beth_{n-1}(\kappa)^+ \to (\kappa^+)^n_\kappa$$

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$$\beth_{n(n+1)}(\kappa)^+ \xrightarrow{\chi-or} (\kappa^+)^n_\kappa$$

(Erdős-Hajnal-Rado, Shelah)

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3. $\underline{\mathcal{K}^{n-tr}}$: Trees of height n (in the appropriate language)
   For $n, m < \omega$

$$\beth_{k(n,m)}(\kappa)^+ \xrightarrow{n-tr} (\kappa^+)^m_\kappa$$

   for some $k(n,m) < \omega$ (Shelah, Grossberg-Shelah)

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- Colorings
- Indiscernibles
- Blueprints
- Applications
- Future Directions

Order Indiscernibles

- Let T be a first order theory and $M \vDash T$

Definition

Given a linear order I, a sequence $\{\mathbf{a}_i \in M \mid i \in I\}$ is an indiscernible sequence iff for any $i_1 < \cdots < i_n; j_1 < \cdots < j_n \in I$ and first-order formula $\phi(\mathbf{x}_1, \ldots, \mathbf{x}_n)$, we have

$$M \vDash \phi\left(\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_n}\right) \leftrightarrow \phi\left(\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_n}\right)$$

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- Let's rephrase this definition a bit

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$$tp_{qf}(i_1, \ldots, i_n; I) = tp_{qf}(j_1, \ldots, j_n; I)$$

then

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This definition suggests two generalizations:

- What if we want to use categories other than LinOrd as the domain/index categories?
- What if we want to have indiscernibles in nonelementary classes $\mathbb{K}$ ($\mathbb{L}_{\infty,\omega}$-axiomatizable, Abstract Elementary Classes, etc.)

Generalized Indiscernibles in Elementary Classes

Definition

Let $\mathcal{K}$ be a class of structures and $I \in \mathcal{K}$. A sequence $\{\mathbf{a}_i \in M \mid i \in I\}$ is a $\mathcal{K}$-indiscernible sequence iff for any $i_1, \ldots, i_n; j_1, \ldots, j_n \in I$,
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Order Indiscernibles in Abstract Elementary Classes

Definition

Fix an AEC $\mathbb{K}$ *and* $M \in \mathbb{K}$.
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Generalized Indiscernibles in Abstact Elementary Classes

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- Okay, so these definitions were pretty straightforward, but for a given $\mathcal{K}$ and $\mathbb{K}$, how do we find them?

Poor notation

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- We are now dealing with two classes at once and, due to a lack of creativity, I call them all K
- 'Script $\mathcal{K}$' or 'calligraphic $\mathcal{K}$' will refer to index/domain categories that are Ramsey or Erdős-Rado that we would like to build indiscernibles based on.
- 'Bold $\mathbb{K}$' will be target categories that are elementary or AECs that we would like to build indiscernibles in.

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- Formally, a blueprint is a collection of quantifier-free types $p_n(x_1, \ldots, x_n)$ for each $n < \omega$ that cohere in a certain way
- $EM(I, \Phi)$ is the model built by insisting that, for every $i_1, \ldots, i_n \in I$,

$$tp_{qf}\left(i_1, \ldots, i_n; EM(I, \Phi)\right) = p_n$$

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  - If $\tau \subset \tau(\Phi)$, then $EM_\tau(I, \Phi) := EM(I, \Phi) \upharpoonright \tau$
- We say Φ is proper for a class $\mathbb{K}$ iff

$$EM_\tau(\cdot, \Phi) : \text{LinOrd} \to \mathbb{K}$$

  is a faithful functor (that preserves directed colimits)
- $\Upsilon^{or}_\kappa[\mathbb{K}]$ are all blueprints that are proper for $\mathbb{K}$ with $|\tau(\Phi)| \leq \kappa$.

Blueprints

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Blueprints

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  - Ramsey's Theorem!

Theorem (Folklore?)

Given infinite $\{\mathbf{a}_i \mid i \in I\} \subset M \vDash T$*, there is a blueprint* $\Phi \in \Upsilon^{or}_{|\tau|}[Mod\ T]$ *such that*

$$\textbf{if } \textit{for all } i_1 < \cdots < i_n \in I,\ M \vDash \phi(\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_n}), \textbf{ then } \phi(\mathbf{x}_1, \ldots, \mathbf{x}_n) \textit{ is in } \Phi$$

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  - Ramsey's Theorem!

Theorem (Folklore?)

Given infinite $\{\mathbf{a}_i \mid i \in I\} \subset M \vDash T$*, there is a blueprint* $\Phi \in \Upsilon^{or}_{|\tau|}[Mod\ T]$ *such that*

if *for all* $i_1 < \cdots < i_n \in I$, $M \vDash \phi(\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_n})$, **then** $\phi(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ *is in* Φ

- This extra bit is called *the (Ramsey) modeling property*.
  - If something happens everywhere in the given sequence, that something happens in the blueprint

Blueprints

- Where do blueprints come from?
  - Ramsey's Theorem!

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If $\Phi \in \Upsilon^{or}[Mod\ T]$*, then* $I \subset EM_\tau(I, \Phi)$ *are order-indiscernibles.*

Generalized blueprints

Let $\mathcal{K}$ be a class of structures.

- A $\mathcal{K}$-*blueprint* in the language τ is a function $\Phi : \mathsf{S}(\mathcal{K}) \to \mathsf{S}(\mathbb{K}^{\tau})$ such that

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- Have the same way of building *EM* models, properness, etc.
- $\Upsilon^{\mathcal{K}}[\mathbb{K}]$ is the collection of $\mathcal{K}$-blueprints proper for $\mathbb{K}$

Generalized Blueprints

- Where do generalized blueprints come from?

Generalized Blueprints

- Where do generalized blueprints come from?
  - Ramsey classes!

Theorem

Let $\mathcal{K}$ be a Ramsey class and $I \in \mathcal{K}$. Given infinite $\{\mathbf{a}_i \mid i \in I\} \subset M \vDash T$, there is a generalized blueprint $\Phi \in \Upsilon^{\mathcal{K}}_{|\tau|}[Mod\ T]$ such that

$$\text{for each } p \in S(\mathcal{K}), \textbf{ if } \text{for all } i_1, \ldots, i_n \in I \text{ realizing } p, \\ M \vDash \phi(\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_n}), \text{ then } \phi(\mathbf{x}_1, \ldots, \mathbf{x}_n) \text{ is in } \Phi$$

- Still have the Ramsey modeling property

Theorem

If $\Phi \in \Upsilon^{\mathcal{K}}[Mod\ T]$, then $I \subset EM_\tau(I, \Phi)$ are $\mathcal{K}$-indiscernibles.

Blueprints in nonelementary classes

- Want to talk about blueprints in nonelementary classes
- Maybe one way to do this would be to have blueprints output Galois types in the appropriate sense, but we don't do this.

Blueprints in nonelementary classes

- Want to talk about blueprints in nonelementary classes
- Maybe one way to do this would be to have blueprints output Galois types in the appropriate sense, but we don't do this.
- Instead, we strengthen the Ramsey modeling property to the *Erdős-Rado modeling property*

Blueprints in nonelementary classes

Definition

Given a blueprint $\Phi \in \Upsilon$ *and* $\{\mathbf{a}_i \in M \mid i \in I\}$*, we say that* Φ *has the Erdős-Rado modeling property iff for every* $n < \omega$*, there is* $i_1 < \cdots < i_n \in I$ *such that* $p_n = tp(\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_n}; M)$*.*

- This says that everything that happens in the blueprint happens somewhere in the given sequence

A translational interlude

- 1. $\mathbb{L}_{\infty,\omega}$-axiomatizable classes
  2. Finitely accessible categories
  3. Elementary classes + type omission
- 1. $\mathcal{K}$-blueprints
  2. Faithful functors from $\mathcal{K}$ that preserve directed colimits

Blueprints in nonelementary classes

- Given $\{\mathbf{a}_i \mid i \in I\} \subset M$ and a blueprint that is Erdős-Rado modeled off this, then every quantifier-free type realized in some $EM_{\tau(M)}(J, \Phi)$ is already realized in M.
  - A new type p would be realized by some term of $j_1, \dots, j_n \in J$. But the type of the j_i's is already realized in M.

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  - A new type p would be realized by some term of $j_1, \ldots, j_n \in J$. But the type of the j_i's is already realized in M.
- By our translation, this means that (after a proper Skolemization) blueprints with the Erdős-Rado modeling property are proper for AECs!!!

Ramsey modeling vs. Erdős-Rado modeling

- Ramsey modeling: If P is true of all of the starting sequence, P is true of the indiscernibles
- Erdős-Rado modeling: If P is true of the indiscernibles, then P is true of some of the starting sequence

Morley's Omitting Types Theorem

- In the spirit of quoting Morley's theorems in a way he never said them:

Theorem (Morley's Omitting Types Theorem)

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- (There's actually a better statement here dealing with many structures and sequences of length cofinal in μ, but I'm suppressig this)
- If $\mathbb{K}$ is an AEC, $M \in \mathbb{K}$, and M^* is the Skolemized/presented version, then applying MOTT to $\underline{M^*}$ gives a blueprint proper for $\mathbb{K}$

Indiscernibles in AECs

Theorem (Folklore in AECs = Shelah)

If $\Phi \in \Upsilon^{or}[\mathbb{K}]$*, then* $I \subset EM_\tau(I, \Phi)$ *are order-Galois indiscernibles.*

Generalized blueprints in AECs

- To get generalized blueprints in AECs, want to find blueprints with the Erdős-Rado modeling property
- Ramsey classes won't work, but this is precisely why the definition of combinatorial Erdős-Rado classes is what it is

Erdős-Rado Classes

Definition

Let $\mathcal{K}$ be a class of structures. We say that $\mathcal{K}$ is a combinatorial Erdős-Rado class iff for every μ, κ and $n < \omega$, there is λ such that

$$(\lambda) \xrightarrow{\mathcal{K}} (\kappa)^n_\mu$$

Definition

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$$\Phi(p) = tp_{qf}(\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_n}; I)$$

Erdős-Rado Classes

Theorem (Generalized Morley's Omitting Types Theorem, B.)

Combinatorial Erdős-Rado classes are Erdős-Rado classes

Proof: If you spend a lot of time looking at proof of the original and the definitions I jut gave, the proof falls out

- Start with $f_\alpha : I_\alpha \to M$ for $\alpha < \chi$ and $I_\alpha \in \mathcal{K}_\alpha$
- Want to build $f^n_\alpha : I^n_\alpha \to M$ and $\Phi_n : S^n(\mathcal{K}) \to S(M)$ such that, for all $i_1, \dots, i_n \in I^n_\alpha$,

$$tp_{qf}\left(f^n_\alpha(i_1), \dots, f^n_\alpha(i_n); M\right) = \Phi_n\left(tp_{\mathcal{K}}(i_1, \dots, i_n; I)\right)$$

- Stage $n = 0$ is given

Erdős-Rado Classes

- Want to build $f_\alpha^n : I_\alpha^n \to M$ and $\Phi_n : S^n(\mathcal{K}) \to S(M)$ such that, for all $i_1, \dots, i_n \in I_\alpha^n$,

$$tp_{qf}\left(f_\alpha^n(i_1), \dots, f_\alpha^n(i_n); M\right) = \Phi_n\left(tp_{\mathcal{K}}(i_1, \dots, i_n; I)\right)$$

- Given stage n and α, find $F(\alpha, n)$ for partition and color $[I^n_{F(\alpha,n)}]^{n+1}$ by the type of the tuple (after applying f_α^n) in M
- Partition gives $I_\alpha^{n+1} \subset I^n_{F(\alpha,n)}$ that are $n+1$-indiscernibles
- Might not be the same across α's, but have enough to thin out until it is
- Then $\Phi = \cup \Phi_n$. †

Erdős-Rado Classes

Theorem (Generalized Morley's Omitting Types Theorem, B.)

Combinatorial Erdős-Rado classes are Erdős-Rado classes

- Key: For every $n < \omega$, there are lots of n-indiscernibles. However, there is no (guarantee of an) infinite indiscernible sequences in M

Erdős-Rado Classes

There is more along these lines:

- As with Morley's original, there better statement is about examples cofinal in the threshold cardinal. Also, should be an 'undefinability of well-order' proof with potentially better bound
- Can also generalize Shelah's Omitting Types Theorem

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Outline

- Colorings
- Indiscernibles
- Blueprints
- Applications
    - Indiscernible Collapse
    - Interpretability Order
- Future Directions

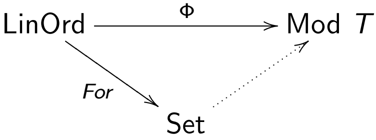
Indiscernible Collapse

- Many dividing lines in model theory can be characterized via indiscernible collapse:

Theorem (Shelah)

T is stable iff every indiscernible sequence in a model of T is actually an indiscernible set

- Categorically, this can be phrased as a lifting property of indiscernible functors along the forgetful functor to Set:



Indiscernible Collapse

- Scow and others (esp. Guingona, Hill, and Scow) have generalized this results based on other dividing lines

Fact

1. *T is NIP iff every ordered-graph indiscernible sequence is actually an order-indiscernible sequence (Scow)*
2. *For NIP theories, the op-dimension is $< n$ iff every $\mathcal{K}^{n-mlo}$-indiscernible sequence is actually a $\mathcal{K}^{(n-1)-mlo}$-indiscernible sequence*

Indiscernible Collapse

- In AECs, the existence of the order property is a bit more subtle:

Definition

$\mathbb{K}$ *has the order property iff there a 'definable' order of size at least* $\beth_{(2^{LS(\mathbb{K})})^+}$

- This is the right size because this allows us to find order-indiscernibles!!
- Having generalized indiscernibles then allows us to formulate dividing lines along the notions of collapse of indiscernibles

Interpretability order

- (In the context of first-order) the interpretability order is $T_0 \triangleleft_1^* T_1$ iff for all large enough regular λ, there is first-order theory T_* of size $\leq |T_0| + |T_1|$ such that
  1. T_* interprets T_ℓ by $\bar{\phi}_\ell$; and
  2. for every model $M_* \vDash T_*$, if $M_*^{\bar{\phi}_1}$ is λ-saturated, then $M_*^{\bar{\phi}_0}$ is λ-saturated.
- Kind of a variant of Keisler order, but doesn't depend on ultrafilters.
- Malliaris and Shelah show, among others,

$$\neg\left(T_{DLO} \triangleleft_1^* T_{RG}\right)$$

Interpretability order

Theorem (Malliaris, Shelah)

$\neg(T_{DLO} \triangleleft_1^* T_{RG})$

- Let T_* be a theory interpreting T_{DLO} and T_{RG}
- For every $\Phi \in \Upsilon^{\lambda-color}[T_*]$, there is $\Psi \geq \Phi$ such that, for any nice (separated) I, $EM(I, \Phi) \upharpoonright RG$ is λ-saturated.
- For every separated I with a (κ, κ)-cut and any Φ, $EM(I, \Phi) \upharpoonright DLO$ is not κ^+-saturated.
- So we can find a model M_* of T_* that satisfies our properties by taking $EM(I, \Psi)$ for I separated with a (κ, κ)-cut

Interpretability order

- Combining with our Erdős-Rado classes, we get stronger result.
- Say $T_0 \triangleleft_1^{*,\mu} T_1$ iff for all large enough regular λ, there is an $\mathbb{L}_{\mu,\omega}$-theory T_* of size $\leq |T_0| + |T_1|$ such that
  1. T_* interprets T_ℓ by $\bar{\phi}_\ell$; and
  2. for every model $M_* \vDash T_*$, if $M_*^{\bar{\phi}_1}$ is λ-saturated, then $M_*^{\bar{\phi}_0}$ is λ-saturated.

Proposition

For every μ,

$$\neg \left(T_{DLO} \triangleleft_1^{*,\mu} T_{RG} \right)$$

Proof: Same, but use that $\mathcal{K}^{\lambda-color}$ is an Erdős-Rado class.

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Compactness?

- I like compactness in nonelementary classes
- 'First' use of blueprints in AECs: compute the Hanf number
    - This is a sort of compactness for model existence

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- Later, used by Shelah [Sh394] to derive weak tameness from amalgamation and categoricity
    - The real result is something like:

Theorem

If $I \subset J$ are 'nice' linear order and $a, b \in EM_\tau(J, \Phi)$ are such that

for all $M_0 \prec EM_\tau(I, \Phi)$ of size $< \beth_\delta$, the types of a and b over M_0 are the same,

then the types of a and b over $EM_\tau(I, \Phi)$ are the same.

- Proof works from deriving a blueprint modeled on an augmented copy of $EM_\tau(I, \Phi)$

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  - Proof works from deriving a blueprint modeled on an augmented copy of $EM_\tau(I, \Phi)$
- So blueprints have historically been used to get compactness in nonelementary classes. Are there other tricks? Is there a general principle underlying them?

Large Cardinals?

- The regular partition relation gives rise to various large cardinals: Ramsey, η-Erdős, etc.

$$\kappa \text{ is Ramsey iff } (\kappa) \to (\kappa)^{<\omega}_{<\kappa}$$

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- Clearly implies Ramsey, but haven't been able to establish an upper bound.
- Moreover, the above cardinals are important because of their connection to $0^{\#}$. Is there a similar construction for, e.g., Ramsey-for-$\mathcal{K}^{2-or}$?

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- Ramsey classes are very rich: many examples, nice necessary and suficient criteria, etc.
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Question

Are $\mathcal{K}^{\omega+1-tr}$ an Erdős-Rado class?

- A positive answer would give an indiscernible collapse/undefinability of [certain combinatorial object] characterization of superstable AECs (cf. Grossberg-Vasey)

Thanks!!

Any questions?